B.A. SHOIMQULOV

T.T. TUYCHIYEV

D.H. DJUMABOYEV

MATEMATIK ANALIZDAN MUSTAQIL ISHLAR

O'ZBEKISTON RESPUBLIKASI OLIY VA
O'RTA MAXSUS TA'LIM VAZIRLIGI

B.A. SHOIMQULOV
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Oliy o'quv yurtlarining bakalavriat ta'lim
yo'nalishi talabalari uchun o'quv qo'llanma

«O'ZBEKISTON FAYLASUFLARI MILLIY JAMIYATI» NASHRIYOTI
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Qo'llanma matematik analiz fani chuqur o'rganiladigan universitetlarning talabalari tomonidan mustaqil ishlarni bajarish uchun mo'ljallangan bo'lib, u bakalavriatning «Matematika», «Tatbiqiy matematika va informatika» va «Mexanika» yo'nalishlari Davlat ta'lim standartlariga mos keladi.

Qo'llanma ketma-ketlik va funksiya limiti, bir o'zgaruvchili funksiyaning differensial va integral hisobi, ko'p o'zgaruvchili funksiyaning limiti, uzluksizligi va differensial hisobi, sonli qatorlar, funksional ketma-ketlik va qatorlar, xosmas integrallar, parametrga bog'liq xos hamda xosmas integrallar, karrali integrallar, egri chiziqli va sirt integrallari, maydonlar nazariyasi elementlari va Furye qatorlari mavzularini o'z ichiga oladi. Qo'llanmada 8 ta mustaqil ish, 2667 ta misol va masalalar keltirilgan bo'lib, ulardan 127 tasi batafsil yechim bilan ta'minlangan.

M u a l l i f l a r:
fizika-matematika fanlari doktori, **dots. B.A. Shoimqulov**
fizika-matematika fanlari nomzodi, **dots. T.T. Tuychiyev**
fizika-matematika fanlari nomzodi **D.H. Djumaboyev**

T a q r i z c h i l a r:
fizika-matematika fanlari nomzodi, **dots. Q. Ataxonov**
fizika-matematika fanlari nomzodi, **dots. A. Karimov**

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I. Tuychiyev Tohir Tursunboyevich.
II. Djumaboyev Davlatboy Xalillayevich.

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MUNDARIJA

SO‘ZBOSHI

Respublikamizda kadrlar tayyorlash Milliy dasturining birinchi (1997-2001 yillar) va ikkinchi bosqichlari (2001-2005-yillar) yakunlandi. O‘tgan vaqt mobaynida Respublika Oliy ta’limi tizimida katta o‘zgarishlar bo‘ldi, xususan, yangi Davlat ta’lim standartlari ishlab chiqildi va tasdiqlandi. Ilm-fan jadal taraqqiy etayotgan, zamonaviy axborot-kommunikatsiya tizimlari vositalari keng joriy etilgan jamiyatda turli fan sohalarida bilimlarning tez yangilanib borishi, ta’lim oluvchilar oldiga ularni jadal egallash bilan bir qatorda, muntazam va mustaqil ravishda bilim olish vazifasini qo‘ymoqda.

Qabul qilingan yangi Davlat ta’lim standartlarida ilg‘or chet el oliy ta’lim muassasalarida keng qo‘llaniladigan va yaxshi samara beradigan mustaqil ta’lim olish usuliga asosiy e’tibor qaratildi. Talabalarda o‘quv adabiyotini mustaqil o‘rganish va undan foydalana bilish malakalarini hosil qilish, mantiqiy fikrlashni o‘stirish va matematikaviy madaniyatning umumiy saviyasini ko‘tarish, tatbiqiy masalalarni matematikaviy tomondan tekshirish malakalarini hosil qilish va bu masalalarni matematikaviy tilda ifodalashga o‘rgatish maqsadida o‘quv dasturlariga matematik analiz fanidan mustaqil ishlar kiritildi va o‘quv rejasida ularga mos soatlar ajratildi.

Ushbu qo‘llanma matematik analiz fani chuqur o‘rganiladigan universitetlarning talabalari tomonidan mustaqil ishlarni bajarishga mo‘ljallangan bo‘lib, u bakalavriatning «Matematika», «Tatbiqiy matematika va informatika» va «Mexanika» yo‘nalishlari Davlat ta`lim standartlariga mos keladi.

Qo‘llanma to‘qqiz paragrafdan iborat bo‘lib, 1-§ da matematik analiz fanidan mustaqil ishlarni bajarish jarayonida kerak bo‘ladigan asosiy formula va qoidalar keltirilgan. Qolgan paragraflarda esa «Ketma-ketlik va funksiya limiti», «Funksiya hosilasi va differensiali, ularning tatbiqlari», «Aniqmas va aniq integrallar, ularning tatbiqlari», «Ko‘p o‘zgaruvchili funksiyalar», «Sonli qatorlar», «Funksional ketma-ketliklar va qatorlar», «Xosmas va parametrga bog‘liq integrallar» va «Karrali va egri chiziqli integrallar, Sirt integrallari va maydonlar nazariyasi elementlari, Furye qatorlari» mavzulari bo‘yicha 8 ta mustaqil ish tavsiya etilgan. Har bir mustaqil ishni berishdan avval shu mustaqil ishni muvaffaqiyatli bajarish uchun lozim bo‘ladigan asosiy tushuncha va tasdiqlar keltirilgan (A bo‘lim). B bo‘limda talaba bajarishi va keyin topshirishi lozim bo‘lgan 21 ta variantdan iborat mustaqil ish vazifalari tavsiya qilingan. D bo‘limda esa talabaning mustaqil ishni bajarishini va undagi materialni o‘zlashtirishini yengillashtirish maqsadida 1 ta variantdagi (21-variant) barcha misol va masalalar to‘liq yechib ko‘rsatilgan.

Qo‘llanmani tayyorlashda mualliflar tomonidan mavzularning oddiy va sodda tilda, tushunarli va ravon bayon etilishiga, faqat zarur, lekin fanni malakali tushunish uchun yetarli ma’lumotlarni berishga, Mirzo Ulug‘bek nomidagi O‘zbekiston Milliy universiteti Mexanika-matematika fakultetida matematik analiz fanining o‘qitilishi jarayonida yig‘ilgan tajribalardan imkon darajasida to‘liq foydalanishga harakat qilindi. Shu munosabat bilan mualliflar o‘quv qo‘llanma talabalarda bilim olishga intilish hissi, mustaqil fikrlash malakalarining shakllanishiga xizmat qiladi, deb umid bildiradilar.

1-§. ASOSIY FORMULA VA QOIDALAR

1⁰. Qisqa ko'paytirish formulalari va Nyuton binomi

1. $(a\pm b)^2 = a^2 \pm 2ab + b^2$.

2. $(a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

3. $(a\pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$.

4. $a^2 - b^2 = (a-b)(a+b)$.

5. $a^3 \pm b^3 = (a\pm b)(a^2 \mp ab + b^2)$.

6. $(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$;

bu yerda $C_n^k = \dfrac{n!}{k!\cdot(n-k)!}$, $n! = 1\cdot 2\cdot \ldots \cdot n$ va $0! = 1$.

7. $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k} b^k = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} =$

$= (a-b)\cdot\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}\right)$ **bu yerda** $n \in N$, $n > 1$.

8. $a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \ldots + b^{n-1}\right)$, **bu yerda** $n-1$ dan katta bo'lgan ixtiyoriy toq natural son.

2⁰. Daraja va ildizning xossalari. Logarifmlar

1. $a^0 = 1$, $a^x \cdot a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$, $(a\cdot b)^x = a^x \cdot b^x$, $a^{-x} = \dfrac{1}{a^x}$

2. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, $\sqrt[m]{ab} = \sqrt[m]{a}\cdot\sqrt[m]{b}$, $\sqrt[m]{\dfrac{a}{b}} = \dfrac{\sqrt[m]{a}}{\sqrt[m]{b}}$, $\left(\sqrt[m]{a}\right)^k = \sqrt[m]{a^k}$,

$$\sqrt[k]{\sqrt[m]{a}} = \sqrt[km]{a}, \quad \sqrt[km]{a^m} = \sqrt[k]{a}.$$

3. $\sqrt[2m]{a^{2m}} = |a| = \begin{cases} a, & \text{agar } a \ge 0 \text{ bo'lsa,} \\ -a, & \text{agar } a \le 0 \text{ bo'lsa.} \end{cases}$ $\sqrt[2m+1]{-a} = -\sqrt[2m+1]{a}$, agar $a \ge 0$ bo'lsa;

$\sqrt[m]{a} < \sqrt[m]{b}$, agar $0 \le a < b$ bo'lsa.

4. Ixtiyoriy x uchun $a^x > 0$;

$$a^x = a^y \Leftrightarrow x = y.$$

5. $a^x > a^y \Leftrightarrow \begin{cases} x > y, & \text{agar } a > 1 \text{ bo‘lsa,} \\ x < y, & \text{agar } 0 < a < 1 \text{ bo‘lsa.} \end{cases}$

6. $(x > 0,\ a > 0,\ y > 0,\ b > 0)$. $b = a^{\log_a b}$, $\log_x x = 1$, $\log_x 1 = 0$,

$\log_b(xy) = \log_b x + \log_b y$, $\log_b \frac{x}{y} = \log_b x - \log_b y$, $\log_{b^k} x^m = \frac{m}{k}\log_b x$,

$$\log_a x = \frac{\log_b x}{\log_b a};\quad \log_a b = \frac{1}{\log_b a}$$

3⁰. Trigonometrik funksiyalar va trigonometriya formulalari

1. Trigonometrik funksiyalarning ishoralari

	$\sin x$	$\cos x$	$\operatorname{tg} x$	$\operatorname{ctg} x$
$0 < x < \frac{\pi}{2}$	+	+	+	+
$\frac{\pi}{2} < x < \pi$	+	–	–	–
$\pi < x < \frac{3\pi}{2}$	–	–	+	+
$\frac{3\pi}{2} < x < 2\pi$	–	+	–	–

2. Trigonometrik funksiyalarning ba’zi bir burchaklardagi qiymatlari

Radianlar	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Graduslar	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
$\operatorname{tg} x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–	0	–	0
$\operatorname{ctg} x$	–	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	–	0	–

3. Asosiy trigonometrik ayniyatlar

1. $\sin^2 x+\cos^2 x=1$.
2. $\operatorname{tg} x=\dfrac{\sin x}{\cos x}$, $\left(x\neq\dfrac{\pi}{2}+\pi n\right)$.
3. $\operatorname{ctg} x=\dfrac{\cos x}{\sin x}$, $(x\neq\pi n)$.
4. $\operatorname{tg} x\cdot\operatorname{ctg} x=1$, $\left(x\neq\dfrac{\pi n}{2}\right)$.
5. $1+\operatorname{tg}^2 x=\dfrac{1}{\cos^2 x}$, $\left(x\neq\dfrac{\pi}{2}+\pi n\right)$.
6. $1+\operatorname{ctg}^2 x=\dfrac{1}{\sin^2 x}$, $(x\neq\pi n)$, $(n\in Z)$.

4. Keltirish formulalari

$y\to$	$\dfrac{\pi}{2}\pm x$	$\pi\pm x$	$\dfrac{3\pi}{2}\pm x$	$2\pi\pm x$
$\sin y$	$\cos x$	$\mp\sin x$	$-\cos x$	$\pm\sin x$
$\cos y$	$\mp\sin x$	$-\cos x$	$\pm\sin x$	$\cos x$
$\operatorname{tg} y$	$\mp\operatorname{ctg} x$	$\pm\operatorname{tg} x$	$\operatorname{ctg} x$	$\pm\operatorname{tg} x$
$\operatorname{ctg} y$	$\mp\operatorname{tg} x$	$\pm\operatorname{ctg} x$	$\operatorname{tg} x$	$\pm\operatorname{ctg} x$

5. Burchak yig‘indisi va ayirmasi uchun formulalar

1. $\sin(x\pm y)=\sin x\cdot\cos y\pm\cos x\cdot\sin y$.
3. $\operatorname{tg}(x\pm y)=\dfrac{\operatorname{tg} x\pm\operatorname{tg} y}{1\mp\operatorname{tg} x\cdot\operatorname{tg} y}$.
2. $\cos(x\pm y)=\cos x\cdot\cos y\mp\sin x\cdot\sin y$
4. $\operatorname{ctg}(x\pm y)=\dfrac{\operatorname{ctg} x\cdot\operatorname{ctg} y\pm 1}{\operatorname{ctg} y\pm\operatorname{ctg} x}$.

6. Ikkilangan va karrali burchak uchun formulalar

1. $\sin 2x=2\sin x\cdot\cos x=\dfrac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x}$.
2. $\cos 2x=\cos^2 x-\sin^2 x=2\cos^2 x-1=1-2\sin^2 x=\dfrac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$.
3. $\operatorname{tg} 2x=\dfrac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}=\dfrac{2}{\operatorname{ctg} x-\operatorname{tg} x}$.
4. $\operatorname{ctg} 2x=\dfrac{\operatorname{ctg}^2 x-1}{2\operatorname{ctg} x}=\dfrac{\operatorname{ctg} x-\operatorname{tg} x}{2}$.
5. $\sin 3x=3\sin x-4\sin^3 x$.
6. $\cos 3x=4\cos^3 x-3\cos x$.

7. $\operatorname{tg}3x=\dfrac{3\operatorname{tg}x-\operatorname{tg}^3x}{1-3\operatorname{tg}^2x}$. 8. $\operatorname{ctg}3x=\dfrac{\operatorname{ctg}^3x-3\operatorname{ctg}x}{3\operatorname{ctg}^2x-1}$.

7. Yarim burchak uchun formulalar

1. $\sin\dfrac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{2}}$. 3. $tg\dfrac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}=\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$.

2. $\cos\dfrac{x}{2}=\pm\sqrt{\dfrac{1+\cos x}{2}}$. 4. $ctg\dfrac{x}{2}=\pm\sqrt{\dfrac{1+\cos x}{1-\cos x}}=\dfrac{\sin x}{1-\cos x}=\dfrac{1+\cos x}{\sin x}$.

Izoh: Tengliklardagi "+" yoki "–" ishora $\dfrac{x}{2}$ burchakning qaysi chorakda joylashganligiga qarab tanlanadi.

8. Trigonometrik funksiyalarning darajalari uchun formulalar

1. $\sin^2 x=\dfrac{1-\cos 2x}{2}$. 2. $\cos^2 x=\dfrac{1+\cos 2x}{2}$.

3. $\sin^3 x=\dfrac{3\sin x-\sin 3x}{4}$. 4. $\cos^3 x=\dfrac{3\cos x+\cos 3x}{4}$.

9. Trigonometrik funksiyalarning yig'indi va ayirmalari uchun formulalar

1. $\sin x+\sin y=2\sin\dfrac{x+y}{2}\cdot\cos\dfrac{x-y}{2}$. 2. $\sin x-\sin y=2\sin\dfrac{x-y}{2}\cdot\cos\dfrac{x+y}{2}$.

3. $\cos x+\cos y=2\cos\dfrac{x+y}{2}\cdot\cos\dfrac{x-y}{2}$. 4. $\cos x-\cos y=-2\sin\dfrac{x+y}{2}\cdot\sin\dfrac{x-y}{2}$.

5. $\cos x\pm\sin x=\sqrt{2}\sin\left(\dfrac{\pi}{4}\pm x\right)=\sqrt{2}\cos\left(\dfrac{\pi}{4}\mp x\right)$.

6. $A\cdot\cos x\_+B\sin x=\sqrt{A^2+B^2}\sin(x+y)$,

bu yerda $A^2+B^2\neq 0$, $\sin y=\dfrac{A}{\sqrt{A^2+B^2}}$, $\cos y=\dfrac{B}{\sqrt{A^2+B^2}}$,

7. $\operatorname{tg}x\pm\operatorname{tg}y=\dfrac{\sin(x\pm y)}{\cos x\cdot\cos y}$. 8. $\operatorname{ctg}x\pm\operatorname{ctg}y=\dfrac{\sin(x\pm y)}{\sin x\cdot\sin y}$.

9. $\operatorname{tg}x+\operatorname{ctg}y=\dfrac{\cos(x-y)}{\cos x\cdot\sin y}$. 10. $\operatorname{ctg}x-\operatorname{tg}y=\dfrac{\cos(x+y)}{\sin x\cdot\cos y}$.

10. Trigonometrik funksiyalarning ko'paytmalari uchun formulalar

1. $\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$.

2. $\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$.

3. $\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$.

4. $\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$.

5. $tgx \cdot tgy = \frac{tgx + tgy}{ctgx + ctgy}$. 6. $ctgx \cdot ctgy = \frac{ctgx + ctgy}{tgx + tgy}$.

7. $\sin(x+y) \cdot \sin(x-y) = \cos^2 y - \cos^2 x$.

8. $\cos(x-y) \cdot \cos(x+y) = \cos^2 y - \sin^2 x$.

Izoh: Yuqorida keltirilgan ayniyatlar va formulalar tenglikning har ikkala tomoni ma'noga ega bo'lgan qiymatlarida o'rinli bo'ladi.

4⁰. Teskari trigonometrik funksiyalar

1. $y = \arcsin x$.

$$D(y) = [-1;\ 1],\quad E(y) = \left[-\frac{\pi}{2}; \frac{\pi}{2}\right],\quad f(-x) = -f(x).$$

2. $y = \arccos x$.

$$D(y) = [-1;\ 1],\quad E(y) = [0; \pi],\quad \arccos(-x) = \pi - \arccos x.$$

3. $y = \mathrm{arctg}\, x$.

$$D(y) = (-\infty; +\infty),\quad E(y) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right),\quad f(-x) = -f(x).$$

4. $y = \mathrm{arcctg}\, x$.

$$D(y) = (-\infty; +\infty),\quad E(y) = (0; \pi),\quad \mathrm{arcctg}(-x) = \pi - \mathrm{arcctg}\, x.$$

5⁰. Trigonometrik tenglamalar

1. $\sin x = a \Rightarrow \begin{cases} |a| > 1 \Rightarrow x \in \varnothing \\ |a| \le 1 \Rightarrow x = (-1)^k \arcsin a + \pi k, \end{cases}$

bu yerda $k \in Z$ va $-\frac{\pi}{2} \le \arcsin a \le \frac{\pi}{2}$.

2. $\cos x = a \Rightarrow \begin{cases} |a| > 1 \Rightarrow x \in \varnothing \\ |a| \le 1 \Rightarrow x = \pm\arccos a + 2\pi k, \end{cases}$ bu yerda $0 \le \arccos a \le \pi$.

3. $\operatorname{tg} x = a \Rightarrow x = \operatorname{arctg} a + \pi k$, bu yerda $\operatorname{arctg} a \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ va $a \in R$.

4. $\operatorname{ctg} x = a \Rightarrow x = \operatorname{arcctg} a + \pi k$, bu yerda $\operatorname{arcctg} a \in (0; \pi)$ va $a \in R$.

6⁰. Eng sodda trigonometrik tenglamalar yechimlari jadvali $(k \in Z)$

a	$\sin x = a$	$\cos x = a$
0	$x = \pi k$	$x = \frac{\pi}{2} + \pi k$
1	$x = \frac{\pi}{2} + 2\pi k$	$x = 2\pi k$
-1	$x = -\frac{\pi}{2} + 2\pi k$	$x = \pi + 2\pi k$
$\frac{1}{2}$	$x = (-1)^k \frac{\pi}{6} + \pi k$	$x = \pm\frac{\pi}{3} + 2\pi k$
$-\frac{1}{2}$	$x = (-1)^{k+1} \frac{\pi}{6} + \pi k$	$x = \pm\frac{2\pi}{3} + 2\pi k$
$\frac{\sqrt{3}}{2}$	$x = (-1)^k \frac{\pi}{3} + \pi k$	$x = \pm\frac{\pi}{6} + 2\pi k$
$-\frac{\sqrt{3}}{2}$	$x = (-1)^{k+1} \frac{\pi}{3} + \pi k$	$x = \pm\frac{5\pi}{6} + 2\pi k$
$\frac{\sqrt{2}}{2}$	$x = (-1)^k \frac{\pi}{4} + \pi k$	$x = \pm\frac{\pi}{4} + 2\pi k$
$-\frac{\sqrt{2}}{2}$	$x = (-1)^{k+1} \frac{\pi}{4} + \pi k$	$x = \pm\frac{3\pi}{4} + 2\pi k$

a	$\operatorname{tg} x = a$	$\operatorname{ctg} x = a$
0	$x = \pi k$	$x = \frac{\pi}{2} + \pi k$
1	$x = \frac{\pi}{4} + \pi k$	$x = \frac{\pi}{4} + \pi k$
-1	$x = -\frac{\pi}{4} + \pi k$	$x = \frac{3\pi}{4} + \pi k$
$\sqrt{3}$	$x = \frac{\pi}{3} + \pi k$	$x = \frac{\pi}{6} + \pi k$
$-\sqrt{3}$	$x = -\frac{\pi}{3} + \pi k$	$x = \frac{5\pi}{6} + \pi k$
$\frac{\sqrt{3}}{3}$	$x = \frac{\pi}{6} + \pi k$	$x = \frac{\pi}{3} + \pi k$
$-\frac{\sqrt{3}}{3}$	$x = -\frac{\pi}{6} + \pi k$	$x = \frac{2\pi}{3} + \pi k$

7⁰. Giperbolik funksiyalar

1. $\operatorname{sh} x := \frac{e^x - e^{-x}}{2}$.

2. $\operatorname{ch} x := \frac{e^x + e^{-x}}{2}$.

3. $\operatorname{th} x := \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

4. $\operatorname{cth} x := \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

5. $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$.

6. $\operatorname{sh}^2 x = 2\operatorname{sh} x \cdot \operatorname{ch} x$.

7. $\operatorname{ch}^2 x = \operatorname{ch}^2 x + \operatorname{sh}^2 x$.

8. $\operatorname{th} x \cdot \operatorname{cth} x = 1$.

8⁰. Arifmetik progressiya

$\{a_n\} : a_1, a_2, \dots a_n, \dots$ – arifmetik progressiya $\Leftrightarrow$ $\forall n \in N$ uchun $a_{n+1} = a_n + d$ (d –ayirma).

1. $a_{n+1} = a_n + d$.

2. $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ $(n > 1)$.

3. $a_n = a_1 + (n-1)d$.

4. $a_n = a_k + d \cdot (n-k)$ $(1 \le k \le n-1)$.

5. $a_n=\dfrac{a_{n-k}+a_{n+k}}{2}$, $(1\le k\le n-1)$. 6. $a_n+a_m=a_k+a_p$, **agar** $n+m=k+p$ **bo'lsa.**

7. $S_n=a_1+a_2+\ldots+a_n=\dfrac{a_1+a_n}{2}\cdot n=\dfrac{2a_1+d(n-1)}{2}\cdot n$.

9⁰. Geometrik progressiya

$\{b_n\}$: $b_1,b_2,\ldots,b_n,\ldots$ $(b_1\ne 0)$ – geometrik progressiya $\Leftrightarrow$ $\forall n\in N$ uchun $b_{n+1}=b_n\cdot q$ (q –maxraj).

1. $b_{n+1}=b_n\cdot q$.
2. $b_n^{\,2}=b_{n-1}\cdot b_{n+1}$, $n>1$.
3. $b_n=b_1\cdot q^{n-1}$.
4. $b_n=b_k\cdot q^{n-k}$ $(1\le k\le n-1)$.
5. $b_n=b_{n-k}\cdot q^k$, $(1\le k\le n-1)$.
6. $b_{n+k}=b_n\cdot q^k$.
7. $b_n^2=b_{n-k}\cdot b_{n+k}$, $(1\le k\le n-1)$.
8. $b_n\cdot b_m=b_k\cdot b_p$, **agar** $n+m=k+p$ **bo'lsa.**
9. $S_n=b_1+b_2+\ldots b_n=\begin{cases} b_1\cdot\dfrac{1-q^n}{1-q}=\dfrac{b_nq-b_1}{q-1}, & q\ne 1\\ b_1\cdot n, & q=1\end{cases}$
10. $S=\lim\limits_{n\to\infty}S_n=\dfrac{b_1}{1-q}$, **agar** $0<|q|<1$ **bo'lsa.**

10⁰. Tenglamalar

1. Chiziqli tenglama $ax=b$:

a) agar $a\ne 0$ bo'lsa, yagona $x=\dfrac{b}{a}$ yechimga ega;

b) agar $a=0$, $b\ne 0$ bo'lsa, yechimga ega emas;

d) agar $a=b=0$ bo'lsa, cheksiz ko'p yechimga ega. Bu holda ixtiyoriy x tenglamaning yechimi bo'ladi.

2. Chiziqli tenglamalar sistemasi

$$\begin{cases} a_1x+b_1y=c_1\\ a_2x+b_2y=c_2\end{cases}$$

Aytaylik, $\Delta=a_1b_2-a_2b_1$, $\Delta x=b_2c_1-b_1c_2$ va $\Delta y=a_1c_2-a_2c_1$ bo'lsin.

a) agar $\Delta\ne 0$ bo'lsa, yagona $x=\dfrac{\Delta x}{\Delta}$, $y=\dfrac{\Delta y}{\Delta}$ yechimga ega;

b) agar $\Delta=0$ bo'lib, Δx va Δy lardan birortasi 0 dan farqli bo'lsa, yechimga ega emas;

d) agar $\Delta = \Delta x = \Delta y = 0$ bo‘lsa, cheksiz ko‘p yechimga ega.

e) Geometrik talqini: $ax + by = c$ tenglama tekislikda **to‘g‘ri chiziqni** aniqlaydi. $\Delta \neq 0$ shart ikkita to‘g‘ri chiziqning kesishishini va ularning yagona umumiy nuqtaga ega bo‘lishini bildiradi. b) dagi shart ikkita to‘g‘ri chiziqlarning **parallel** bo‘lib, ularning umumiy nuqtaga ega bo‘lmasligini anglatadi. Va nihoyat, $\Delta = \Delta x = \Delta y = 0$ shart ikkita to‘g‘ri chiziqning **ustma-ust** tushishini va ularning cheksiz umumiy nuqtaga ega bo‘lishini bildiradi.

3. Kvadrat tenglama $ax^2 + bx + c = 0$: $D = b^2 - 4ac$ bo‘lsin.

a) $a = 0$ bo‘lsa kvadrat tenglama yuqorida ko‘rilgan chiziqli tenglamaga aylanadi; $a \neq 0$ bo‘lib,

b) $D = 0$ bo‘lsa, yagona $x = -\dfrac{b}{2a}$ yechimga ega;

d) $D > 0$ bo‘lsa, ikkita $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$ yechimga ega;

e) $D < 0$ bo‘lsa, yechimga ega emas.

4. Viyet teoremasi:

a) agar x_1 va x_2 lar $x^2 + px + q = 0$ tenglamaning yechimi bo‘lsa, unda

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}$$

bo‘ladi;

b) agar x_1, x_2 va x_3 lar $x^3 + px^2 + qx + r = 0$ tenglamaning yechimi bo‘lsa, unda

$$\begin{cases} x_1 + x_2 + x_3 = -p, \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 x_3 = q \\ x_1 \cdot x_2 \cdot x_3 = -r \end{cases}$$

bo‘ladi.

5. $a^x = b$, $a > 0$ **tenglama**

a) $b > 0$ bo‘lganda $x = \log_a b$ yechimga ega;

b) $b < 0$ bo‘lganda yechimga ega emas.

6. $\log_a x = b$ **tenglama** $a > 0$ **bo‘lganda** $x = a^b$ **yechimga ega.**

7. Trigonometrik tenglamalar:

n—ixtiyoriy butun son bo‘lsin.

a) $\sin x = a$ tenglama $|a| \leq 1$ bo‘lganda $x = (-1)^n \arcsin a + n\pi$ yechimga ega, $|a| > 1$ bo‘lganda esa yechimga ega emas;

b) $\cos x = a$ tenglama $|a| \le 1$ bo‘lganda $x = \pm\arccos a + 2n\pi$ yechimga ega, $|a| > 1$ bo‘lganda esa yechimga ega emas;

d) $\operatorname{tg} x = a$ tenglamaning yechimi $x = \operatorname{arctg} a + n\pi$ bo‘ladi.

e) $\operatorname{ctg} x = a$ tenglamaning yechimi $x = \operatorname{arcctg} a + n\pi$ bo‘ladi.

11°. Tengsizliklar

1. Tengsizliklarning xossalari:

a) $a \ge b \Leftrightarrow$ ixtiyoriy c uchun $a + c \ge b + c$;

b) $a \ge b$ va $c \ge d \Rightarrow a + c \ge b + d$;

d) $a \ge b$ va $c > 0 \Rightarrow ac \ge bc$;

e) $a \ge b$ va $c < 0 \Rightarrow ac \le bc$.

2. Chiziqli tengsizlik $ax > b$**:**

a) $a = 0$ va $b \ge 0$ bo‘lsa, yechimga ega emas;

b) $a = 0$ va $b < 0$ bo‘lsa, $x \in (-\infty; +\infty)$ bo‘ladi;

d) $a > 0$ bo‘lsa, $x \in \left(\frac{b}{a}; +\infty\right)$ va $a < 0$ bo‘lsa, $x \in \left(-\infty; \frac{b}{a}\right)$ bo‘ladi.

3. $ax < b$ **tengsizlik** (-1) ga ko‘paytirilish yordamida $-ax > -b$ tengsizlikka keltiriladi.

4. Kvadrat tengsizlik $ax^2 + bx + c > 0$: $D = b^2 - 4ac$ bo‘lsin.

a) $a = 0$ bo‘lsa kvadrat tengsizlik chiziqli tengsizlikka aylanadi;

b) $a < 0$ bo‘lib, $D \le 0$ bo‘lsa yechimga ega emas;

d) $a < 0$ bo‘lib, $D > 0$ bo‘lsa, $x \in \left(\frac{-b + \sqrt{D}}{2a}; \frac{-b - \sqrt{D}}{2a}\right)$ bo‘ladi;

e) $a > 0$ bo‘lib, $D \ge 0$ bo‘lsa,

$$x \in \left(-\infty; \frac{-b - \sqrt{D}}{2a}\right) \cup \left(\frac{-b + \sqrt{D}}{2a}; +\infty\right) \text{ bo‘ladi;}$$

f) $a > 0$ va $D > 0$ bo‘lsa, $x \in (-\infty; +\infty)$ bo‘ladi.

5. $ax^2 + bx + c < 0$ **kvadrat tengsizlik** (-1) ga ko‘paytirilish yordamida $-ax^2 - bx - c > 0$ tengsizlikka keltiriladi.

6. a) $a > 1$ bo‘lganda $a^{f(x)} > a^{g(x)}$ tengsizlik $f(x) > g(x)$ tengsizlikka teng kuchli;

b) $0<a<1$ bo'lganda $a^{f(x)}>a^{g(x)}$ tengsizlik $f(x)<g(x)$ tengsizlikka teng kuchli.

7. a) $b>1$ bo'lganda $\log_b f(x)>\log_b g(x)$ tengsizlik $f(x)>g(x)>0$ tengsizlikka ekvivalent;

b) $0<b<1$ bo'lganda $\log_b f(x)>\log_b g(x)$ tengsizlik $0<f(x)<g(x)$ tengsizlikka ekvivalent.

8. Ratsional tengsizliklar intervallar usuli yordamida yechiladi: ratsional kasr surat va maxrajining barcha ildizlari butun sonlar o'qini intervallarga ajratadi. Har bir intervalda ratsional kasr o'z ishorasini o'zgartirmaydi. Kerakli intervallar tekshirish yordamida topiladi.

9. Trigonometrik tengsizliklar:

a) $\sin x>a$ **tengsizlik**:

1) $a\geq 1$ bo'lsa, yechimga ega emas;

2) $a<-1$ bo'lsa, $x\in(-\infty;\ +\infty)$ bo'ladi;

3) $-1\leq a<1$ bo'lganda, $x\in(\arcsin a+2n\pi;\ \pi-\arcsin a+2n\pi)$ bo'ladi.

b) $\sin x<a$ **tengsizlik:**

1) $a\leq -1$ bo'lganda yechimga ega emas;

2) $a>1$ bo'lsa, $x\in(-\infty;\ +\infty)$ bo'ladi;

3) $-1<a\leq 1$ bo'lsa, $x\in(-\pi-\arcsin a+2n\pi;\ \arcsin a+2n\pi)$ bo'ladi.

d) $\cos x>a$ **tengsizlik:**

1) $a\geq 1$ bo'lganda yechimga ega emas;

2) $a<-1$ bo'lsa, $x\in(-\infty;\ +\infty)$ bo'ladi;

3) $-1\leq a<1$ bo'lganda, $x\in(-\arccos a+2n\pi;\ \arccos a+2n\pi)$ bo'ladi.

e) $\cos x<a$ **tengsizlik:**

1) $a\leq -1$ bo'lganda yechimga ega emas;

2) $a>1$ bo'lsa, $x\in(-\infty;\ +\infty)$ bo'ladi;

3) $-1<a\leq 1$ bo'lsa, $x\in(\arccos a+2n\pi;\ 2\pi\text{-}\ \arccos a+2n\pi)$ bo'ladi.

f) $tgx>a$ **tengsizlik** $x\in\left(arctga+n\pi;\ \frac{\pi}{2}+n\pi\right)$ yechimga ega.

g) $tgx<a$ **tengsizlik** $x\in\left(-\frac{\pi}{2}+n\pi;\ arctga+n\pi\right)$ yechimga ega.

h) $ctgx > a$ **tengsizlik** $x \in (n\pi;\ arctga + n\pi)$ yechimga ega.

i) $ctgx < a$ **tengsizlik** $x \in (arcctga + n\pi;\ \pi + n\pi)$ yechimga ega.

10. Modul qatnashgan tenglama va tengsizlikni yechish uchun modul ostida qatnashgan funksiyalarning barcha nollari topiladi, ular yordamida sonlar o'qi oraliqlarga ajratiladi va har bir oraliqda moduldan qutulinadi.

12⁰. Ajoyib va muhim limitlar

1. $\lim\limits_{n\to\infty} \frac{n}{2^n} = 0$.

2. $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$.

3. $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = 0 \quad (a > 1)$.

4. $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0 \quad (\forall a > 0)$.

5. $\lim\limits_{n\to\infty} nq^n = 0, \quad |q| < 1$.

6. $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1 \quad (a > 0)$.

7. $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0 \quad (a > 1)$.

8. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

9. $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

10. $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \approx 2{,}71828$.

11. $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{tgx}{x} = \lim\limits_{x\to 0} \frac{shx}{x} = \lim\limits_{x\to 0} \frac{thx}{x} = 1$.

12. $\lim\limits_{x\to 0} \frac{\sin \alpha x}{x} = \alpha \quad (\alpha \in R)$.

13. $\lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e$.

14. $\lim\limits_{x\to 0} \frac{\ln(1 + x)}{x} = 1$.

15. $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$.

16. $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln a \quad (a > 0)$.

17. $\lim\limits_{x\to 0} \frac{(1 + x)^\alpha - 1}{x} = \alpha \quad (\alpha \in R)$.

18. $\lim\limits_{x\to +0} x^a \ln x = \lim\limits_{x\to +\infty} x^{-a} \ln x = \lim\limits_{x\to +\infty} x^a e^{-x} = 0 \quad (a > 0)$.

13⁰. Differensiallashning umumiy qoidalari

1. $y = c = const, \quad y' = 0$.

2. $y = c \cdot u \quad (c = const), \quad y' = c \cdot u'$.

3. $y = u \pm v, \quad y' = u' \pm v'$.

4. $y = u \cdot v, \quad y' = u' \cdot v + u \cdot v'$.

5. $y=\dfrac{u}{v}$ $\left(v(x)\neq 0\right)$, $y'=\dfrac{u'\cdot v-u\cdot v'}{v^2}$. 6. $y=f(u)$ $\left(u=u(x)\right)$, $y'=f'_u\cdot u'_x$.

7. $y=f(x)$, $x=f^{-1}(y)$ $y'_x=\dfrac{1}{x'_y}$. 8. $y=u^v$, $y'=u^v\cdot v'\cdot\ln u+u^{v-1}\cdot v\cdot u'$.

14⁰. Asosiy elementar funksiyalarning hosilalari

1. $\left(x^n\right)'=nx^{n-1}$.

2. $\left(x^x\right)'=x^x\cdot(1+\ln x)$.

3. $(\sin x)'=\cos x$.

4. $(\cos x)'=-\sin x$.

5. $(\operatorname{tg} x)'=\dfrac{1}{\cos^2 x}$.

6. $(\operatorname{ctg} x)'=-\dfrac{1}{\sin^2 x}$.

7. $(\ln x)'=\dfrac{1}{x}$.

8. $(\log_a x)'=\dfrac{1}{x\ln a}$ $(a>0,\ a\neq 1)$.

9. $\left(e^x\right)'=e^x$.

10. $\left(a^x\right)'=a^x\ln a$ $(a>0)$.

11. $(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$.

12. $(\arccos)'=-\dfrac{1}{\sqrt{1-x^2}}$.

13. $(\operatorname{arctg} x)'=\dfrac{1}{1+x^2}$.

14. $(\operatorname{arcctg} x)'=-\dfrac{1}{1+x^2}$.

15. $(\operatorname{sh} x)'=\operatorname{ch} x$.

16. $(\operatorname{ch} x)'=shx$.

17. $(\operatorname{th} x)'=\dfrac{1}{ch^2 x}$.

18. $(\operatorname{cth} x)'=-\dfrac{1}{\operatorname{sh}^2 x}$.

19. $(\operatorname{arcsh} x)'=\dfrac{1}{\sqrt{x^2+1}}$.

20. $(\operatorname{arcch} x)'=\dfrac{1}{\sqrt{x^2-1}}$.

21. $(\operatorname{arcth} x)'=\dfrac{1}{1-x^2}$.

22. $(\operatorname{arccth} x)'=-\dfrac{1}{1-x^2}$.

15⁰. Integrallashning umumiy qoidalari

1. $d\left[\int f(x)dx\right]=f(x)dx$.

2. $\int dF(x)=F(x)+c$.

3. $\int cf(x)dx = c\int f(x)dx \quad (c = const \neq 0)$.

4. $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$.

16⁰. Aniqmas integrallar jadvali

1. $\int 0 \cdot dx = \tilde{n}$.

2. $\int dx = x + c$.

3. $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1, \ \alpha \in R)$.

4. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$.

5. $\int \frac{dx}{x^2} = -\frac{1}{x} + c$.

6. $\int \frac{dx}{x} = \ln|x| + c$.

7. $\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c \quad (a \neq 0)$.

8. $\int e^x dx = e^x + c$.

9. $\int a^x dx = \frac{a^x}{\ln a} + c \quad (a > 0, \ a \neq 1)$.

10. $\int \sin x dx = -\cos x + c$.

11. $\int \cos x dx = \sin x + c$.

12. $\int \frac{dx}{\sin^2 x} = -ctgx + c$.

13. $\int \frac{dx}{\cos^2 x} = tgx + c$.

14. $\int \text{sh}\, x dx = \text{ch}\, x + c$.

15. $\int \text{ch}\, x dx = \text{sh}\, x + c$.

16. $\int \frac{dx}{\text{sh}^2 x} = -\text{cth}\, x + c$.

17. $\int \frac{dx}{ch^2 x} = thx + c$. $\int \frac{dx}{ch^2 x} = thx + c$.

18. $\int \frac{dx}{1+x^2} = arctgx + c$.

19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctg \frac{x}{a} + c \quad (a \neq 0)$.

20. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$.

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + c \quad (a > 0)$.

22. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c \quad (a \neq 0)$.

23. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c \quad (a \neq 0)$. $\lim_{n\to\infty} z_n = a$

24. $\int\frac{dx}{\sqrt{x^2+a}}=\ln\left|x+\sqrt{x^2+a}\right|+c \quad (a\neq 0)$.

25. $\int\frac{xdx}{\sqrt{a^2\pm x^2}}=\pm\sqrt{a^2\pm x^2}+c$.

26. $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+c \quad (a>0)$.

27. $\int\sqrt{x^2\pm a^2}dx=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$.

28. $\int\frac{dx}{\sin x}=\ln\left|tg\frac{x}{2}\right|+c$.

29. $\int\frac{dx}{\cos x}=\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+c$.

30. $\int\frac{dx}{\operatorname{tg} x}=\ln|\sin x|+c$.

31. $\int\frac{dx}{\operatorname{ctg} x}=-\ln|\cos x|+c$.

17⁰. Aniq integralning tatbiqlari

1. Aniq integral yordamida tekis shaklning yuzasini hisoblash.

a) Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

Agar $f_1(x)\in C[a,b]$, $f_2(x)\in C[a,b]$ bo'lib,

$$D=\begin{cases}a\leq x\leq b,\\ f_1(x)\leq y\leq f_2(x)\end{cases}$$

bo'lsa, u holda

$$S=\int_a^b\left[f_2(x)-f_1(x)\right]dx$$

bo'ladi.

b) Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

Agar

$$D=\begin{cases}\alpha\leq\varphi\leq\beta,\\ 0\leq r\leq r(\varphi)\end{cases}$$

bo'lib, $r(\varphi)\in C[\alpha,\beta]$ bo'lsa,

$$S=\frac{1}{2}\int_{\alpha}^{\beta}r^2(\varphi)d\varphi$$

bo'ladi.

2. Aniq integral yordamida yoy uzunligini hisoblash.

a) Dekart koordinatalar sistemasida berilgan yoy uzunligini hisoblash.

$\breve{AB}$: $\{(x,f(x)):x\in[a,b]\}$ bo'lib, $f'(x)\in C[a,b]$ bo'lsa, unda $\breve{AB}$ egri chiziq uzunligi l ushbu

$$l=\int_a^b\sqrt{1+[f'(x)]^2}dx$$

formula yordamida hisoblanadi.

b) Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligini hisoblash.

Agar $\breve{AB}$: $\begin{cases}x=\varphi(t),\\ y=\psi(t)\end{cases}$ $\alpha\le t\le\beta$ bo'lib, $\varphi'(t)\in C[\alpha,\beta]$ va $\psi'(t)\in C[\alpha,\beta]$ bo'lsa,

$$l=\int_{\alpha}^{\beta}\sqrt{[\varphi'(t)]^2+[\psi'(x)]^2}dt$$

bo'ladi.

d) Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligini hisoblash.

Agar $\breve{AB}$: $\begin{cases}\alpha\le\varphi\le\beta,\\ r=r(\varphi)\end{cases}$ bo'lib, $r'(\varphi)\in C[\alpha,\beta]$ bo'lsa, unda

$$l=\int_{\alpha}^{\beta}\sqrt{r^2(\varphi)+[r'(\varphi)]^2}d\varphi$$

bo'ladi.

3. Aylanma sirtning yuzasi.

$\breve{AB}$: $\{(x,f(x)):x\in[a,b]\}$ bo'lib, $f(x)\ge 0$ va $f'(x)\in C[a,b]$ bo'lsin. $\breve{AB}$ yoyni OX o'qi atrofida aylantirish natijasida hosil bo'lgan aylanma sirtning yuzasi ushbu

$$S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx$$

formula yordamida hisoblanadi.

4. Aylanma jismning hajmi.

Ushbu $D=\begin{cases} a\le x\le b, \\ 0\le y\le f(x) \end{cases}$ egri chiziqli trapetsiyani OX o'qi atrofida aylantirishdan hosil bo'lgan aylanma jismning hajmi

$$V=\pi\int_a^b [f(x)]^2 dx$$

formula yordamida hisoblanadi.

5. O'zgaruvchi kuchning bajargan ishi.

OX o'qida shu o'q bo'ylab biror jism $F=F(x)$ kuch ta'sirida harakat qilayotgan bo'lsin. Agar $F(x)\in C[a,b]$ bo'lsa, $F=F(x)$ kuch ta'sirida jismni a nuqtadan b nuqtaga o'tkazishda bajarilgan ish ushbu

$$A=\int_a^b F(x)dx$$

formula yordamida hisoblanadi.

6. Statik moment. Og'irlik markazi.

Egri chiziqning OX va OY o'qlariga nisbatan statik momentlari M_x va M_y lar

$$M_x=\int_0^l ydl \quad \text{va} \quad M_y=\int_0^l xdl$$

formulalar yordamida hisoblanadi. Bu yerda $dl=\sqrt{(dx)^2+(dy)^2}$ – yoy differensiali, l esa berilgan egri chiziq uzunligi.

Berilgan egri chiziq og'irlik markazining koordinatalari esa ushbu

$$x_0=\frac{M_y}{l}; \quad y_0=\frac{M_x}{l}$$

formulalar yordamida hisoblanadi.

7. Geometrik figuralarning statik momentlari va og'irlik markazi.

Agar geometrik figura

$$D=\begin{cases} a\le x\le b \\ 0\le y\le f(x) \end{cases}$$

egri chiziqli trapetsiyadan iborat bo'lsa, unda

$$M_x = \frac{1}{2}\int_a^b y^2 dx, \quad M_y = \frac{1}{2}\int_a^b xydx \text{ va } (x_0; y_0) = \left(\frac{M_y}{S}, \frac{M_x}{S}\right)$$

bo'ladi. Bu yerda $S = \int_a^b y(x)dx$ –trapetsiyaning yuzi.

18⁰. Matematik belgilar

Formula, ta'rif va tasdiqlarni yozishda quyidagi matematik belgilardan foydalanish qulay bo'lib, yozuvni ancha ixchamlashtiradi:

$\in$ –tegishli,
$\notin$ –tegishli emas,
$\subset$ –qism,
$\forall$ –ixtiyoriy,
$\exists$ –mavjud,
$\exists!$ –mavjud va yagona,
$\Rightarrow$ –kelib chiqadi, "...bo'lsa, ...bo'ladi",
$\Leftrightarrow$ –teng kuchli,
$:=$ -ta'rifga ko'ra teng,
: –shunday,
$\wedge$ –va,
$\vee$ –yoki,
$\lhd$ –isbotning boshlanishi,
$\rhd$ –isbotning oxiri.

2-§. 1-MUSTAQIL ISH

Ketma-ketlik va funksiya limiti

Sonli ketma-ketlik va uning limiti.
Cheksiz kichik va cheksiz katta ketma-ketliklar.
Monoton ketma-ketliklar va ularning limiti.
Fundamental ketma-ketliklar.
Ketma-ketlikning yuqori va quyi limitlari.
Funksiyaning limiti.
Funksiyaning uzluksizligi va uzilish nuqtalari.
Funksiyaning tekis uzluksizligi.

–A–

Asosiy tushuncha va teoremalar

1°. Sonli ketma-ketlik va uning limiti

1-ta'rif. *Agar har bir* $n \in N$ *natural songa biror qonun yoki qoidaga ko'ra bitta* x_n *haqiqiy son mos qo'yilgan bo'lsa,* $x_1, x_2, \ldots, x_n, \ldots$ *sonli ketma-ketlik berilgan deyiladi va u* $\{x_n\}$ *kabi belgilanadi.*

x_n $(n = 1, 2, \ldots)$ miqdorlar $\{x_n\}$ ketma-ketlikning hadlari deyiladi.

$\{x_n\}$ va $\{y_n\}$ ketma-ketliklar berilgan bo'lsa,

$$\{x_n + y_n\} = \{x_1 + y_1,\ x_2 + y_2, \ldots\},$$

$$\{x_n - y_n\} = \{x_1 - y_1,\ x_2 - y_2, \ldots\},$$

$$\{x_n \cdot y_n\} = \{x_1 \cdot y_1,\ x_2 \cdot y_2, \ldots\},$$

$$\left\{\frac{x_n}{y_n}\right\} = \left\{\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots\right\} \qquad (y_n \neq 0,\ n = 1, 2, \ldots)$$

ketma-ketliklarga mos ravishda $\{x_n\}$ va $\{y_n\}$ ketma-ketliklarning **yig'indisi, ayirmasi, ko'paytmasi va nisbati** deyiladi.

2-ta'rif. *Agar* $\exists M$ $(\exists m)$ *son mavjud bo'lsaki,* $\forall n \in N$ *uchun* $x_n \le M$ $(x_n \ge m)$ *tengsizlik o'rinli bo'lsa,* $\{x_n\}$ *ketma-ketlik yuqoridan (quyidan) chegaralangan deyiladi. Aks holda esa, ya'ni* $\forall M$ $(\forall m)$ *son olinganda ham* $\exists n \in N$ *son mavjud bo'lsaki,* $x_n > M$ $(x_n < m)$ *bo'lsa,* $\{x_n\}$ *ketma-ketlik* ***yuqoridan (quyidan) chegaralanmagan*** *deyiladi.*

3-ta'rif. *Agar* $\exists M>0$ *son mavjud bo'lsaki,* $\forall n\in N$ *uchun* $|x_n|\le M$ *bo'lsa,* $\{x_n\}$ *ketma-ketlik* ***chegaralangan*** *deyiladi. Aks holda esa, ya'ni* $\forall M>0$ *son olinganda ham* $\exists n\in N$ *son topilsaki* $|x_n|>M$ *bo'lsa,* $\{x_n\}$ ***chegaralanmagan ketma-ketlik*** *deyiladi.*

4-ta'rif. *Berilgan* $\{x_n\}$ *ketma-ketlik uchun shunday* a *son topilib,* $\forall\varepsilon>0$ *son olinganda ham* $\exists n_0=n_0(\varepsilon,a)\in N$ *son mavjud bo'lsaki,* $n>n_0$ *tengsizlikni qanoatlantiruvchi barcha natural sonlar uchun* $|x_n-a|<\varepsilon$ *tengsizlik o'rinli bo'lsa,* a *son* $\{x_n\}$ *ketma-ketlikning* ***limiti*** *deyiladi va* $\lim\limits_{n\to\infty}x_n=a$ *ko'rinishda belgilanadi.*

Agar 4-ta'rifdagi shartni qanoatlantiruvchi a son mavjud bo'lmasa, $\{x_n\}$ ketma-ketlik **limitga ega emas** deyiladi.

5-ta'rif (4-ta'rifning inkori). *Agar* $\forall n_0\in N$ *son olinganda ham* $\exists\varepsilon>0,\ \exists n>n_0$ *son topilsaki,* $|x_n-a|\ge\varepsilon$ *bo'lsa,* a *son* $\{x_n\}$ *ketma-ketlikning* ***limiti emas*** *deyiladi va* $\lim\limits_{n\to\infty}x_n=a$ *ko'rinishda belgilanadi.*

6-ta'rif. *Agar* $\{x_n\}$ *ketma-ketlik chekli limitga ega bo'lsa, bu ketma-ketlik* ***yaqinlashuvchi*** *deyiladi. Aks holda bu ketma-ketlik* ***uzoqlashuvchi*** *deyiladi.*

2⁰. Cheksiz kichik va cheksiz katta ketma-ketliklar

1-ta'rif. *Agar* $\{x_n\}$ *ketma-ketlikning limiti nolga teng* ($\lim\limits_{n\to\infty}x_n=0$) *bo'lsa,* $\{x_n\}$ *ketma-ketlik* ***cheksiz kichik*** *ketma-ketlik deyiladi.*

2-ta'rif. *Agar* $\forall M>0$ *son olinganda ham* $\exists n_0\in N$ *son mavjud bo'lsaki,* $\forall n>n_0$ *natural sonlar uchun* $|x_n|>M$ *tengsizlik o'rinli bo'lsa,* $\{x_n\}$ *ketma-ketlik* ***cheksiz katta*** *ketma-ketlik deyiladi.*

Agar $\{x_n\}$ cheksiz katta ketma-ketlik bo'lsa, $\lim\limits_{n\to\infty}x_n=\infty$ ko'rinishda yoziladi. Agar $\{x_n\}$ cheksiz katta ketma-ketlik bo'lib, biror nomerdan boshlab barcha hadlari **musbat (manfiy)** bo'lsa, $\lim\limits_{n\to\infty}x_n=+\infty$ ($\lim\limits_{n\to\infty}x_n=-\infty$) ko'rinishda yoziladi.

Har qanday cheksiz katta ketma-ketlik chegaralanmagan bo'ladi, lekin bu tasdiqning teskarisi har doim ham o'rinli bo'lavermaydi.

1-teorema. *Chekli sondagi cheksiz kichik ketma-ketliklar yig'indisi cheksiz kichik ketma-ketlik bo'ladi.*

2-teorema. *Chegaralangan ketma-ketlik bilan cheksiz kichik ketma-ketlik ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.*

3-teorema. *Agar* $\forall n \in N$ *uchun* $x_n \neq 0$ *bo'lib,* $\{x_n\}$ *– cheksiz katta (cheksiz kichik) ketma-ketlik bo'lsa, u holda* $\left\{\frac{1}{x_n}\right\}$ *cheksiz kichik (cheksiz katta) ketma-ketlik bo'ladi.*

4-teorema. $\lim\limits_{n\to\infty} x_n = a$ *bo'lishi uchun* $\{\alpha_n\} = \{x_n - a\}$ *ketma-ketlikning cheksiz kichik ketma-ketlik bo'lishi zarur va yetarlidir.*

3⁰. Yaqinlashuvchi ketma-ketliklarning xossalari

1-teorema. *Agar* $\{x_n\}$ *ketma-ketlik yaqinlashuvchi bo'lsa, uning limiti yagona bo'ladi.*

2-teorema. *Agar* $\{x_n\}$ *ketma-ketlik yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.*

3-teorema. *Agar* $\{x_n\}$ *va* $\{y_n\}$ *ketma-ketliklar yaqinlashuvchi bo'lsa, u holda* $\{x_n \pm y_n\}$, $\{x_n \cdot y_n\}$ *ketma-ketliklar ham yaqinlashuvchi bo'ladi va*

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n,$$

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

formulalar o'rinli bo'ladi.

4-teorema. *Agar* $\{x_n\}$ *va* $\{y_n\}$ *ketma-ketliklar yaqinlashuvchi bo'lib,* $\forall n \in N$ *uchun* $y_n \neq 0$ *va* $\lim\limits_{n\to\infty} y_n \neq 0$ *bo'lsa,* $\left\{\frac{x_n}{y_n}\right\}$ *ketma-ketlik ham yaqinlashuvchi bo'ladi va*

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

formula o'rinli bo'ladi.

5-teorema. *Agar* $\lim\limits_{n\to\infty} x_n = a$ *bo'lib, biror nomerdan boshlab* $x_n \geq c$ $(x_n \leq c)$ *bo'lsa, u holda* $a \geq c$ $(a \leq c)$ *bo'ladi.*

6-teorema. ***("Ikki mirshab haqidagi teorema").*** *Agar* $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = a$ *bo'lib, biror nomerdan boshlab* $x_n \leq z_n \leq y_n$ *tengsizlik o'rinli bo'lsa, u holda* $\lim\limits_{n\to\infty} z_n = a$ *bo'ladi.*

Agar $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} y_n = 0$ bo'lsa, $\lim_{n\to\infty}\frac{x_n}{y_n}$ ga $\frac{0}{0}$ ko'rinishdagi aniqmaslik deyiladi. $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$ va boshqa ko'rinishdagi aniqmasliklar ham shu kabi ta'riflanadi.

4⁰. Monoton ketma-ketliklar va ularning limiti

1-ta'rif. *Agar $\{x_n\}$ ketma-ketlikning hadlari $\forall n\in N$ uchun $x_n \le x_{n+1}$ $(x_n \ge x_{n+1})$ tengsizlikni qanoatlantirsa $\{x_n\}$* ***o'suvchi (kamayuvchi)*** *ketma-ketlik deyiladi.*

2-ta'rif. *O'suvchi va kamayuvchi ketma-ketliklar* ***monoton*** *ketma-ketliklar deb ataladi.*

1-teorema. *Agar $\{x_n\}$ ketma-ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, u holda u yaqinlashuvchi bo'ladi.*

2-teorema. *Agar $\{x_n\}$ ketma-ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, u holda u yaqinlashuvchi bo'ladi.*

5⁰. Fundamental ketma-ketliklar

1-ta'rif. *Agar $\forall\varepsilon>0$ son olinganda ham $\exists n_0 = n_0(\varepsilon)\in N$ son mavjud bo'lsaki, $\forall n>n_0$ va $p\in N$ sonlar uchun $|x_{n+p}-x_n|<\varepsilon$ tengsizlik bajarilsa, $\{x_n\}$* ***fundamental ketma-ketlik deyiladi.***

2-ta'rif. *(1-ta'rifning inkori). $\forall n_0\in N$ son olinganda ham shunday $n>n_0$, $p\in N$, $\varepsilon>0$ sonlar mavjud bo'lib, $|x_{n+p}-x_n|\ge\varepsilon$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ ketma-ketlik* ***fundamental emas*** *deyilali.*

Teorema (Koshi). *Ketma-ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarlidir.*

6⁰. Qismiy ketma-ketliklar. Ketma-ketlikning yuqori va quyi limitlari

$\{x_n\}$ ketma-ketlik berilgan bo'lib, $k_1,k_2,\ldots,k_n,\ldots$ $(k_n\ge n)$ o'suvchi natural sonlar ketma-ketligi bo'lsin. $\{x_n\}$ ketma-ketlikning $k_1,k_2,\ldots,k_n,\ldots$ nomerli hadlaridan $x_{k_1},x_{k_2},\ldots,x_{k_n},\ldots$ ketma-ketlikni tuzamiz. Hosil bo'lgan $\{x_{k_n}\}$ sonli ketma-ketlik $\{x_n\}$ ketma-ketlikning **qismiy ketma-ketligi** deb ataladi.

1-teorema. *Agar* $\lim\limits_{n\to\infty} x_n = a$ *bo'lsa, u holda uning har qanday qismiy ketma-ketligining limiti ham* a *ga teng bo'ladi.*

2-teorema. ***(Bolsano-Veyershtrass).*** *Agar* $\{x_n\}$ *ketma-ketlik chegaralangan bo'lsa, u holda bu ketma-ketlikdan yaqinlashuvchi bo'lgan qismiy ketma-ketlik ajratish mumkin.*

1-ta'rif. $\{x_n\}$ *ketma-ketlikning qismiy ketma-ketligi limiti* $\{x_n\}$ *ketma-ketlikning* ***qismiy limiti*** *deb ataladi.*

2-ta'rif. *Yuqoridan (quyidan) chegaralangan ketma-ketlik qismiy limitlarining eng kattasi (eng kichigi) berilgan ketma-ketlikning* ***yuqori (quyi)*** *limiti deyiladi va* $\overline{\lim\limits_{n\to\infty}} x_n$ $\left(\underline{\lim\limits_{n\to\infty}} x_n\right)$ *ko'rinishda belgilanadi.*

3-teorema. $\lim\limits_{n\to\infty} x_n = a$ *bo'lishi uchun* $\overline{\lim\limits_{n\to\infty}} x_n = \underline{\lim\limits_{n\to\infty}} x_n = a$ *bo'lishi zarur va yetarli.*

7°. Funksiya tushunchasi. Funksiya limiti

Bizga biror $X \subset R$ to'plam berilgan bo'lib, x o'zgaruvchi miqdor X to'plamdan olingan bo'lsin. Agar har bir $x \in X$ songa biror qonun yoki qoidaga ko'ra bitta y son mos qo'yilsa, u holda X to'plamda **funksiya** aniqlangan deyiladi va $y = f(x)$ kabi belgilanadi, x o'zgaruvchiga **erkli o'zgaruvchi** (yoki funksiyaning argumenti), X to'plam $f(x)$ funksiyaning **aniqlanish sohasi**, x soniga mos keluvchi y soniga esa funksiyaning x nuqtadagi **xususiy qiymati** deb ataladi. $f(x)$ funksiyaning barcha xususiy qiymatlar to'plami Y ga $f(x)$ funksiyaning **qiymatlar to'plami** (yoki **o'zgarish sohasi**) deyiladi. Shunday qilib,

$$Y = \{y \in R:\ y = f(x),\ x \in X\}.$$

Agar a $(a \in X$,êè $a \notin X)$ nuqtaning ixtiyoriy atrofida X to'plamning a dan farqli kamida bitta nuqtasi bo'lsa, u holda a nuqta X to'plamning **limit nuqtasi** deyiladi.

Bundan keyin butun paragraf davomida $X - f(x)$ funksiyaning aniqlanish sohasi, a nuqta X to'plamning limit nuqtasi deb tushuniladi.

1-ta'rif. (Koshi). *Agar* $\forall\varepsilon>0$ *uchun* $\exists\delta=\delta(\varepsilon,a)>0$ *topilsaki,* $0<|x-a|<\delta$ *tengsizlikni qanoatlantiruvchi* $\forall x\in X$ *uchun* $|f(x)-b|<\varepsilon$ *tengsizlik bajarilsa, u holda* b *soni* $f(x)$ *funksiyaning* a ***nuqtadagi limiti*** *deyiladi va* $\lim\limits_{x\to a}f(x)=b$ *kabi belgilanadi.*

2-ta'rif. (Geyne). *Agar* X *to'plamning nuqtalaridan tuzilgan,* a *ga intiluvchi* $\forall\{x_n\}$ $(x_n\neq a,\ n=1,\ 2,\ \ldots)$ *ketma-ketlik uchun* $\{f(x_n)\}$ *ketma-ketlik hamma vaqt yagona* b *soniga intilsa, shu* b *soni* $f(x)$ *funksiyaning* a ***nuqtadagi limiti deb*** *ataladi.*

Keltirilgan ta'riflardan ko'rinib turibdiki, funksiyaning a nuqtadagi limiti mavjud bo'lishi uchun funksiya a nuqtada aniqlangan bo'lishi, ya'ni $a\in X$ bo'lishi, mutlaqo shart emas (a nuqtaning X to'plam uchun limit nuqta bo'lishi yetarli, ya'ni, umuman olganda, $a\notin X$).

Endi 1- va 2-ta'riflarga teskari ta'riflarni keltiramiz.

1-ta'rifning inkori. *Agar* $\exists\varepsilon>0$ *topilsaki,* $\forall\delta>0$ *uchun* $0<|x-a|<\delta$ *tengsizlikni qanoatlantiruvchi* $\exists x\in X$ *mavjud bo'lib,* $|f(x)-b|\geq\varepsilon$ *tengsizlik bajarilsa,* b *soni* $f(x)$ *funksiyaning* a ***nuqtadagi limiti emas*** *deyiladi* $\left(\lim\limits_{x\to a}f(x)\neq b\right)$.

2-ta'rifning inkori. *Agar* a *nuqtaga intiluvchi* $\exists\{x_n\}$ $(x_n\in X,\ x_n\neq a,\ n=1,\ 2,\ \ldots)$ *ketma-ketlik topilsaki, unga mos* $\{f(x_n)\}$ *ketma-ketlik* b *ga intilmasa, u holda* b *son* $f(x)$ *funksiyaning* a ***nuqtadagi limiti emas*** *deyiladi.*

1-teorema. *Funksiya limitining 1- va 2-ta'riflari ekvivalentdir.*

Biz 1-teoremadan quyidagi xulosani chiqaramiz: funksiyaning limitini hisoblayotganda qaysi ta'rif bo'yicha hisoblash oson va qulay bo'lsa, shu ta'rifdan foydalanish kerak.

Ba'zi bir hollarda $f(x)$ funksiyaning a nuqtadagi limiti mavjud bo'lmaydi. Ana shunday hollarda funksiyaning nuqtadagi **bir tomonli (o'ng va chap)** limitlari to'g'risida gap yuritiladi.

3-ta'rif ***(Koshi).*** $\forall\varepsilon>0$ *uchun* $\exists\delta=\delta(a,\varepsilon)>0$ *topilsaki,* $a<x<a+\delta$ $(a-\delta<x<a)$ *tengsizlikni qanoatlantiruvchi* $\forall x\in X$

uchun $|f(x)-b|<\varepsilon$ *tengsizlik bajarilsa,* b *son* $f(x)$ *funksiyaning* a *nuqtadagi* ***o'ng (chap) limiti*** *deb ataladi va*

$$\lim_{x\to a+0} f(x)=f(a+0)=b \quad \left(\lim_{x\to a-0} f(x)=f(a-0)=b\right)$$

kabi belgilanadi.

4-ta'rif ***(Geyne).*** a *nuqtaga intiluvchi* $\forall\{x_n\}$, $x_n\in X$, $x_n>a$ $(x_n<a)$ *ketma-ketlik olinganda ham unga mos* $\{f(x_n)\}$ *ketma-ketlik* b *soniga intilsa,* b *soni* $f(x)$ *funksiyaning* a *nuqtadagi* ***o'ng (chap) limiti*** *deyiladi.*

2-teorema. $\lim\limits_{x\to a} f(x)=b$ *bo'lishi uchun* $f(a+0)=f(a-0)=b$ *tenglikning bajarilishi zarur va yetarli.*

Endi funksiyaning $x\to+\infty$ dagi limiti ta'rifini beramiz. $f(x)$ funksiya $(c,+\infty)$ cheksiz oraliqda aniqlangan bo'lsin.

5-ta'rif. (Koshi). $\forall\varepsilon>0$ *uchun* $\exists A>0$ $(A\ge c)$ *topilsaki,* $\forall x>A$ *uchun* $|f(x)-b|<\varepsilon$ *tengsizlik bajarilsa,* b *son* $f(x)$ *funksiyaning* $x\to+\infty$ ***dagi limiti*** *deyiladi va* $\lim\limits_{x\to+\infty} f(x)=b$ *kabi belgilanadi.*

6-ta'rif. (Geyne). $+\infty$ *ga intiluvchi* $\forall\{x_n\}$ $(x_n>c)$ *ketma-ketlik uchun unga mos* $\{f(x_n)\}$ *ketma-ketlik* b *soniga intilsa,* b *soni* $f(x)$ *funksiyaning* $x\to+\infty$ ***dagi limiti*** *deb ataladi.*

3- va 4-ta'riflar hamda 5- va 6-ta'riflar bir-biriga ekvivalent. $\lim\limits_{x\to-\infty} f(x)=b$ ning ta'rifi ham yuqoridagiga o'xshash aniqlanadi. Agar $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to-\infty} f(x)=b$ bo'lsa, u holda $\lim\limits_{x\to\infty} f(x)=b$ deb yoziladi.

7-ta'rif. *Agar* $\lim\limits_{x\to a} f(x)=\infty$ $\left(\lim\limits_{x\to a} f(x)=0\right)$ *bo'lsa,* $f(x)$ *funksiya* a *nuqtada* ***cheksiz katta (cheksiz kichik) funksiya*** *deyiladi.*

Cheksiz katta va cheksiz kichik funksiyalar ham cheksiz katta va cheksiz kichik ketma-ketliklar uchun 2^0-punktda keltirilgan xossalarga ega.

8^0. Limitga ega bo'lgan funksiyalarning xossalari

1-ta'rif. *Ushbu* $\dot{U}_\delta(a)=\{x\in R:\ 0<|x-a|<\delta\}$ *to'plam* a *nuqtaning* ***o'yilgan*** δ ***atrofi*** *deb ataladi.*

1-teorema. *$f(x)$ va $g(x)$ funksiyalar a nuqtaning biror o'yilgan atrofida aniqlangan bo'lib, $\lim_{x\to a} f(x)=b$ va $\lim_{x\to a} g(x)=c$ bo'lsin. U holda*

1) $\lim_{x\to a}\left[f(a)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)=b\pm c$,

2) $\lim_{x\to a}\left[f(a)\cdot g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)=b\cdot c$,

3) *agar* $c\neq 0$ *bo'lsa,* $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{b}{c}$ *bo'ladi.*

2-teorema. («Ikki mirshab haqidagi teorema»). *Agar $f(x)$, $g(x)$ va $h(x)$ funksiyalar a nuqtaning biror o'yilgan atrofida aniqlangan bo'lib, shu atrofda $f(x)\le g(x)\le h(x)$ tengsizlikni qanoatlantirsa va $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=b$ tenglik bajarilsa, u holda $\lim_{x\to a} g(x)=b$ bo'ladi.*

Funksiya limitini hisoblashda quyidagi ajoyib limitlar katta ahamiyatga ega.

Birinchi ajoyib limit:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{1}$$

Ikkinchi ajoyib limit:

$$\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^{x}=e \tag{2}$$

9^0. Funksiya limiti uchun Koshi teoremasi

$f(x)$ funksiya X to'plamda berilgan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin.

Ta'rif. *Agar $\forall\varepsilon>0$ uchun $\exists\delta>0$ topilsaki, argument x ning $0<|x'-a|<\delta$, $0<|x''-a|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x', x''$ $(x'\in X, x''\in X)$ qiymatlarida $|f(x'')-f(x')|<\varepsilon$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya uchun a* ***nuqtada Koshi sharti bajariladi*** *deyiladi.*

Ta'rifning inkori. Agar $\exists\varepsilon>0$ son topilsaki, $\forall\delta>0$ son uchun, $0<|x'-a|<\delta$, $0<|x''-a|<\delta$ tengsizlikni qanoatlantiruvchi

$\forall x', x'' \in X$ lar mavjud bo'lib, $|f(x'') - f(x')| \geq \varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya uchun a **nuqtada Koshi sharti bajarilmaydi** deyiladi.

Teorema. (Koshi). $f(x)$ *funksiya* a *nuqtada chekli limitga ega bo'lishi uchun bu funksiyaning* a *nuqtada Koshi shartini bajarishi zarur va yetarlidir.*

10^0. Funksiyaning uzluksizligi va uzilishi

$f(x)$ funksiya a nuqtaning biror to'liq atrofida aniqlangan bo'lsin.

1-ta'rif. *Agar*

$$\lim_{x \to a} f(x) = f(a) \qquad (3)$$

bo'lsa, $f(x)$ funksiya a **nuqtada uzluksiz** deyiladi.

Funksiya uzluksizligi ta'rifini Koshi va Geyne ta'riflari yordamida ham berish mumkin. Biz ularga to'xtalib o'tirmaymiz.

Endi $f(x)$ funksiya a nuqtaning biror o'ng (chap) yarim atrofida, ya'ni $[a, a+\delta)$ (mos ravishda, $(a-\delta, a]$) yarim intervalda aniqlangan bo'lsin.

2-ta'rif. *Agar*

$$\lim_{x \to a+0} f(x) = f(a) \quad \left(\lim_{x \to a-0} f(x) = f(a)\right)$$

bo'lsa, $f(x)$ funksiya a **nuqtada o'ngdan (chapdan) uzluksiz** deyiladi.

Teorema. $f(x)$ *funksiyaning* a *nuqtada uzluksiz bo'lishi uchun uning shu nuqtada o'ngdan va chapdan uzluksiz bo'lishi zarur va yetarlidir.*

Faraz qilaylik, $f(x)$ funksiya a nuqtada uzluksiz bo'lsin. U holda $\lim_{x \to a} f(x) = f(a)$ bo'ladi. $\Rightarrow \lim_{x-a \to 0}[f(x) - f(a)] = 0$. Agar $\Delta x := x - a$ – argument orttirmasi va $\Delta y := \Delta f(a) = f(x) - f(a)$ – funksiyaning a nuqtadagi orttirmasi belgilashlarini kiritsak, $x = a + \Delta x$ va $\Delta y = \Delta f(a) = f(a + \Delta x) - f(a)$ bo'ladi. Natijada, biz

$$\lim_{x-a \to 0}[f(x) - f(a)] = \lim_{\Delta x \to 0}[f(a + \Delta x) - f(a)] = \lim_{\Delta x \to 0} \Delta y = 0$$

ekanligini hosil qilamiz. Shunday qilib,

$$\lim_{\Delta x \to 0} \Delta y = 0 \qquad (4)$$

tenglik bajarilsa, $f(x)$ funksiya a nuqtada uzluksiz bo'ladi.

3-ta'rif. *$f(x)$ funksiya (c,d) intervalning har bir nuqtasida uzluksiz bo'lsa, funksiya (c,d) intervalda uzluksiz deyiladi.*

$f(x)$ funksiya (c,d) da uzluksiz bo'lib, s nuqtada o'ngdan, d nuqtada chapdan uzluksiz bo'lsa, unda u $[c,d]$ kesmada uzluksiz deyiladi.

X to'plamda uzluksiz funksiyalar sinfi $C(X)$ kabi belgilanadi.

4-ta'rif. *Agar*

$$\lim_{x \to a} f(x) = b \neq f(a) \qquad (1-\text{hol})$$

$$\lim_{x \to a} f(x) - \nexists \qquad (2-\text{hol})$$

$$\lim_{x \to a} f(x) = \infty \qquad (3-\text{hol})$$

bo'lsa, unda $f(x)$ funksiya a ***nuqtada uzilishga ega*** *deyiladi.*

Funksiyaning a nuqtada uzilishga ega bo'ladigan hollarini alohida-alohida ko'rib chiqaylik.

a) $\lim_{x \to a} f(x) = b \neq f(a)$ **bo'lsin.**

Bu holda $\lim_{x \to a+0} f(x) = f(a+0)$ va $\lim_{x \to a-0} f(x) = f(a-0)$ lar mavjud bo'lib, $f(a+0) = f(a-0) \neq f(a)$ bo'ladi. Bunday nuqta **bartaraf qilish mumkin bo'lgan uzilish nuqtasi** deb ataladi.

Misollar.

1. $f(x) = \begin{cases} x^2, \text{ agar } x \neq 0 \text{ bo'lsa,} \\ 1, \text{ agar } x = 0 \text{ bo'lsa} \end{cases}$

funksiya uchun $x = 0$ nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi bo'ladi, chunki

$$\lim_{x \to a+0} f(x) = \lim_{x \to -0} f(x) = 0 \text{ va } f(0) = 1.$$

Agar $f(0) = 0$ deb qabul qilsak, funksiya uzluksiz bo'lib qoladi.

2. $f(x)=\begin{cases} 1-x\sin\dfrac{1}{x}, \text{àgar } x\neq 0 \text{ bo'lsa}, \\ 2, \text{agar } x=0 \text{ bo'lsa} \end{cases}$

funksiya uchun ham $x=0$ nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi bo'ladi, chunki

$$\lim_{x\to a+0} f(x)=\lim_{x\to -0} f(x)=1 \quad \text{va} \quad f(0)=2.$$

b) $\lim\limits_{x\to a+0} f(x)-\nexists$ **bo'lsin.**

Bunda quyidagi uchta hol bo'lishi mumkin.

1) $\lim\limits_{x\to a-0} f(x)=f(a-0)$ va $\lim\limits_{x\to a+0} f(x)=f(a+0)$ lar $\exists$ va $f(a-0)\neq f(a+0)$.

Funksiyaning bunday nuqtadagi uzilishi birinchi tur uzilish va $|f(a+0)-f(a-0)|$ ayirmaga funksiyaning a nuqtadagi sakrashi deyiladi.

Masalan,

$$f(x)=\begin{cases} \dfrac{1}{1+2^{\frac{1}{x}}}, \text{àgar } x\neq 0 \text{ bo'lsa}, \\ 0, \text{àgar } x=0 \text{ bo'lsa} \end{cases}$$

funksiya uchun $x=0$ nuqta 1-tur uzilish nuqtasi bo'ladi va funksiyaning bu nuqtadagi sakrashi 1 ga teng:

$$|f(a+0)-f(a-0)|=|f(+0)-f(-0)|=|0-1|=1;$$

2) $x\to a$ da $f(x)$ funksiyaning o'ng va chap limitlaridan hech bo'lmaganda biri $\nexists$. Funksiyaning a nuqtadagi bunday uzilishi ikkinchi tur uzilish deyiladi.

Misollar.

1. $f(x)=\begin{cases} \sin\dfrac{1}{x}, \text{agar } x>0 \text{ bo'lsa}, \\ -x, \text{agar } x\leq 0 \text{ bo'lsa} \end{cases}$

funksiya $x=0$ nuqtada ikkinchi tur uzilishga ega, chunki

$$\lim_{x\to -0} f(x) = \lim_{x\to -0}(-x) = 0 = f(0), \text{ lekin } \lim_{x\to +0} f(x) = \lim_{x\to 0}\sin\frac{1}{x} - \nexists.$$

2. $$D(x) = \begin{cases} 0, & \text{agar } x - \text{irratsional bo'lsa,} \\ 1, & \text{agar } x - \text{ratsional bo'lsa} \end{cases}$$

funksiya $\forall a \in R$ nuqtada ikkinchi tur uzilishga ega, chunki $x \to a$ da $D(x)$ funksiyaning o'ng limiti ham, chap limiti ham $\nexists$.

3). $x \to a$ da $f(x)$ funksiyaning o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlar turli ishorali cheksiz. Funksiyaning a nuqtadagi bunday uzilishi ham **ikkinchi tur uzilish** deyiladi.

d) $\lim\limits_{x\to a} f(x) = \infty$ bo'lsa, $f(x)$ funksiya $x = a$ nuqtada **ikkinchi tur uzilishga** ega deyiladi.

11^0. Uzluksiz funksiyalarning xossalari

1-teorema. *Agar $f(x)$ va $g(x)$ funksiyalar $X \subset R$ to'plamda aniqlangan bo'lib, ularning har biri $a \in X$ nuqtada uzluksiz bo'lsa, u holda*

1) $f(x) \pm g(x)$,

2) $f(x) \cdot g(x)$,

3) $\dfrac{f(x)}{g(x)}$ ($\forall x \in X$ uchun $g(x) \neq 0$)

funksiyalar ham shu nuqtada uzluksiz bo'ladi.

Izoh: 1-teoremaning aksi har doim ham o'rinli bo'lavermaydi. Masalan, $f(x) = x$ va

$$g(x) = \begin{cases} \sin\frac{1}{x}, & \text{agar } x \neq 0 \text{ bo'lsa,} \\ 0, & \text{agar } x = 0 \text{ bo'lsa} \end{cases}$$

funksiyalar ko'paytmasi $f(x)\cdot g(x) = x\cdot\sin\frac{1}{x}$ funksiya R da uzluksiz, lekin $g(x)$ funksiya $x=0$ nuqtada uzilishga ega.

Aytaylik, $y = f(x)$ funksiya X to'plamda, $z = \varphi(y)$ funksiya esa $Y=\{y=f(x):x\in X\}$ to'plamda aniqlangan bo'lib, ular yordamida X to'plamda aniqlangan $z=\varphi[f(x)]$ murakkab funksiya tuzilgan bo'lsin.

2-teorema. *Agar $y=f(x)$ funksiya $a\in X$ nuqtada, $z=\varphi(y)$ funksiya esa, unga mos $y_a=f(a)$ nuqtada uzluksiz bo'lsa, $z=\varphi[f(x)]$ murakkab funksiya a nuqtada uzluksiz bo'ladi.*

Bu teorema limit hisoblashda juda muhim rol o'ynaydi va uning yordamida 1–§ ning 9^0 –punktidagi muhim limitlar keltirib chiqariladi.

3-teorema. *Agar $\lim\limits_{x\to a} f(x)=b$ $(b>0)$ va $\lim\limits_{x\to a} g(x)=c$ bo'lsa, $\lim\limits_{x\to a}[f(x)]^{g(x)}=c$ bo'ladi.*

$[f(x)]^{g(x)}$ ko'rinishdagi funksiyaga **darajali-ko'rsatkichli funksiya** deb ataladi.

12^0. Funksiyaning tekis uzluksizligi

Biror $y=f(x)$ funksiya X to'plamda berilgan bo'lsin.

Ta'rif. *Agar $\forall\varepsilon>0$ son uchun $\exists\delta=\delta(\varepsilon)>0$ son topilsaki, X to'plamning $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x'$ va x'' $(x',x''\in X)$ nuqtalarida $|f(x'')-f(x')|<\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda **tekis uzluksiz** deb ataladi.*

Ta'rifning inkori. $\exists\varepsilon>0$ son topilsaki, $\forall\delta>0$ son olinganda ham $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi shunday $\forall x'\ x''\in X$ nuqtalar mavjud bo'lib $|f(x'')-f(x')|\geq\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda **tekis uzluksiz** emas deyiladi.

Kantor teoremasi. *Agar $f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lsa, u shu kesmada tekis uzluksiz bo'ladi.*

NAZORAT SAVOLLARI

1. Sonli ketma-ketlik tushunchasi.
2. Ketma-ketlik limitining ta'rifi va uning inkori.
3. Yaqinlashuvchi va uzoqlashuvchi ketma-ketliklarning ta'riflari.
4. Cheksiz kichik ketma-ketliklar va ularning xossalari.
5. Cheksiz katta ketma-ketliklar va ularning xossalari.
6. Cheksiz kichik va cheksiz katta ketma-ketliklar orasidagi bog'lanish.
7. Yaqinlashuvchi ketma-ketliklarning xossalari.
8. Monoton ketma-ketlikning ta'rifi.
9. Monoton ketma-ketliklar haqidagi Veyershtrass teoremasi.
10. Fundamental ketma-ketliklar va Koshi teoremasi.
11. Qismiy ketma-ketliklar. Ketma-ketlikning yuqori va quyi limitlari.
12. Funksiya tushunchasi. Funksiyaning aniqlanish sohasi va qiymatlar to'plami.
13. Funksiya limitining Koshi ta'rifi va uning inkori.
14. Funksiya limitining Geyne ta'rifi va uning inkori.
15. Funksiya limiti Koshi va Geyne ta'riflarining ekvivalentligi.
16. Funksiyaning bir tomonli limitlari.
17. Limitga ega bo'lgan funksiyalarning xossalari.
18. Ikki mirshab haqidagi teorema.
19. Birinchi ajoyib limit.
20. Ikkinchi ajoyib limit.
21. Funksiya limiti haqidagi Koshi teoremasi.
22. Funksiyaning nuqtadagi va to'plamdagi uzluksizligi ta'riflari.
23. Bir tomonlama uzluksizlik.
24. Bartaraf qilish mumkin bo'lgan uzilish nuqtasi.
25. Birinchi tur uzilish nuqtasi.
26. Ikkinchi tur uzilish nuqtasi.
27. Uzluksiz funksiyalarning xossalari.
28. Funksiyaning tekis uzluksizligi va Kantor teoremasi.

–B–
Mustaqil yechish uchun misol va masalalar

1-masala. $\lim_{n\to\infty} x_n = a$ **ekanligi ta'rif yordamida ko'rsatilsin** $(n_0(\varepsilon)-?)$.

1.1 $x_n = \dfrac{3n-2}{2n-1}$, $a=\dfrac{3}{2}$.

1.2 $x_n = \dfrac{4n-1}{2n+1}$, $a=2$.

1.3 $x_n = \dfrac{4n^2+1}{3n^2+2}$, $a=\dfrac{4}{3}$.

1.4 $x_n = \dfrac{9-n^3}{1+2n^3}$, $a=-\dfrac{1}{2}$.

1.5 $x_n = \dfrac{1-2n^2}{2+4n^2}$, $a=-\dfrac{1}{2}$.

1.6 $x_n = -\dfrac{5n}{n+1}$, $a=-5$.

1.7 $x_n = \dfrac{n+1}{1-2n}$, $a=-\dfrac{1}{2}$.

1.8 $x_n = \dfrac{2n+1}{3n-5}$, $a=\dfrac{2}{3}$.

1.9 $x_n = \dfrac{1-2n^2}{n^2+3}$, $a=-2$.

1.10 $x_n = \dfrac{3n^2+2}{4n^2-1}$, $a=\dfrac{3}{4}$.

1.11 $x_n = \dfrac{4+2n}{1-3n}$, $a=-\dfrac{2}{3}$.

1.12 $x_n = \dfrac{5n+15}{6-n}$, $a=-5$.

1.13 $x_n = \dfrac{13-n^2}{1+2n^2}$, $a=-\dfrac{1}{2}$.

1.14 $x_n = \dfrac{2n-1}{2-3n}$, $a=-\dfrac{2}{3}$.

1.15 $x_n = \dfrac{3n-1}{5n+1}$, $a=\dfrac{3}{5}$.

1.16 $x_n = \dfrac{5n+1}{10n-3}$, $a=\dfrac{1}{2}$.

1.17 $x_n = \dfrac{1+3n}{6-n}$, $a=-3$.

1.18 $x_n = \dfrac{2n+3}{n+5}$, $a=2$.

1.19 $x_n = \dfrac{3n^2+2}{4n^2-1}$, $a=\dfrac{3}{4}$.

1.20 $x_n = \dfrac{2-3n^2}{4+5n^2}$, $a=-\dfrac{3}{5}$.

1.21 $x_n = \dfrac{2n^3}{n^3-2}$, $a=2$.

2-masala. a **soni** $\{x_n\}$ **ketma-ketlikning limiti emasligi ta'rif yordamida ko'rsatilsin.**

2.1 $x_n=(-1)^n+1,\ a=0$.

2.2 $x_n=\cos\frac{\pi n}{3},\ a=\frac{1}{2}$.

2.3 $x_n=\sin\frac{\pi n}{6},\ a=\frac{1}{2}$.

2.4 $x_n=\cos\frac{\pi n}{100},\ a=-1$.

2.5 $x_n=2^{(-1)^n n},\ a=0$.

2.6 $x_n=n\cdot\left[1+(-1)^n\right],\ a=0$.

2.7 $x_n=(-1)^n,\ a=-1$.

2.8 $x_n=(-1)^{n+1},\ a=1$.

2.9 $x_n=\frac{2-\cos\pi n}{2+\cos\pi n},\ a=3$.

2.10 $x_n=\left(\frac{1}{2}\right)^n,\ a=1$.

2.11 $x_n=\frac{n^2-1}{n^3},\ a=1$.

2.12 $x_n=\frac{2n+3}{n^2},\ a=2$.

2.13 $x_n=\frac{1}{2n+1},\ a=\frac{1}{2}$.

2.14 $x_n=\sin\frac{\pi n}{2},\ a=1$.

2.15 $x_n=\frac{(-1)^2}{n},\ a=-1$.

2.16 $x_n=\sin\frac{\pi n}{3},\ a=0$.

2.17 $x_n=\frac{n+1}{3-2n^2},\ a=-\frac{1}{2}$.

2.18 $x_n=(-1)^n n,\ a=-1$.

2.19 $x_n=n^{(-1)^n},\ a=0$.

2.20 $x_n=\frac{n^2-2n}{n+1},\ a=1$.

2.21 $x_n=\sqrt{n^2+1}-n,\ a=1$.

3-masala.

Yaqinlashuvchi ketma-ketlikning chegaralanganligi haqidagi teoremadan foydalanib $\{x_n\}$ **ketma-ketlikning uzoqlashuvchi ekanligi ko"rsatilsin.**

3.1 $x_n=n^{(-1)^n}$

3.2 $x_n=n^2\sin\frac{\pi n}{4}$

3.3 $x_n=\sqrt{n}\cos\frac{\pi n}{2}$

3.4 $x_n=(-1)^n\ln n$

3.5 $x_n=(-1)^n\ln\frac{1}{n}$

3.6 $x_n=\frac{1}{n}+n^{(-1)^n}$

Cheksiz kichik ketma-ketlikning chegaralangan ketma-ketlikka ko'paytmasi haqidagi teoremadan foydalanib $\{x_n\}$ **ketma-ketlikning yaqinlashuvchi ekanligi ko'rsatilsin.**

3.7 $x_n=\frac{\operatorname{sgn}(\operatorname{tg} n)}{n}$.

3.8 $x_n=\frac{1}{n(8+\sin n)}$.

3.9 $x_n=\frac{1}{n^2\cdot\left[2+(-1)^n\right]}$.

3.10 $x_n = \frac{\cos \pi n}{\sqrt{n}}$. 3.11 $x_n = \frac{\sin \frac{\pi n}{4}}{n^2}$. 3.12 $x_n = \frac{\frac{n}{2} - \left[\frac{n}{2}\right]}{\ln(n+1)}$.

Qismiy ketma-ketlikning limiti haqidagi teoremadan foydalanib $\{x_n\}$ ketma-ketlikning uzoqlashuvchi ekanligi ko'rsatilsin.

3.13 $x_n = (0{,}5)^{(-1)^n n}$.

3.14 $x_n = 2^{(-1)^n n}$.

3.15 $x_n = \left[2 + (-1)^n\right]^n$.

3.16 $x_n = \sin \frac{\pi n}{2002}$.

«Ikki mirshab haqidagi teorema» dan foydalanib $\{x_n\}$ ketma-ketlikning yaqinlashuvchiligi ko'rsatilsin.

3.17 $x_n = \left(\frac{5}{n}\right)^n$.

3.18 $x_n = \left(\frac{n+10}{2n-1}\right)^n$.

3.19 $x_n = \frac{2^n}{(2n)!}$.

3.20 $x_n = \frac{-1 + \sqrt{n} + \sin n}{n}$.

3.21 $x_n = \left(\frac{2n+3}{n^2}\right)^n$.

4-masala. Koshi kriteriyasi, monoton ketma-ketlikning limiti haqidagi teorema yoki limitlar ustidagi amallar haqidagi teoremalardan foydalanib $\{x_n\}$ ketma-ketlik yaqinlashishga tekshirilsin.

4.1 $x_n = n^{(-1)^n} + \frac{\operatorname{sgn}(tgn)}{n}$.

4.2 $x_n = \frac{1}{n} - \frac{\frac{n}{2} - \left[\frac{n}{2}\right]}{\ln(n+1)}$.

4.3 $x_n = \frac{n^2 + n}{n - n^2}$.

4.4 $x_n = (-1)^n \cdot \left(1 + \frac{\cos \pi n}{\sqrt{n}}\right)$.

4.5 $x_n = \frac{n + \sin\left(\frac{\pi n}{4}\right)}{n^2}$.

4.6 $x_n = \frac{n+1}{n^2 \cdot (8 + \sin n)}$.

4.7 $x_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{2^n}\right)$.

4.8 $x_n = \left(1 - \frac{1}{\lg 11}\right)\left(1 - \frac{1}{\lg 12}\right) \cdot \ldots \cdot \left(1 - \frac{1}{\lg(10+n)}\right)$.

4.9 $x_n = \frac{\sin\alpha}{2} + \frac{\sin 2\alpha}{2^2} + \ldots + \frac{\sin n\alpha}{2^n}$.

4.10 $x_n = \frac{|\sin 1|}{2} + \frac{|\sin 2|}{2^2} + \ldots + \frac{|\sin n|}{2^n}$.

4.11 $x_n = \frac{1}{2^2} + \frac{2}{3^2} + \ldots + \frac{n}{(n+1)^2}$.

4.12 $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

4.13 $x_n = \frac{n!}{n^n}$.

4.14 $x_n = \left(1 + \frac{1}{n}\right)^n$.

4.15 $x_n = \cos 1 + \frac{\cos 2}{2^2} + \ldots + \frac{\cos n}{n^2}$.

4.16 $x_n = 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$.

4.17 $x_n = \frac{\lg n}{n}$.

4.18 $x_n = \frac{2^n}{n!}$.

4.19 $x_n = 0,\underbrace{77\ldots7}_{n \text{ ta}}$.

4.20 $x_n = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \ldots + \frac{(-1)^{n-1}}{n(n+1)}$.

4.21 $x_n = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$.

5-masala. Sonli ketma-ketlikning limiti hisoblansin $\left(\lim\limits_{n\to\infty} x_n - ?\right)$.

5.1 $x_n = n\left[\sqrt{n(n-2)} - \sqrt{n^2-3}\right]$.

5.2 $x_n = \left(n - \sqrt[3]{n^3-5}\right)n\sqrt{n}$.

5.3 $x_n = \sqrt{(n^2+1)(n^2-4)} - \sqrt{n^4-9}$.

5.4 $x_n = \frac{\sqrt{n^5-8} - n\sqrt{n(n^2+5)}}{\sqrt{n}}$.

5.5 $x_n = \sqrt{n^2-3n+2} - n$.

5.6 $x_n = n + \sqrt[3]{4-n^2}$.

5.7 $x_n = \sqrt{n(n+2)} - \sqrt{n^2-2n+3}$.

5.8 $x_n = \sqrt{(n+2)(n+1)} - \sqrt{(n-1)(n+3)}$.

5.9 $x_n = n^2\left[\sqrt{n(n^4-1)} - \sqrt{n^5-8}\right]$.

5.10 $x_n = n^2\left(\sqrt[3]{5+n^3} - \sqrt[3]{3+n^3}\right)$.

5.11 $x_n = \sqrt{n^2+3n-2} - \sqrt{n^2-3}$.

5.12 $x_n = \sqrt{n}\left(\sqrt{n+2} - \sqrt{n-3}\right)$.

5.13 $x_n = \sqrt{n(n+5)} - n$.

5.14 $x_n = \sqrt{n^3+8}\left(\sqrt{n^3+2} - \sqrt{n^3-1}\right)$.

5.15 $x_n = \frac{\sqrt{(n^3+1)(n^2+3)} - \sqrt{n(n^4+2)}}{2\sqrt{n}}$.

5.16 $x_n = n - \sqrt{n(n-1)}$.

5.17 $x_n = n^3\left[\sqrt[3]{n^2\left(n^6+4\right)}-\sqrt[3]{n^8-1}\right]$. **5.18** $x_n = \sqrt[3]{n}\left[\sqrt[3]{n^2}-\sqrt[3]{n(n-1)}\right]$.

5.19 $x_n = \sqrt{n+2}\left(\sqrt{n+3}-\sqrt{n-4}\right)$. **5.20** $x_n = n\left(\sqrt{n^4+3}-\sqrt{n^4-2}\right)$.

5.21 $x_n = n\left(\sqrt[3]{5+8n^3}-2n\right)$.

6-masala $\lim\limits_{n\to\infty} x_n - ?$

6.1 $x_n = \frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n-1}{n^2}$. **6.2** $x_n = \frac{(2n+1)!+(2n+2)!}{(2n+3)!}$.

6.3 $x_n = \frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}$. **6.4** $x_n = \frac{2^{n+1}+3^{n+1}}{2^n+3^n}$.

6.5 $x_n = \frac{1+2+\ldots+n}{\sqrt{9n^4+1}}$. **6.6** $x_n = \frac{1+3+5+\ldots+(2n-1)}{1+2+3+\ldots+n}$.

6.7 $x_n = \frac{1+3+5+\ldots+(2n-1)}{n+3}-n$. **6.8** $x_n = \frac{1+4+7+\ldots+(3n-2)}{\sqrt{5n^4+n+1}}$.

6.9 $x_n = \frac{(n+4)!-(n+2)!}{(n+3)!}$. **6.10** $x_n = \frac{(3n-1)!+(3n+1)!}{3n!\cdot(n-1)}$.

6.11 $x_n = \frac{2^n-5^{n+1}}{2^{n+1}+5^{n+2}}$. **6.12** $x_n = \frac{1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}}{1+\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^n}}$.

6.13 $x_n = \frac{\sqrt[3]{n^3+5}-\sqrt{3n^4+2}}{1+3+5+\ldots+(2n-1)}$. **6.14** $x_n = \frac{3^n-2^n}{3^{n-1}+2^n}$.

6.15 $x_n = \frac{n+2}{1+2+3+\ldots+n}-\frac{2}{3}$. **6.16** $x_n = \frac{5}{6}+\frac{13}{36}+\ldots+\frac{3^n+2^n}{6^n}$.

6.17 $x_n = \frac{2-5+4-7+\ldots+2n-(2n+3)}{n+3}$. **6.18** $x_n = \frac{(2n+1)!+(2n+2)!}{(2n+3)!-(2n+2)!}$.

6.19 $x_n=\dfrac{n^2+\sqrt{n}-1}{2+7+12+...+(5n-3)}$. **6.20** $x_n=\dfrac{2+4+...+2n}{n+3}-n$.

6.21 $x_n=\dfrac{3}{4}+\dfrac{5}{16}+\dfrac{9}{64}+...+\dfrac{1+2^n}{4^n}$.

7-masala. $\lim\limits_{n\to\infty}x_n-?$

7.1 $x_n=\left(\dfrac{3n^2-6n+7}{3n^2+20n-1}\right)^{-n+1}$. **7.2** $x_n=\left(\dfrac{2n^2+2}{2n^2+1}\right)^{n^2}$.

7.3 $x_n=\left(\dfrac{n-1}{n+3}\right)^{n+2}$. **7.4** $x_n=\left(\dfrac{n^2-1}{n^2}\right)^{n^4}$.

7.5 $x_n=\left(\dfrac{2n+3}{2n+1}\right)^{n+1}$. **7.6** $x_n=\left(\dfrac{n+1}{n-1}\right)^{n}$.

7.7 $x_n=\left(\dfrac{n^2-3n+6}{n^2+5n+1}\right)^{\frac{n}{2}}$. **7.8** $x_n=\left(\dfrac{n-10}{n+1}\right)^{3n+1}$.

7.9 $x_n=\left(\dfrac{6n-7}{6n+4}\right)^{3n+2}$. **7.10** $x_n=\left(\dfrac{3n^2+4n-1}{3n^2+2n+7}\right)^{2n+5}$.

7.11 $x_n=\left(\dfrac{n^2+n+1}{n^2+n-1}\right)^{-n^2}$. **7.12** $x_n=\left(\dfrac{2n^2+5n+7}{2n^2+5n+3}\right)^{n}$.

7.13 $x_n=\left(\dfrac{n-1}{n+1}\right)^{n^2}$. **7.14** $x_n=\left(\dfrac{5n^2+3n-1}{5n^2+3n+3}\right)^{n^3}$.

7.15 $x_n=\left(\dfrac{3n+1}{3n-1}\right)^{2n+3}$. **7.16** $x_n=\left(\dfrac{2n^2+7n-1}{2n^2+3n-1}\right)^{-n^2}$.

7.17 $x_n=\left(\dfrac{n+3}{n+5}\right)^{n+4}$. **7.18** $x_n=\left(\dfrac{n^3+1}{n^3-1}\right)^{2n-n^3}$.

7.19 $x_n=\left(\dfrac{10n-3}{10n-1}\right)^{5n}$. **7.20** $x_n=\left(\dfrac{3n^2-5n}{3n^2-5n+7}\right)^{n+1}$.

7.21 $x_n=\left(\dfrac{2n^2+21n-7}{2n^2+18n+9}\right)^{2n+1}$.

8-masala. $\{x_n\}$ **ketma-ketlikning yuqori va quyi limitlari topilsin** $\left(\overline{\lim_{n\to\infty}} x_n - ?,\ \underline{\lim_{n\to\infty}} x_n - ?\right)$.

8.1 $x_n = \left(\cos\frac{\pi n}{2}\right)^{n+1}$.

8.2 $x_n = \frac{1+(-1)^n n}{n}$.

8.3 $x_n = (-n)^{\sin\frac{\pi n}{2}}$.

8.4 $x_n = \frac{1}{n} + \sin\frac{\pi n}{3}$.

8.5 $x_n = \frac{\left[1-(-1)^n\right]\cdot 2^n + 1}{2^n + 3}$.

8.6 $x_n = \sqrt[n]{4^{(-1)^n} + 2}$.

8.7 $x_n = 2^{(-1)^n n}$.

8.8 $x_n = (-1)^{n-1}\left(1 + \frac{2}{n+1}\right)$.

8.9 $x_n = 2 + \frac{n}{n+1}\cos\frac{n\pi}{2}$.

8.10 $x_n = \frac{n}{n+1}\sin^2\frac{\pi n}{4}$.

8.11 $x_n = \frac{n-1}{n+1}\cos n\pi$.

8.12 $x_n = \frac{n^2+1}{n^2-1}\sin^n\frac{\pi n}{2}$.

8.13 $x_n = \frac{3n-2}{n}\sin\frac{\pi n}{2}$.

8.14 $x_n = \left(1{,}5\cdot\cos\frac{2\pi n}{3}\right)^n$.

8.15 $x_n = \frac{n^2\sin\frac{\pi n}{2} + 1}{n+1}$.

8.16 $x_n = (-1)^{\frac{n(n+1)}{2}}\frac{n^2+3n-1}{1-n-n^2}$.

8.17 $x_n = \frac{n^2-n+1}{2n^2-1}\cos\frac{2\pi n}{3}$.

8.18 $x_n = \frac{4n^2+3n-2}{2-n+n^2}\sin\left(\frac{\pi}{6} + \frac{2\pi n}{3}\right)$.

8.19 $x_n = \left(\frac{n+1}{n}\right)^n(-1)^n + \sin\frac{\pi n}{2}$.

8.20 $x_n = \frac{n+2}{n+1}\sin\frac{\pi n}{3}$.

8.21 $x_n = \sin n^0$.

9-masala. $y=f(x)$ **funksiyaning aniqlanish sohasi topilsin** $(D(f)-?)$.

9.1 $y=\ln\left[1-\lg\left(x^2-5x+16\right)\right]$.

9.2 $y=\log_x \log_{0,5}\left(\frac{4}{5}-2^{x-1}\right)$.

9.3 $y=\log_{x+1}\left(x^2-3x+2\right)$.

9.4 $y=\frac{\sqrt{x^2-4}}{\log_2\left(x^2+2x-2\right)}$.

9.5 $y=\frac{\sqrt{x+5}}{\lg(9-5x)}$.

9.6 $y=\sqrt{\log_3\frac{2x-3}{x-1}}$.

9.7 $y=\lg\frac{3x-x^2}{x-1}$.

9.8 $y=\sqrt[3]{\frac{x+2}{\lg\cos x}}$.

9.9 $y=\lg\left(16-x^2\right)+ctgx$.

9.10 $y=\left(8-2x-x^2\right)^{-\frac{3}{2}}$.

9.11 $y=\sqrt{x^2-|x|-2}$.

9.12 $y=\sqrt[3]{\frac{x}{1-|x|}}$.

9.13 $y=\sqrt{3-5x-2x^2}$.

9.14 $y=\sqrt{\frac{x}{6-x}}$.

9.15 $y=\frac{\operatorname{tg} x}{\cos 2x}$.

9.16 $y=\sqrt{\frac{\sin x+\cos x}{\sin x-\cos x}}$.

9.17 $y=\arccos(0,5x-1)$.

9.18 $y=\arccos x-\arcsin(3-x)$.

9.19 $y=\operatorname{arctg}\frac{x}{x^2-9}$.

9.20 $y=\arcsin\frac{x^2-1}{x}$.

9.21 $y=\frac{\arcsin(0,5x-1)}{\sqrt{x^2-3x+1}}$.

10-masala. Quyidagi tengliklar ta'rif yordamida isbotlansin ($\delta(\varepsilon)$**—topilsin).**

10.1 $\lim\limits_{x\to-3}\frac{2x^2+5x-3}{x+3}=-7$.

10.2 $\lim\limits_{x\to1}\frac{5x^2-4x-1}{x-1}=6$.

10.3 $\lim\limits_{x \to -2} \dfrac{3x^2 + 5x - 2}{x + 2} = -7$.

10.4 $\lim\limits_{x \to 3} \dfrac{4x^2 - 14x + 6}{x - 3} = 10$.

10.5 $\lim\limits_{x \to -\frac{1}{2}} \dfrac{6x^2 + x - 1}{x + \frac{1}{2}} = -5$.

10.6 $\lim\limits_{x \to \frac{1}{2}} \dfrac{6x^2 - x - 1}{x - \frac{1}{2}} = 5$.

10.7 $\lim\limits_{x \to -\frac{1}{3}} \dfrac{9x^2 - 1}{x + \frac{1}{3}} = -6$.

10.8 $\lim\limits_{x \to 2} \dfrac{3x^2 - 5x - 2}{x - 2} = 7$.

10.9 $\lim\limits_{x \to -\frac{1}{3}} \dfrac{3x^2 - 2x - 1}{x + \frac{1}{3}} = -4$.

10.10 $\lim\limits_{x \to -1} \dfrac{7x^2 + 8x + 1}{x + 1} = -6$.

10.11 $\lim\limits_{x \to 3} \dfrac{x^2 - 4x + 3}{x - 3} = 2$.

10.12 $\lim\limits_{x \to \frac{1}{2}} \dfrac{2x^2 + 3x - 2}{x - \frac{1}{2}} = 5$.

10.13 $\lim\limits_{x \to \frac{1}{3}} \dfrac{6x^2 - 5x + 1}{x - \frac{1}{3}} = -1$.

10.14 $\lim\limits_{x \to -\frac{7}{5}} \dfrac{10x^2 + 9x - 7}{x + \frac{7}{5}} = -19$.

10.15 $\lim\limits_{x \to -\frac{7}{2}} \dfrac{2x^2 + 13x + 21}{2x + 7} = -\dfrac{1}{2}$.

10.16 $\lim\limits_{x \to \frac{5}{2}} \dfrac{2x^2 - 9x + 10}{2x - 5} = \dfrac{1}{2}$.

10.17 $\lim\limits_{x \to \frac{1}{2}} \dfrac{6x^2 + x - 1}{x - \frac{1}{3}} = 5$.

10.18 $\lim\limits_{x \to -\frac{1}{2}} \dfrac{6x^2 - 75x - 39}{x + \frac{1}{2}} = -81$.

10.19 $\lim\limits_{x \to 11} \dfrac{2x^2 - 21x - 11}{x - 11} = 23$.

10.20 $\lim\limits_{x \to 5} \dfrac{5x^2 - 24x - 5}{x - 5} = 26$.

10.21 $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 - 3x} = 2$.

11-masala. Limitlar hisoblansin.

11.1 $\lim\limits_{x\to\frac{1}{3}}\dfrac{\sqrt[3]{\dfrac{x}{9}}-\dfrac{1}{3}}{\sqrt{\dfrac{1}{3}+x}-\sqrt{2x}}$.

11.2 $\lim\limits_{x\to\frac{1}{2}}\dfrac{\sqrt[3]{\dfrac{x}{4}}-\dfrac{1}{2}}{\sqrt{\dfrac{1}{2}+x}-\sqrt{2x}}$.

11.3 $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x^2}-4}$.

11.4 $\lim\limits_{x\to 4}\dfrac{\sqrt[3]{16x}-4}{\sqrt{4+x}-\sqrt{2x}}$.

11.5 $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{x-6}+2}{x+2}$.

11.6 $\lim\limits_{x\to 3}\dfrac{\sqrt[3]{9x}-3}{\sqrt{3+x}-\sqrt{2x}}$.

11.7 $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x^2-1}$.

11.8 $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{4x}-2}{\sqrt{2+x}-\sqrt{2x}}$.

11.9 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$.

11.10 $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{1+x}-\sqrt{2x}}$.

11.11 $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\cdot\sqrt[3]{x^4}}$.

11.12 $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8+3x+x^2}-2}{x+x^2}$.

11.13 $\lim\limits_{x\to 0}\dfrac{\sqrt{1-2x+x^2}-(1+x)}{x}$.

11.14 $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$.

11.15 $\lim\limits_{x\to 16}\dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$.

11.16 $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{x-6}+2}{x^3+8}$.

11.17 $\lim\limits_{x\to 3}\dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$.

11.18 $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x^2}-1}$.

11.19 $\lim\limits_{x\to -8}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$.

11.20. $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.

11.21 $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$.

12-masala. Limitlar hisoblansin.

12.1 $\lim\limits_{x\to 1}\dfrac{x^2-1}{\ln x}$.

12.2 $\lim\limits_{x\to 1}\dfrac{\sqrt{x^2-x+1}-1}{\ln x}$.

12.3 $\lim\limits_{x\to \pi}\dfrac{1+\cos 3x}{\sin^2 7x}$.

12.4 $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{1-\sin 2x}{(\pi-4x)^2}$.

12.5 $\lim\limits_{x\to 1}\dfrac{1+\cos \pi x}{tg^2\pi x}$.

12.6 $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{tg3x}{tgx}$.

12.7 $\lim\limits_{x\to \pi}\dfrac{\sin^2 x-tg^2x}{(x-\pi)^4}$.

12.8 $\lim\limits_{x\to 1}\dfrac{\sqrt{x^2-x+1}-1}{tg\pi x}$.

12.9 $\lim\limits_{x\to \pi}\dfrac{\cos 5x-\cos 3x}{\sin^2 x}$.

12.10 $\lim\limits_{x\to 2\pi}\dfrac{\sin 7x-\sin 3x}{e^{x^2}-e^{4\pi^2}}$.

12.11 $\lim\limits_{x\to 2}\dfrac{\ln(5-2x)}{\sqrt{10-3x}-2}$.

12.12 $\lim\limits_{x\to 1}\dfrac{\sqrt{x^2-3x+3}-1}{\sin \pi x}$.

12.13 $\lim\limits_{x\to \pi}\dfrac{x^2-\pi^2}{\sin x}$.

12.14 $\lim\limits_{x\to 1}\dfrac{3^{5x-3}-3^{2x^2}}{tg\pi x}$.

12.15 $\lim\limits_{x\to 4}\dfrac{2^x-16}{\sin \pi x}$.

12.16 $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\ln 2x-\ln \pi}{\sin\dfrac{5x}{2}\cos x}$.

12.17 $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\ln tgx}{\cos 2x}$.

12.18 $\lim\limits_{x\to 2}\dfrac{\ln(9-2x^2)}{\sin 2\pi x}$.

12.19 $\lim\limits_{x\to 2}\dfrac{1-2^{4-x^2}}{2\left(\sqrt{2x}-\sqrt{3x^2-5x+2}\right)}$.

12.20 $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{1-2\cos x}{\pi-3x}$.

12.21 $\lim\limits_{x\to \pi}\dfrac{e^\pi-e^x}{\sin 5x-\sin 3x}$.

13-masala. Limitlar hisoblansin.

13.1 $\lim\limits_{x\to 1}\left(\frac{3x-1}{x+1}\right)^{\frac{1}{\sqrt[3]{x}-1}}$.

13.2 $\lim\limits_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}$.

13.3 $\lim\limits_{x\to 1}\left(\frac{2x-1}{x}\right)^{\frac{1}{\sqrt[3]{x}-1}}$.

13.4 $\lim\limits_{x\to 2}\left(\frac{\cos x}{\cos 2}\right)^{\frac{1}{x-2}}$.

13.5 $\lim\limits_{x\to 8}\left(\frac{2x-7}{x+1}\right)^{\frac{1}{\sqrt[3]{x}-2}}$.

13.6 $\lim\limits_{x\to \frac{\pi}{4}}(tgx)^{1/\cos(3\pi/4-x)}$.

13.7 $\lim\limits_{x\to 1}\left(\frac{2x-1}{x}\right)^{\frac{1}{\sqrt[5]{x}-1}}$.

13.8 $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{tg\frac{\pi x}{2a}}$.

13.9 $\lim\limits_{x\to 2\pi}(\cos x)^{ctg 2x/\sin 2x}$.

13.10 $\lim\limits_{x\to 2\pi}(\cos x)^{1/\sin^2 2x}$.

13.11 $\lim\limits_{x\to 3}\left(\frac{6-x}{3}\right)^{tg\frac{\pi x}{6}}$.

13.12 $\lim\limits_{x\to 4\pi}(\cos x)^{ctgx/\sin 4x}$.

13.13 $\lim\limits_{x\to 1}(3-2x)^{tg\frac{\pi x}{2}}$.

13.14 $\lim\limits_{x\to 4\pi}(\cos x)^{5/tg5x\cdot\sin 2x}$.

13.15 $\lim\limits_{x\to 3}\left(\frac{6-x}{3}\right)^{tg\frac{\pi x}{6}}$.

13.16 $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{6tgx\cdot tg3x}$.

13.17 $\lim\limits_{x\to 1}\left(2e^{x-1}-1\right)^{\frac{x}{x-1}}$.

13.18 $\lim\limits_{x\to \frac{\pi}{2}}\left(tg\frac{x}{2}\right)^{\frac{1}{x-\frac{\pi}{2}}}$.

13.19 $\lim\limits_{x\to 1}\left(2e^{x-1}-1\right)^{\frac{3x-1}{x-1}}$.

13.20 $\lim\limits_{x\to \frac{\pi}{2}}(1+\cos 3x)^{\sec x}$.

13.21 $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{18\sin x/ctgx}$.

14-masala. Limitlar hisoblansin.

14.1 $\lim_{x\to 0}\frac{7^{2x}-5^{3x}}{2x-arctg3x}$.

14.2 $\lim_{x\to 0}\frac{e^{3x}-e^{-2x}}{2\arcsin x-\sin x}$.

14.3 $\lim_{x\to 0}\frac{6^{2x}-7^{-2x}}{\sin 3x-2x}$.

14.4 $\lim_{x\to 0}\frac{e^{5x}-e^{3x}}{\sin 2x-\sin x}$.

14.5 $\lim_{x\to 0}\frac{3^{2x}-5^{3x}}{arctgx+x^3}$.

14.6 $\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{arctgx-x^2}$.

14.7 $\lim_{x\to 0}\frac{3^{5x}-2^{x}}{x-\sin 9x}$.

14.8 $\lim_{x\to 0}\frac{e^{4x}-e^{-2x}}{2arctgx-\sin x}$.

14.9 $\lim_{x\to 0}\frac{12^{x}-5^{-3x}}{2\arcsin x-x}$.

14.10 $\lim_{x\to 0}\frac{e^{7x}-e^{-2x}}{\sin x-2x}$.

14.11 $\lim_{x\to 0}\frac{3^{5x}-2^{7x}}{\arcsin 2x-x}$.

14.12 $\lim_{x\to 0}\frac{e^{5x}-e^{x}}{\arcsin x+x^3}$.

14.13 $\lim_{x\to 0}\frac{4^{x}-2^{7x}}{tg3x-x}$.

14.14 $\lim_{x\to 0}\frac{e^{x}-e^{-x}}{tg2x-\sin x}$.

14.15 $\lim_{x\to 0}\frac{10^{2x}-7^{-x}}{2tgx-arctgx}$.

14.16 $\lim_{x\to 0}\frac{e^{2x}-e^{x}}{\sin 3x-\sin 5x}$.

14.17 $\lim_{x\to 0}\frac{7^{3x}-3^{2x}}{tgx+x^3}$.

14.18 $\lim_{x\to 0}\frac{e^{4x}-e^{2x}}{2tgx-\sin x}$.

14.19 $\lim_{x\to 0}\frac{3^{2x}-7^{x}}{\arcsin 3x-5x}$.

14.20 $\lim_{x\to 0}\frac{e^{2x}-e^{-5x}}{2\sin x-tgx}$.

14.21 $\lim_{x\to 0}\frac{9^{x}-2^{3x}}{arctg2x-7x}$.

15-masala. $y=f(x)$ **funksiyaning** $x=x_0$ **nuqtadagi o'ng va chap limitlari topilsin** $\left(f(x_0+0)-?,\ f(x_0-0)-?\right)$.

15.1 $f(x)=\text{arctg}\frac{1}{1-x},\ x_0=1$.

15.2 $f(x)=\frac{1}{1+e^{\frac{1}{x}}},\ x_0=0$.

15.3 $f(x)=\frac{x-|x|}{2x}$, $x_0=0$.

15.4 $f(x)=\arccos(x-1)$, $x_0=0$.

15.5 $f(x)=2^{ctgx}$, $x_0=0$.

15.6 $f(x)=\frac{2(1-x^2)+|1-x^2|}{3(1-x^2)-|1-x^2|}$, $x_0=1$.

15.7 $f(x)=\text{sign}(\cos x), x_0=\frac{\pi}{2}$.

15.8 $f(x)=\text{arctg}(\text{tg}\,x), x_0=\frac{\pi}{2}$.

15.9 $f(x)=\frac{1}{x+3^{\frac{1}{3-x}}}$, $x_0=3$.

15.10 $f(x)=\frac{1}{x-[x]}$, $x_0=-1$.

15.11 $f(x)=x+[x^2], x_0=10$.

15.12 $f(x)=\lim\limits_{n\to\infty}\frac{x^n}{1+x^n}$, $x_0=1$.

15.13 $f(x)=\lim\limits_{n\to\infty}\frac{2x^n-3}{x^n-1}$, $x_0=1$.

15.14 $f(x)=e^{-\frac{1}{2x}}, x_0=0$.

15.15 $f(x)=\frac{\sin x}{|x|}$, $x_0=0$.

15.16 $f(x)=\frac{x}{(x-3)^3}, x_0=3$.

15.17 $f(x)=\frac{x^3-1}{|x-1|}$, $x_0=1$.

15.18 $f(x)=\frac{2}{1+2^{\frac{1}{2-x}}}$, $x_0=2$.

15.19 $f(x)=\frac{\sqrt{1-\cos 2x}}{x}$, $x_0=0$.

15.20 $f(x)=\frac{\cos x}{3-2^{\frac{1}{\sin x}}}$, $x_0=0$.

15.21 $f(x)=\begin{cases} x+1, & x\le 2, \\ -2x+1, & x>2 \end{cases}, x_0=2$.

16-masala. $y=f(x)$ **funksiya** $x=x_0$ **nuqtada uzluksiz ekanligi ta'rif yordamida isbotlansin (** $\delta(\varepsilon)$ **–topilsin).**

16.1 $f(x)=5x^2-1$, $x_0=6$.

16.2 $f(x)=4x^2-2$, $x_0=5$.

16.3 $f(x)=3x^2-3$, $x_0=4$.

16.4 $f(x)=2x^2-4$, $x_0=3$.

16.5 $f(x)=-2x^2-5$, $x_0=2$.

16.6 $f(x)=-3x^2-6$, $x_0=1$.

16.7 $f(x)=-4x^2-7$, $x_0=1$.

16.8 $f(x)=-5x^2-8$, $x_0=2$.

16.9 $f(x)=-5x^2-9$, $x_0=3$.

16.10 $f(x)=-4x^2+9$, $x_0=4$.

16.11 $f(x)=-3x^2+8,\ x_0=5$.

16.12 $f(x)=-2x^2+7,\ x_0=6$.

16.13 $f(x)=2x^2+6,\ x_0=7$.

16.14 $f(x)=3x^2+5,\ x_0=8$.

16.15 $f(x)=4x^2+4,\ x_0=9$.

16.16 $f(x)=5x^2+3,\ x_0=8$.

16.17 $f(x)=5x^2+1,\ x_0=7$.

16.18 $f(x)=4x^2-1,\ x_0=6$.

16.19 $f(x)=3x^2-2,\ x_0=5$.

16.20 $f(x)=2x^2-3,\ x_0=4$.

16.21 $f(x)=-2x^2-4,\ x_0=3$.

17-masala.

Quyidagi funksiyalar *a* ning qanday qiymatlarida +uzluksiz bo'lishi aniqlansin.

17.1 $y=\begin{cases} x\operatorname{ctg}2x, & x\neq 0,\ |x|<\dfrac{\pi}{2}, \\ a, & x=0 \end{cases}$

17.2 $y=\begin{cases} ax^2+1, & x>0, \\ -x, & x\leq 0 \end{cases}$

17.3 $y=\begin{cases} \cos x, & x\leq 0, \\ a(x-1), & x>0 \end{cases}$

17.4 $y=\begin{cases} x^2+a, & x>0, \\ 1-x^2, & x\leq 0 \end{cases}$

17.5 $y=\begin{cases} 2^x, & x\geq 0, \\ a(x-1), & x<0 \end{cases}$

17.6 $y=\begin{cases} (\pi+2x)\operatorname{tg}x, & -\pi<x<\dfrac{\pi}{2},\ x\neq-\dfrac{\pi}{2}, \\ a, & x=-\dfrac{\pi}{2} \end{cases}$

17.7 $y=\begin{cases} (\arcsin x)ctgx, & x\neq 0, \\ a, & x=0 \end{cases}$

17.8 $y=\begin{cases} \dfrac{c^x-1}{x}, & x\neq 0, \\ a, & x=0,\ c>0, \end{cases}$

17.9 $y=\begin{cases} \dfrac{x}{\ln(1+2x)}, & x\neq 0 \\ a, & x=0 \end{cases}$

17.10 $y=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0, \\ a, & x=0 \end{cases}$

Quyidagi funksiyalar uzluksizlikka tekshirilsin va grafiklari chizilsin.

17.11 $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}$.

17.12 $\lim\limits_{t\to+\infty}\dfrac{\ln\left(1+e^{xt}\right)}{\ln\left(1+e^{t}\right)}$.

17.13 $f(x)=\lim\limits_{n\to\infty}\dfrac{1}{1+x^{2n+1}}$.

17.14 $f(x)=sign\left(\cos\dfrac{1}{x}\right)$.

17.15 $f(x)=\lim_{n\to\infty}\cos^{2n}x$.

17.16 $f(x)=[x]\sin\pi x$.

17.17 $f(x)=\lim_{n\to\infty}\frac{x+e^{nx}}{1+xe^{nx}}$.

17.18 $f(x)=\lim_{n\to\infty}\sqrt[2n]{\cos^{2n}x+\sin^{2n}x}$.

17.19 $f(x)=\lim_{n\to\infty}\sqrt[n]{1+x^{2n}}$.

17.20 $f(x)=x^2-[x^2]$.

17.21 $f(x)=\lim_{n\to\infty}\frac{x}{1+(2\sin x)^{2n}}$.

18-masala.
Quyidagi funksiyalar berilgan oraliqda tekis uzluksizlikka tekshirilsin

18.1 $f(x)=\frac{x}{4-x^2}$, $-1\le x\le 1$.

18.2 $f(x)=\ln x$, $0<x<1$.

18.3 $f(x)=\frac{\sin x}{x}$, $0<x<\pi$.

18.4 $f(x)=e^x\cos\frac{1}{x}$, $0<x<1$.

18.5 $f(x)=\operatorname{arctg}x$, $-\infty<x<+\infty$.

18.6 $f(x)=x\sin x, 0\le x<+\infty$.

18.7 $f(x)=\begin{cases}1-x^2, & -1\le x\le 0,\\ 1+x, & 0<x\le 1,\end{cases}$ $-1\le x\le 1$

18.8 $f(x)=\begin{cases}x+1, & x\le 0,\\ e^{-x}, & x>0,\end{cases}$ $-\infty<x<+\infty$

$y=f(x)$ **funksiya X to'plamda tekis uzluksiz emasligi isbotlansin.**

18.9 $f(x)=\cos\frac{1}{x}$, $X=(0,\ 1)$.

18.10 $f(x)=\frac{1}{x}$, $X=(0,\ 1)$.

18.11 $f(x)=\sin x^2$, $X=R$.

18.12 $f(x)=\sin\frac{1}{x}$, $X=(0,\ 1)$.

18.13 $f(x)=x^2$, $X=R$.

18.14 $f(x)=\frac{1}{x-2}$, $X=(2,\ 3)$.

$y=f(x)$ **funksiya X to'plamda tekis uzluksiz ekanligi ta'rif yordamida ko'rsatilsin (** $\delta=\delta(\varepsilon)$ **topilsin).**

18.15 $f(x)=-x+1$, $X=(-\infty,+\infty)$.

18.16 $f(x)=\sqrt[3]{x}$, $X=[0;\ 2]$.

18.17 $f(x)=\frac{1}{x}$, $X=[0{,}1;\ 1]$. **18.18** $f(x)=2x^2+5$, $X=[-1;\ 7]$.

18.19 $f(x)=x^2-2x-1$, $X=[-2;\ 5]$. **18.20** $f(x)=x^3+1$, $X=[-2;\ 3]$.

18.21 $f(x)=2\sin x-\cos x$, $X=R$.

–D–

Namunaviy variant yechimi

Namunaviy variant sifatida 21-variantni olib, shu variantdagi misol va masalalarning yechimlarini keltiramiz.

1.21-masala. $\lim\limits_{n\to\infty} x_n = a$ **ekanligi ta'rif yordamida ko'rsatilsin** $(n_0(\varepsilon)\ \ -?)$.

$$x_n=\frac{2n^3}{n^3-2},\quad a=2$$

$\triangleleft$ $(\lim\limits_{n\to\infty} x_n = a) \Leftrightarrow \left(\forall\varepsilon>0\ \exists n_0=n_0(\varepsilon)\in N:\ \forall n>n_0\ \ |x_n-a|<\varepsilon\right)$.

$$|x_n-a|=\left|\frac{2n^3}{n^3-3}-2\right|=\left|\frac{2n^3-2n^3+6}{n^3-3}\right|=\frac{6}{|n^3-3|}=$$

$$=\frac{6}{\left(n-\sqrt[3]{3}\right)\left(n^2+\sqrt[3]{3}n+\sqrt[3]{3^2}\right)}<\frac{6}{n^2+\sqrt[3]{3}n+\sqrt[3]{9}}<\frac{6}{\sqrt[3]{3}n}<$$

$$<\frac{6}{n}<\varepsilon\Rightarrow n>\frac{6}{\varepsilon}\Rightarrow n_0=\left[\frac{6}{\varepsilon}\right]$$

Demak, $\forall\varepsilon>0$ son olinganda ham $n_0=\max\left\{2,\ \left[\frac{6}{\varepsilon}\right]\right\}$ deb olsak, $\forall n>n_0$ uchun $|x_n-a|<\varepsilon$ bo'ladi. $\Rightarrow \lim\limits_{n\to\infty} x_n = a$ $\triangleright$

2.21-masala. a **soni** $\{x_n\}$ **ketma-ketlikning limiti emasligi ta'rif yordamida ko'rsatilsin.**

$$x_n=\sqrt{n^2+1}-n,\quad a=1$$

$\lhd$ $(\lim\limits_{n\to\infty} x_n \neq a) \Leftrightarrow$ $\left(\forall n_0 \in N\ \exists\varepsilon>0,\ \exists n>n_0:\ |x_n-a|\geq\varepsilon\right)$ $|x_n-a|=\left|\left(\sqrt{n^2+1}-n\right)-1\right|=$

$$=\left|\sqrt{n^2+1}-(n+1)\right|=\left|\frac{n^2+1-(n+1)^2}{\sqrt{n^2+1}+n+1}\right|=\frac{\left|n^2+1-n^2-2n-1\right|}{\sqrt{n^2+1}+n+1}=\frac{2n}{\sqrt{n^2+1}+n+1}\geq$$

$$\geq\frac{2n}{\sqrt{n^2+3n^2}+n+n}=\frac{2n}{2n+2n}=\frac{2n}{4n}=\frac{1}{2}$$

Demak, $\varepsilon=\frac{1}{2}$ deb olsak, $\forall n\in N$ uchun $|x_n-a|\geq\varepsilon$ tengsizlik bajarilar ekan. Bu esa $\lim\limits_{n\to\infty} x_n \neq a$ ekanligini anglatadi. $\rhd$

3.21-masala. "Ikki mirshab haqidagi teorema"dan foydalanib $\{x_n\}$ ketma-ketlikning yaqinlashuvchiligi ko'rsatilsin.

$$x_n=\left(\frac{2n+3}{n^2}\right)^n$$

$\lhd$ $\frac{2n+3}{n^2}\leq\frac{2n+3n}{n^2}=\frac{5n}{n^2}=\frac{5}{n}\leq\frac{1}{2}$, agar $n\geq10$ bo'lsa,

$\frac{2n+3}{n^2}\geq\frac{2+3}{n^2}=\frac{5}{n^2}\geq\frac{1}{20}$ agar $n\geq10$ bo'lsa.

Agar $y_n=\frac{1}{20^n}$ va $z_n=\frac{1}{2^n}$ deb belgilasak, unda $\forall\, n\geq10$ uchun $y_n\leq x_n\leq z_n$ qo'sh tengsizlik bajariladi. $\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty} z_n=0$ va "ikki mirshab haqidagi teorema" ga ko'ra $\lim\limits_{n\to\infty} x_n=0$ bo'ladi. $\rhd$

4.21-masala. $x_n=1+\frac{1}{2!}+\ldots+\frac{1}{n!}$ **ketma-ketlik Koshi kriteriyasi, monoton ketma-ketlikning limiti haqidagi teorema yoki limitlar ustidagi amallar haqidagi teoremalardan foydalanib, yaqinlashishga tekshirilsin.**

$\lhd$ Monoton ketma-ketlikning limiti haqidagi teoremadan foydalanib, berilgan $\{x_n\}$ ketma-ketlikning yaqinlashuvchi ekanligini ko'rsatamiz.

$$x_{n+1}=x_n+\frac{1}{(n+1)!}>x_n\Rightarrow\{x_n\}\uparrow.$$

Endi bu ketma-ketlikning yuqoridan chegaralanganligini ko'rsatamiz:

$$x_n = 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}...+\frac{1}{n!}<1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}...+\frac{1}{2^{n-1}}<$$

$$<1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}...+\frac{1}{2^{n-1}}+\frac{1}{2^n}+...=\frac{1}{1-\frac{1}{2}}=2$$

Shunday qilib, $\{x_n\}\uparrow$ va $\forall n\in N$ uchun $x_n\le 2 \Rightarrow \lim\limits_{n\to\infty} x_n - \exists \Rightarrow \{x_n\}$ –yaqinlashuvchi ▷

5.21-masala. Sonli ketma-ketlikning limiti hisoblansin

$$(\lim_{n\to\infty} x_n - ?)$$

$$x_n = n\left(\sqrt[3]{5+8n^3}-2n\right)$$

$$\lhd \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} n\left(\sqrt[3]{5+8n^3}-2n\right) = \lim_{n\to\infty}\frac{n\left[\left(5+8n^3\right)-(2n)^3\right]}{\sqrt[3]{\left(5+8n^3\right)^2}+2n\cdot\sqrt[3]{5+8n^3}+4n^2}=$$

$$=\lim_{n\to\infty}\frac{5n}{n^2\left[\sqrt[3]{\left(\frac{5}{n^3}+8\right)^2}+2\cdot\sqrt[3]{\frac{5}{n^3}+8}+4\right]}=\lim_{n\to\infty}\frac{5}{n\cdot\left[\sqrt[3]{\left(\frac{5}{n^3}+8\right)^2}+2\cdot\sqrt[3]{\frac{5}{n^3}+8}+4\right]}=0. \rhd$$

6.21-masala. $\lim\limits_{n\to\infty} x_n - ?$

$$x_n = \frac{3}{4}+\frac{5}{16}+\frac{9}{64}+...+\frac{1+2^n}{4^n}$$

$$\lhd \quad x_n = \frac{3}{4}+\frac{5}{16}+\frac{9}{64}+...+\frac{1+2^n}{4^n}=\frac{1+2}{4}+\frac{1+2^2}{4^2}+\frac{1+2^3}{4^3}+...+\frac{1+2^n}{4^n}=$$

$$=\underbrace{\left(\frac{1}{4}+\frac{1}{4^2}+...+\frac{1}{4^n}\right)}_{=y_n}+\underbrace{\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}\right)}_{=z_n}=y_n+z_n \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty}(y_n+z_n)=\lim_{n\to\infty} y_n + \lim_{n\to\infty} z_n = \frac{\frac{1}{4}}{1-\frac{1}{4}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{3}+1=\frac{4}{3}. \rhd$$

7.21-masala. $\lim\limits_{n\to\infty} x_n - ?$

$$x_n = \left(\frac{2n^2+21n-7}{2n^2+18n+9}\right)^{2n+1}$$

◁ $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\left(\frac{2n^2+21n-7}{2n^2+18n+9}\right)^{2n+1} = (1^\infty) = \lim\limits_{n\to\infty}\left(\frac{2n^2+18n+9+3n-16}{2n^2+18n+9}\right)^{2n+1} =$

$= \lim\limits_{n\to\infty}\left(1+\frac{3n-16}{2n^2+18n+9}\right)^{2n+1} = \left(\left(\lim\limits_{n\to\infty}(1+\alpha)^{\frac{1}{\alpha}} = e\right.\right.$ tenglikdan foydalanamiz.

Bizning holda $\left.\left.\alpha = \frac{3n-16}{2n^2+18n+9}\right)\right) = e^{\lim\limits_{n\to\infty}\frac{(3n-16)(2n+1)}{2n^2+18n+9}} =$

$$= e^{\lim\limits_{n\to\infty}\frac{n^2\left(3n-\frac{16}{n}\right)\left(2+\frac{1}{n}\right)}{n^2\left(2+\frac{18}{n}+\frac{9}{n^2}\right)}} = e^{\frac{3\cdot 2}{2}} = e^3 \quad ▷$$

8.21-masala. $\{x_n\}$ **ketma-ketlikning yuqori va kuyi limitlari topilsin** $\left(\overline{\lim\limits_{n\to\infty}} x_n - ?,\ \underline{\lim\limits_{n\to\infty}} x_n - ?\right)$.

$$x_n = \sin n^0$$

◁ Berilgan ketma-ketlikning qiymatlari to‘plamida

$$0,\ \pm\sin 1^0,\ \pm\sin 2^0, \ldots,\ \pm\sin 89^0,\ \pm 1$$

sonlari cheksiz ko‘p uchraydi, chunki $\forall n \in N$ va $\forall p \in N$ uchun keltirish formulasiga ko‘ra $\sin n^0 = \sin\left(360^0 p + n^0\right)$ va $\sin\left(180^0 \pm n^0\right) = \mp\sin n^0$. Demak, yuqoridagi sonlarning har biri berilgan ketma-ketlikning qismiy limiti bo‘ladi. Shu bilan birgalikda $\{x_n\}$ ketma-ketlikning yuqorida ko‘rsatilgan 181 ta sondan boshqa qismiy limiti yo‘q. $\Rightarrow \overline{\lim\limits_{n\to\infty}} x_n = 1$ va $\underline{\lim\limits_{n\to\infty}} x_n = -1$ ▷

9.21-masala. $y = f(x)$ **funksiyaning aniqlanish sohasi topilsin** $(D(f)-?)$.

$$y = \frac{\arcsin(0,5x-1)}{\sqrt{x^2-3x+1}}$$

$\lhd$ $D(f)$: $\begin{cases} -1 \le 0,5x-1 \le 1 \\ x^2-3x+1>0 \end{cases} \Rightarrow \begin{cases} 0 \le 0,5x \le 2 \\ \left(x-\frac{3-\sqrt{5}}{2}\right)\left(x-\frac{3+\sqrt{5}}{2}\right)>0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 0 \le x \le 4 \\ x \in \left(-\infty, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, +\infty\right) \end{cases} \Rightarrow D(f) = \left[0; \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}; 4\right]$. $\rhd$

10.21-masala. Quyidagi

$$\lim_{x\to 3}\frac{x^2-9}{x^2-3x}=2$$

tenglik ta'rif yordamida isbotlansin ($\delta(\varepsilon)$ topilsin).

$\lhd$ $\left(\lim\limits_{x\to a} f(x)=b\right) \Leftrightarrow \left(\forall\varepsilon>0\ \exists\delta(\varepsilon)>0:\ 0<|x-a|<\delta \Rightarrow |f(x)-b|<\varepsilon\right)$.

$f(x)$ funksiyani $x=3$ nuqtaning biror atrofida, masalan (2; 4) intervalda, qaraymiz.

$\forall\varepsilon>0$ son olamiz va $|f(x)-2|$ ayirmani $x\neq 3$ da quyidagi ko'rinishga keltiramiz:

$$|f(x)-2|=\left|\frac{x^2-9}{x^2-3x}-2\right|=\left|\frac{x+3}{x}-2\right|=\left|\frac{3-x}{x}\right|=\frac{|x-3|}{|x|}<\frac{|x-3|}{2},$$

chunki $x\in(2;\ 4)$ edi.

Oxirgi tengsizlikdan ko'rinib turibdiki, agar $\delta=2\varepsilon$ deb olsak, $0<|x-3|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x\in(2;\ 4)$ uchun

$$|f(x)-2|=\frac{|x-3|}{2}<\frac{\delta}{2}=\frac{2\varepsilon}{2}=\varepsilon$$

bo'ladi. Bu yerdan ta'rifga ko'ra $\lim\limits_{x\to 3}\frac{x^2-9}{x^2-3x}=2$ ekanligini hosil qilamiz $\rhd$

11.21-masala $\lim\limits_{x\to 8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$ **hisoblansin.**

$\lhd$ $\lim\limits_{x\to 8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 8}\left[\frac{(9+2x)-5^2}{x-2^3}\cdot\frac{\sqrt[3]{x^2}+2\cdot\sqrt[3]{x}+2^2}{\sqrt{9+2x}+5}\right]=$

$$= \lim_{x\to 8} \frac{2(x-8)\left(\sqrt[3]{x^2}+2\cdot\sqrt[3]{x}+4\right)}{(x-8)\left(\sqrt{9+2x}+5\right)} = 2\cdot\lim_{x\to 8} \frac{\sqrt[3]{x^2}+2\cdot\sqrt[3]{x+4}}{\sqrt{9+2x}+5} = 2,4. \ \triangleright$$

12.21-masala. $\lim\limits_{x\to\pi} \dfrac{e^{\pi}-e^{x}}{\sin 5x - \sin 3x}$ **hisoblansin.**

$$\triangleleft \quad \lim_{x\to\pi} \frac{e^{\pi}-e^{x}}{\sin 5x-\sin 3x} = \left(\frac{0}{0}\right) = \left(\begin{pmatrix} x=\pi+t \text{ almashtirish bajaramiz} \\ x\to\pi \Rightarrow t\to 0 \end{pmatrix}\right) =$$

$$= \lim_{t\to 0} \frac{e^{\pi}-e^{\pi+1}}{\sin(5\pi+5t)-\sin(3\pi+3t)} = e^{\pi}\lim_{t\to 0}\frac{1-e^{t}}{-\sin 5t+\sin 3t} = e^{\pi}\lim_{t\to 0}\frac{e^{t}-1}{\sin 5t-\sin 3t} =$$

$$= e^{\pi}\lim_{t\to 0}\frac{\dfrac{e^{t}-1}{t}}{5\cdot\dfrac{\sin 5t}{5t}-3\cdot\dfrac{\sin 3t}{3t}} = \left(\begin{pmatrix} \lim\limits_{t\to 0}\dfrac{e^{t}-1}{t}=1 \\ \lim\limits_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha}=1 \end{pmatrix}\right) = e^{\pi}\,\frac{1}{5-3} = \frac{e^{\pi}}{2}. \ \triangleright$$

13.21-masala. $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\frac{18\sin x}{ctgx}}$ **hisoblansin.**

$$\triangleleft \quad \lim_{x\to\frac{\pi}{2}} (\sin x)^{\frac{18\sin x}{ctgx}} = \left(1^{\infty}\right) = \left(\begin{pmatrix} x=\dfrac{\pi}{2}+t \\ x\to\dfrac{\pi}{2}\Rightarrow t\to 0 \text{ almashtirish bajaramiz} \end{pmatrix}\right) =$$

$$= \lim_{t\to 0}(\cos t)^{\frac{18\cos t}{-tgt}} = \lim_{t\to 0}\left[1+(\cos t-1)\right]^{-\frac{18\cos t}{tgt}} = \left(\left(\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e \text{ dan foydalanamiz}\right)\right) =$$

$$= e^{-\lim\limits_{t\to 0}\frac{18\cos t}{tgt}(\cos t-1)} = e^{\lim\limits_{t\to 0}\frac{18\cos^2 t}{\sin t}2\sin^2\frac{t}{2}} = e^{\lim\limits_{t\to 0}\frac{36\cos^2 t\cdot\sin^2\frac{t}{2}}{2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}} = e^{\lim\limits_{t\to 0}\left[\frac{18\cos^2 t}{\cos\frac{t}{2}}\cdot\sin\frac{t}{2}\right]} = e^{0} = 1. \ \triangleright$$

14.21-masala. Ushbu limit hisoblansin.

$$\lim_{x\to 0}\frac{9^{x}-2^{3x}}{arctg 2x-7x}$$

$$\triangleleft \quad \lim_{x\to 0}\frac{9^{x}-2^{3x}}{\operatorname{arctg}2x-7x} = \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\dfrac{9^{x}-1}{x}-3\cdot\dfrac{2^{3x}-1}{3x}}{2\cdot\dfrac{\operatorname{arctg}2x}{2x}-7} =$$

$$\left(\left(\lim_{t\to 0}\frac{a^t-1}{t}=\ln a \text{ ââ } \lim_{t\to 0}\frac{\operatorname{arctg} t}{t}=1\right)\right)=$$

$$=\frac{\ln 9-3\ln 2}{2-7}=-\frac{1}{5}\ln\frac{9}{8}. \ \triangleright$$

15.21-masala. $y=f(x)$ **funksiyaning** $x=x_0$ **nuqtadagi o'ng va chap limitlari topilsin** $(f(x_0+0)-?,\ f(x_0-0)-?)$

$$f(x)=\begin{cases}x+1, & x\le 2,\\ -2x+1, & x>2,\end{cases}\quad x_0=2$$

$$\triangleleft\ f(x_0+0)=f(2+0)=\lim_{x\to 2+0}f(x)=\lim_{x\to 2+0}(-2x+1)=-3$$

$$f(x_0-0)=f(2-0)=\lim_{x\to 2-0}f(x)=\lim_{x\to 2-0}(x+1)=3. \ \triangleright$$

16.21-masala. $y=f(x)$ **funksiya** $x=x_0$ **nuqtada uzluksiz ekanligi ta'rif yordamida isbotlansin (** $\delta(\varepsilon)$ **topilsin).**

$$f(x)=-2x^2-4,\quad x_0=3$$

$\triangleleft$ $f(x)$ funksiyani $x_0=3$ nuqtaning biror atrofida, masalan, (2; 4) intervalda qaraymiz. $\forall\varepsilon>0$ son olamiz va $|f(x)-f(x_0)|=|f(x)-f(3)|$ ayirmani baholaymiz:

$$|f(x)-f(3)|=|-2x^2-4-(-22)|=|-2x^2+18|=2|x^2-9|=$$
$$=2|x+3|\cdot|x-3|<14\cdot|x-3|.$$

Bu tenglikdan ko'rinib turibdiki, agar $\delta=\frac{\varepsilon}{14}$ deb olsak, $|x-3|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x\in(2;\ 4)$ uchun $|f(x)-f(3)|<14|x-3|<14\delta=14\frac{\varepsilon}{14}=\varepsilon$ bo'ladi. $\Rightarrow$ $f(x)=-2x^2-4$ funksiya $x_0=3$ nuqtada uzluksiz. $\triangleright$

17.21-masala. $f(x)=\lim_{n\to\infty}\frac{x}{1+(2\sin x)^{2n}}$ **funksiya uzluksizlikka tekshirilsin va grafigi chizilsin.**

◁ $f(x)=\lim\limits_{n\to\infty}\dfrac{x}{1+(2\sin x)^{2n}}=\begin{cases}x, & |2\sin x|<1,\\ \dfrac{x}{2}, & |2\sin x|=1,\\ 0, & |2\sin x|>1\end{cases}=\begin{cases}x, & -\dfrac{\pi}{6}+\pi k<x<\dfrac{\pi}{6}+\pi k,\\ \dfrac{x}{2}, & x=\pm\dfrac{\pi}{6}+\pi k,\\ 0, & \dfrac{\pi}{6}+\pi k<x<\dfrac{5\pi}{6}+\pi k,\ k\in Z.\end{cases}$

Bu tenglikdan ko'rinib turibdiki $f(x)$ funksiya $\left(-\dfrac{\pi}{6}+\pi k;\ \dfrac{\pi}{6}+\pi k\right)$ va $\left(\dfrac{\pi}{6}+\pi k;\ \dfrac{5\pi}{6}+\pi k\right)$, $k\in Z$ oraliqlarda uzluksiz hamda $x=\pm\dfrac{\pi}{6}+\pi k,\ k\in Z$ nuqtalar funksiyaning 1-tur uzilish nuqtalari bo'ladi. Yuqoridagi ma'lumotlardan foydalanib funksiyaning grafigini chizish qiyin emas. ▷

18.21-masala. $y=f(x)$ **funksiya** X **to'plamda tekis uzluksiz ekanligi ta'rif yordamida ko'rsatilsin (** $\delta=\delta(\varepsilon)$ **topilsin).**

$$f(x)=2\sin x-\cos x,\quad X=R$$

◁ ($f(x)$ funksiya X to'plamda tekis uzluksiz) $\Leftrightarrow$ $\left(\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0,\ \forall x',x''\in X:\ |x''-x'|<\delta\ \Rightarrow\ |f(x'')-f(x')|<\varepsilon\right)$

$\forall\varepsilon>0$ son olib $|f(x'')-f(x')|$ ni baholaymiz:

$$|f(x'')-f(x')|=|(2\sin x''-\cos x'')-(2\sin x'-\cos x')|=$$

$$=|2(\sin x''-\sin x'')-(\cos x''-\cos x')|=\left(\left(\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\right.\right.\text{ va}$$

$$\left.\left.\cos\alpha-\cos\beta=-2\sin\frac{\alpha-\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}\ \text{formulalardan foydalanamiz}\right)\right)=$$

$$=\left|4\sin\frac{x''-x'}{2}\cdot\cos\frac{x''+x'}{2}+2\sin\frac{x''-x'}{2}\cdot\sin\frac{x''+x'}{2}\right|=$$

$$=2\cdot\left|\sin\frac{x''-x'}{2}\right|\cdot\left|2\cdot\cos\frac{x''+x'}{2}+\sin\frac{x''+x'}{2}\right|\le 2\cdot\frac{|x''-x'|}{2}\cdot\left(2\cdot\left|\cos\frac{x''+x'}{2}\right|+\left|\sin\frac{x''+x'}{2}\right|\right)\le$$

$$\le|x''-x'|\cdot(2+1)=3\cdot|x''-x'|$$

Bu tenglikdan ko'rinib turibdiki $\delta=\dfrac{\varepsilon}{3}$ deb olsak, $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x',x''\in R$ uchun $|f(x'')-f(x')|<\varepsilon$ bo'ladi. $\Rightarrow$ $f(x)$ funksiya R da tekis uzluksiz. ▷

3-§. 2-MUSTAQIL ISH

Funksiya hosilasi va differensiali. Ularning tatbiqlari.
Funksiya hosilasi va differensialining ta'riflari.
Hosilaning geometrik va mexanik ma'nolari.
Turli usulda berilgan funksiyalarning hosilalari.
Yuqori tartibli hosila va differensiallar.
Differensial hisobning asosiy teoremalari.
Lopital qoidasi.
O-simvolika.
Teylor formulasi.
Funksiyani to'liq tekshirish.

-A-
Asosiy tushuncha va teoremalar

1^0. Hosila va differensial ta'riflari. Hosilaning geometrik va mexanik ma'nolari

$y=f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lib, $x\in(a,b)$ bo'lsin. Bu x nuqtaga shunday Δx orttirma beraylikki, $x+\Delta x\in(a,b)$ bo'lsin.

1-ta'rif. $f'(x):=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ *(1) – funksiyaning x nuqtadagi hosilasi.*

2-ta'rif. $f'(x+0):=\lim\limits_{\Delta x\to +0}\dfrac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to +0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ - *o'ng hosila.* $f'(x-0):=\lim\limits_{\Delta x\to -0}\dfrac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to -0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ - *chap hosila.*

1 va 2-ta'riflardan quyidagilar chiqib keladi:

1) Agar $y=f(x)$ funksiya x nuqtada $f'(x)$ hosilaga ega bo'lsa, u holda $f'(x+0)$ va $f'(x-0)$ lar mavjud va $f'(x+0)=f'(x-0)=f'(x)$ bo'ladi.

2) Agar $f'(x+0)$ va $f'(x-0)$ lar mavjud bo'lib, $f'(x+0)=f'(x-0)$ bo'lsa, unda $f'(x)$ ham mavjud va $f'(x)$q$f'(x+0)=f'(x-0)$ bo'ladi.

1-teorema. *Agar* x_0 *nuqtada* $f'(x_0)$ *mavjud bo'lsa, u holda* $y=f(x)$ *funksiya grafigining* $(x_0, f(x_0))$ *nuqtasiga urinma o'tkazish mumkin va bu urinmaning burchak koeffitsienti* $f'(x_0)$ *ga teng bo'ladi.*

$y=f(x_0)+f'(x_0)\cdot(x-x_0)$ **-(2) - urinma tenglamasi.**

$y=f(x_0)-\dfrac{1}{f'(x_0)}\cdot(x-x_0)$ **-(3) - normal tenglamasi.**

Agar $S=f(t)$ moddiy nuqtaning sonlar o'qidagi t vaqtga mos keluvchi o'rnini bildirsa, unda $\Delta f=f(t+\Delta t)-f(t)$ - nuqtaning Δt vaqt oralig'idagi ko'chishi, $\dfrac{f(t+\Delta t)-f(t)}{\Delta t}$ - o'rtacha tezlik, $f'(t)$ esa t momentdagi ***oniy tezlik*** bo'ladi.

3-ta'rif. *Agar* Δy *ni ushbu*

$$\Delta y=f(x+\Delta x)-f(x)=A(x)\cdot\Delta x+\alpha(x,\Delta x)\cdot\Delta x, \qquad (4)$$

bu yerda $\Delta x\to 0$ da $\alpha(x,\Delta x)\to 0$ ko'rinishda ifodalash mumkin bo'lsa, unda $y=f(x)$ funksiya x nuqtada **differensiallanuvchi** deyiladi.

$A(x)\cdot\Delta x$ ifoda funksiya orttirmasining ***chiziqli bosh qismi*** yoki *funksiya* ***differensiali*** deb ataladi va **dy** kabi belgilanadi.

$\alpha(x,\Delta x)$ ifoda funksiya orttirmasining ***qoldiq hadi*** deb ataladi. Agar 0-simvolikadan foydalansak, $\Delta x\to 0$ da $\Delta y=A\cdot\Delta x+\hat{\imath}(\Delta x)$ tenglikni xosil qilamiz.

2-teorema. $y=f(x)$ *funksiya* x *nuqtada differensiallanuvchi bo'lishi uchun shu nuqtada chekli* $f'(x)$ *mavjud bo'lishi zarur va yetarli.*

3-teorema. *Differensiallanuvchi funksiya uzluksiz bo'ladi.*

Agar 2-teorema shartlari bajarilsa $df(x)=f'(x)\cdot\Delta x=f'(x)\cdot dx$ bo'ladi. Differensiallashning asosiy qoidalari va elementar funksiyalar uchun hosilalar jadvali 1-§ ning 13^0 va 14^0 punktlarida keltirilgan.

2^0. Turli ko'rinishda berilgan funksiyalarning hosilalari

a) Murakkab funksiyaning hosilasi

Aytaylik, $y=f(u)$ va $u=\text{j}(x)$ funksiyalar berilgan bo'lib, ular yordamida $y=f\,[\text{j}(x)]$ murakkab funksiya tuzilgan bo'lsin. Agar $u=\text{j}(x)$ funksiya x nuqtada va $y=f(u)$ funksiya x nuqtaga mos keluvchi u nuqtada hosilaga ega bo'lsa, unda

$$y'_x = y'_u \cdot u'_x \tag{5}$$

tenglik o‘rinli bo‘ladi.

b) Teskari funksiyaning hosilasi

Agar $y = f(x)$ funksiya x nuqtada $f'(x) \neq 0$ hosilaga ega bo‘lsa, bu funksiyaga teskari $x = f^{-1}(y)$ funksiya x nuqtaga mos bo‘lgan $\acute{o}$ nuqtada hosilaga ega va

$$x'_y = \frac{1}{y'_x} \tag{6}$$

bo‘ladi.

d) Parametrik ko‘rinishda berilgan funksiyaning hosilasi

Faraz qilaylik, $y = y(x)$ funksiya parametrik ko‘rinishda.

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad \alpha < t < \beta \tag{7}$$

sistema yordamida aniqlangan bo‘lsin. Agar $\varphi(t)$ va $\psi(t)$ funksiyalar differensiallanuvchi bo‘lib, $\varphi'(t) \neq 0$ bo‘lsa, unda (7)-sistema differensiallanuvchi $y = \psi\left[\varphi^{-1}(x)\right]$ funksiyani aniqlaydi va

$$y'_x = \frac{y'_t}{x'_t} = \frac{\psi'(t)}{\varphi'(t)} \tag{8}$$

tenglik o‘rinli bo‘ladi.

e) Oshkormas funksiyaning hosilasi

Agar biror oraliqda differensiallanuvchi bo‘lgan $y = y(x)$ funksiya $F(x,y) = 0$ tenglik yordamida aniqlansa, unda oshkormas ko‘rinishda berilgan funksiyaning $y' = y'(x)$ hosilasini ushbu

$$\frac{d}{dx} F(x,y) = 0 \tag{9}$$

tenglikdan topish mumkin.

Masalan, ushbu $y^5 + y^3 + y - x = 0$ tenglik yordamida oshkormas ko‘rinishda berilgan $y = y(x)$ funksiyaning y' hosilasini topaylik.

◁ (9)-tenglikka ko‘ra

$$\left(y^5 + y^3 + y - x \right)'_x = 0 \Rightarrow 5y^4 \cdot y' + 3y^2 \cdot y' + y' - 1 = 0 \Rightarrow y' \frac{1}{5y^4 + 3y^2 + 1}. \triangleright$$

3^0. Differensialning taqribiy hisoblashga tatbiqi

Ma'lumki, $y=f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsa, unda

$$\Delta f(x_0)=df(x_0)+o(\Delta x)$$

tenglik o'rinli bo'ladi. Agar $df(x_0)\neq 0$ bo'lsa, bu tenglikdan yetarlicha kichik Δx lar uchun

$$\Delta f(x_0)\approx df(x_0)$$

yoki

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\cdot\Delta x \qquad (10)$$

taqribiy hisoblash formulasini hosil qilamiz.

4^0. Yuqori tartibli hosila va differensiallar

a) $y=f(x)$ funksiyaning yuqori tartibli hosila va differensiallari ushbu

$$f^{(n)}(x)=\left\{f^{(n-1)}(x)\right\}' \qquad (n=2,3,...);$$

$$d^n y=d\left(d^{n-1}y\right) \qquad (n=2,3,...);$$

tengliklar yordamida aniqlanadi.

b) Asosiy formulalar

1) $\left(a^x\right)^n=a^x\cdot\ln^n a$ (a>0); $\left(e^x\right)^{(n)}=e^x$

2) $(\sin x)^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)$

3) $(\cos x)^{(n)}=\cos\left(x+\frac{n\pi}{2}\right)$

4) $\left(x^\alpha\right)^{(n)}=\alpha(\alpha-1)...(\alpha-n+1)x^{\alpha-n}$, $\alpha\in R$

5) $(\ln x)^{(n)}=\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$

d) Leybnis formulasi

Agar $u=u(x)$ va $v=v(x)$ funksiyalar n-tartibli hosilalarga ega bo'lsa, unda $y=u(x)\cdot v(x)$ funksiya ham n-tartibli hosilaga ega bo'ladi va

$$y^{(n)}=(u\cdot v)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(k)}v^{(n-k)} \qquad (11)$$

tenglik o'rinli bo'ladi. Bu yerda $u^{(0)}=u$, $v^{(0)}=v$ va $C_n^k=\frac{n!}{k!\cdot(n-k)!}$.

(11)-formulaga n-tartibli hosilani hisoblash uchun ***Leybnis formulasi*** deyiladi.

$u(x)\cdot v(x)$ funksiyaning n-tartibli differensiali $d^n(u\cdot v)$ uchun ham Leybnis formulasi o'rinli.

5^0. Differensial hisobning asosiy teoremalari

Aytaylik $y=f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lsin.

1-teorema. (Ferma teoremasi). Agar

1) $f(x)\in C[a,b]$,

2) $\forall x\in(a,b)$ uchun chekli $f'(x)-\exists$,

3) ichki $c\in(a,b)$ nuqtada $f(x)$ funksiya eng katta (yoki eng kichik) qiymatga erishsa, unda $f'(c)=0$ bo'ladi.

2-teorema. (Roll teoremasi). *Agar*

1) $f(x)\in C[a,b]$,

2) $\forall x\in(a,b)$ uchun chekli $f'(x)-\exists$,

3) $f(a)=f(b)$

bo'lsa, $\exists x_0\in(a,b)$ *nuqta topiladiki,* $f'(x_0)=0$ *bo'ladi.*

3-teorema. (Lagranj teoremasi). *Agar*

1) $f(x)\in C[a,b]$

2) $\forall x\in(a,b)$ uchun chekli $f'(x)-\exists$

bo'lsa $\exists x_0\in(a,b)$ nuqta topiladiki

$$f(b)-f(a)=f'(x_0)\cdot(b-a)$$

bo'ladi.

1-natija. *Agar* $\forall x\in(a,b)$ *uchun* $f'(x)=0$ *bo'lsa, unda* (a,b)*da* $f(x)\equiv const$ *bo'ladi.*

2-natija. *Agar* $f(x)$ *funksiya* (a,b) *intervalda chegaralangan* $f'(x)$ *hosilaga ega bo'lsa, u holda* $f(x)$ (a,b) *da tekis uzluksiz bo'ladi.*

Lagranj teoremasini ba'zi bir tengsizliklarni isbotlashda qo'llash mumkin. Masalan, $(1+x)^\alpha\geq 1+\alpha x$ **Bernulli tengsizligi** $\forall x>-1$ va $\alpha>1$ da o'rinli ekanligi isbotlansin.

◁ **1-hol.** $x>0$ ***bo'lsin.*** Unda $f(u)=(1+u)^{\alpha}$, $u\in[0,x]$ funksiya uchun Lagranj teoremasiga ko'ra $\exists x_0\in(0,x)$ nuqta topiladiki $f(x)-f(0)=(1+x)^{\alpha}-1=\alpha\cdot(1+x_0)^{\alpha-1}\cdot x>\alpha x$ bo'ladi $\Rightarrow(1+x)^{\alpha}>1+\alpha x$

2-hol. $-1<x<0$ ***bo'lsin.*** Unda $f(u)=(1+u)^{\alpha}$, $u\in[x,0]$ funksiya uchun Lagranj teoremasini qo'llaymiz. $\Rightarrow\exists x_0\in(x;0)$ $f(0)-f(x)=1-(1+x)^{\alpha}=\alpha\cdot(1+x_0)^{\alpha-1}\cdot(0-x)=((1+x_0<1))<-\alpha x\Rightarrow(1+\alpha x)^{\alpha}>1+\alpha x$.

3-hol $x=0$ ***bo'lsin.*** Unda $(1+x)^{\alpha}=1+\alpha x=1$ bo'ladi. Endi 3 ta holni umumlashtirsak, isbot qilishimiz kerak bo'lgan Bernulli tengsizligini hosil qilamiz. ▷

4-teorema_(Koshi teoremasi). *Agar*

1) $f(x),g(x)\in C[a,b]$,

2) $\forall x\in(a,b)$ uchun chekli $f'(x)$va $g'(x)$ - $\exists$ hamda $g'(x)\neq 0$ bo'lsa, unda $\exists x_0\in(a,b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_0)}{g'(x_0)}$$

tenglik o'rinli bo'ladi.

6^0. Aniqmasliklarni ochish. Lopital qoidalari

2-§ da ko'rganimizdek funksiya limitini hisoblashda biz $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 1^{∞}, 0^0 va shu kabi aniqmasliklarga duch keldik. Bu aniqmasliklarni ochishda Lopital qoidalari katta yordam beradi.

Teorema. $f(x)$ *va* $g(x)$ *funksiyalar uchun quyidagi shartlar o'rinli bo'lsin.*

1) $f(x)$va $g(x)$ funksiyalar a nuqtaning biror atrofida aniqlangan va chekli hosilaga ega,

2) $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=0$,

3) a nuqtaning shu atrofida $[f'(x)]^2+[g'(x)]^2\neq 0$,

4) $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ -chekli yoki cheksiz.

U holda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

tenglik o'rinli bo'ladi.

Izoh: Agar bu teoremaning shartlari a nuqtaning chap (yoki o'ng) yarim atrofida bajarilsa, unda teorema $\frac{f(x)}{g(x)}$ ning a nuqtadagi chap (yoki o'ng) limitiga nisbatan o'rinli bo'ladi.

Yuqoridagi $\frac{0}{0}$ ko'rinishidagi aniqmasliklar uchun keltirilgan Lopital teoremasi $\frac{\infty}{\infty}$ ko'rinishidagi aniqmasliklar uchun ham o'rinli bo'ladi. Boshqa ko'rinishdagi aniqmasliklar esa $\frac{0}{0}$ va $\frac{\infty}{\infty}$ ko'rinishidagi aniqmasliklarga keltiriladi.

7°. O-simvolika

Funksiya limitini hisoblashda va funksiyaning asimptotik xarakterini o'rganishda «o-kichik» va «O-katta» tushunchalari muhim ahamiyatga ega. Biz a nuqta deganda chekli son yoki ∞ ni tushunamiz. a chekli bo'lgan holda nuqtaning atrofi deganda quyidagi to'plamlardan biri tushuniladi: $(a-\delta;a)$, $(a;a+\delta)$, $(a-\delta;a+\delta)$, bu yerda $\delta>0$. Agar $a=\infty$ bo'lsa, u holda a nuqtaning atrofi deganda quyidagi to'plamlardan biri nazarda tutiladi: $(-\infty;-\Delta)$, $(\Delta,+\infty)$ yoki $(-\infty;-\Delta)\cup(\Delta,+\infty)$, bu yerda $\Delta>0$. Aytaylik, berilgan funksiyalar a nuqtaning biror atrofida aniqlangan bo'lsin.

1-ta'rif. Agar shunday o'zgarmas K son topilsaki, a nuqtaning biror atrofida

$$|\varphi(x)|\le K\cdot|\psi(x)|$$

tengsizlik bajarilsa, u holda shu atrofda $\varphi(x)$ funksiya $\psi(x)$ ga nisbatan O -katta deyiladi va $\varphi(x)=O(\varphi(x))$ kabi belgilanadi.

2-ta'rif. Agar a nuqtaning biror atrofida $\varphi(x)=\alpha(x)\cdot\psi(x)$ tenglik o'rinli bo'lib, $\lim\limits_{x\to a}\alpha(x)=0$ bo'lsa, unda $x\to a$ da $\varphi(x)$ funksiya $\psi(x)$ ga nisbatan o -kichik deyiladi va $\varphi(x)=o(\psi(x))$ kabi belgilanadi.

1-ta'rifdan ko'rinadiki, agar $\psi(x)\neq 0$ bo'lsa, unda $\lim\limits_{x\to a}\frac{\varphi(x)}{\psi(x)}=0$ bo'lganda $\varphi(x)=o(\psi(x))$ bo'ladi.

Izoh. Quyidagi tengliklar o'rinli:

1) $o(f(x))+o(f(x))=o(f(x))$,
2) $K\cdot o(f(x))=o(f(x))$,
3) $o(f(x))\cdot o(f(x))=o(f(x))$,
4) $o(f(x))\cdot O(f(x))=o(f(x))$,
5) $x\to 0$ da $x^m=o(x^n)\Leftrightarrow m>n$
6) $x\to\infty$ da $x^m=o(x^n)\Leftrightarrow m<n$.

3-ta'rif. Agar $x\to a$ da $\varphi(x)-\psi(x)=o(\psi(x))$ bo'lsa, unda $x\to a$ da $\varphi(x)$ va $\psi(x)$ funksiyalar ekvivalent deyiladi hamda $\varphi(x)\sim\psi(x)$ kabi belgilanadi.

Bu ta'rifdan ko'rinadiki, agar $\psi(x)\neq 0$ bo'lsa, unda $\lim\limits_{x\to a}\frac{\varphi(x)}{\psi(x)}=1$ bo'lganda $\varphi(x)\sim\psi(x)$ bo'ladi.

1-teorema. Agar ushbu

$$\lim_{x\to a}\frac{\varphi(x)+o(\varphi(x))}{\psi(x)+o(\psi(x))}\quad \text{yoki}\quad \lim_{x\to a}\frac{\varphi(x)}{\psi(x)}$$

limitlardan birortasi mavjud bo'lsa, unda

$$\lim_{x\to a}\frac{\varphi(x)+o(\varphi(x))}{\psi(x)+o(\psi(x))}=\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}$$

tenglik o'rinli bo'ladi.

1-teoremadan foydalanish samaradorligi Teylor formulasi yordamida yanada oshadi.

2-teorema. Agar $f(x)$ funksiya a nuqtada $f'(a)$, $f''(a),\ldots,f^{(n)}(a)$ hosilalarga ega bo'lsa, u holda a nuqtaning biror atrofida ushbu

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+o\left((x-a)^n\right).$$

Peano ko'rinishidagi qoldiq hadli Teylor formulasi o'rinli bo'ladi.

Natija. $x \to 0$ da quyidagi tengliklar o'rinli bo'ladi.

1. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + ... + \frac{m(m-1)...(m-n+1)}{n!}x^n + o(x^n)$

2. $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + o(x^n)$

3. $\ln(1+x) = x - \frac{x^2}{2} + ... + (-1)^{n-1} \cdot \frac{x^n}{n} + o(x^n)$

4. $\sin x = x - \frac{x^3}{3!} + ... + (-1)^{n-1} \cdot \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n})$

5. $\cos x = 1 - \frac{x^2}{2!} + ... + (-1)^n \cdot \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$

6. $\operatorname{tg} x = x + \frac{1}{3}x^3 + o(x^4)$

7. $\operatorname{arctg} x = x - \frac{1}{3}x^3 + o(x^4)$

Misol. $\lim\limits_{x\to 0} \frac{\ln\cos x + x^2}{\sin x \cdot tgx}$ hisoblansin.

$$\triangleleft \ \lim_{x\to 0} \frac{\ln\cos x + x^2}{\sin x \cdot tgx} = \lim_{x\to 0} \frac{\ln\left(1 - \frac{1}{2}x^2 + o(x^2)\right) + x^2}{\left(x + o(x^2)\right)\left(x + o(x^2)\right)} =$$

$$= \lim_{x\to 0} \frac{\left(-\frac{1}{2}x^2 + o(x^2)\right) + o\left(-\frac{1}{2}x^2 + o(x^2)\right) + x^2}{x^2 + o(x^2)} =$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}x^2 + o(x^2) + x^2}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2} \ \triangleright$$

Izoh. Limitni hisoblash jarayonida biz natijada keltirilgan 5, 4, 6, 3 tengliklardan va 1-teoremadan foydalandik.

8^0.Funksiyalarni tekshirish

a) Funksiyaning monotonligi

Faraz qilaylik , $y=f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin.

1-ta'rif. $x_2>x_1$ *tengsizlikni qanoatlantiruvchi* $\forall x_1,x_2\in(a,b)$ *uchun* $f(x_2)\geq f(x_1)$ $(f(x_2)\leq f(x_1))$ *bo'lsa,* $f(x)$ *funksiya* (a,b) *oraliqda* ***o'suvchi*** $\uparrow$ ***(kamayuvchi*** $\downarrow$*) deyiladi.*

Agar funksiya o'suvchi yoki kamayuvchi bo'lsa, bunday funksiyaga **monoton** funksiya deyiladi.

1-teorema. $f(x)$ *funksiya* (a,b) *intervalda chekli* $f'(x)$ *hosilaga ega bo'lsin. Bu funksiya shu intervalda o'suvchi (kamayuvchi) bo'lishi uchun* (a,b) *da* $f'(x)\geq 0$ $(f'(x)\leq 0)$ *bo'lishi zarur va yetarli.*

b) Funksiyaning ekstremumlari

$y=f(x)$ funksiya (a,b) intervalda berilgan bo'lib, $x_0\in(a,b)$ bo'lsin.

2-ta'rif. *Agar* x_0 *nuqtaning* $\exists$ $\bigcup_\delta(x_0)$ *atrofi mavjud bo'lsaki,* $\forall x\in\bigcup_\delta(x_0)$ *uchun*

$$f(x)\leq f(x_0) \qquad (f(x)\geq f(x_0))$$

tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada **maksimumga** **(minimumga)** erishadi deyiladi. $f(x_0)$ qiymat $f(x)$ ning maksimum (minimum) qiymati deyiladi va

$$f(x_0)=\max_{x\in\bigcup_\delta(x_0)}\{f(x)\} \quad \left(f(x_0)=\min_{x\in\bigcup_\delta(x_0)}\{f(x)\}\right)$$

kabi belgilanadi.

Funksiyani maksimum va minimumi umumiy nom bilan uning ***ekstremumi*** deyiladi.

2-teorema.*(Ekstremumning zaruriy sharti).* *Agar* $f(x)$ *funksiya* x_0 *nuqtada (* $x_0\in(a,b)$ *) chekli* $f'(x_0)$ *hosilaga ega bo'lib, bu nuqtada* $f(x)$ *funksiya ekstremumga erishsa, u holda* $f'(x_0)=0$ *bo'ladi.*

Endi funksiya ekstremumga erishishining yetarli shartlarini keltiramiz.

Faraz qilaylik, $y=f(x)$ funksiya x_0 nuqtada uzliksiz bo'lib, $\bigcup_\delta(x_0)$ g' $\{x_0\}$ da chekli $f'(x\ )$ hosilaga ega bo'lsin.

3-teorema. *Agar $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini musbatdan (manfiydan) manfiydan (musbatdan) o'zgartirsa, unda $f(x)$ funksiya x_0 nuqtada maksimumga (minimumi) erishadi. Agar $f'(x)$ ishorasini o'zgartirmasa, u holda $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.*

4-teorema. *$f(x)$ funksiya x_0 nuqtada $f', f'', ... f^{(n)}$ hosilalarga ega bo'lib,*

$$f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0$$

bo'lsin. Unda

1) agar n juft son bo'lib,

$$f^{(n)}(x_0) < 0 \quad \left(f^{(n)}(x_0) > 0\right)$$

bo'lsa, $f(x)$ funksiya x_0 nuqtada **maksimumga (minimumga)** erishadi.

2) agar n toq son bo'lsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.

Funksiyaning hosilasi nolga aylanadigan yoki hosilasi mavjud bo'lmagan nuqtalariga uning ***kritik*** nuqtalari deyiladi.

Izoh: Funksiya hosilasi mavjud bo'lmagan nuqtalarda ham funksiya ekstremumga erishishi mumkin. Masalan, $f(x) = |x|$ funksiya uchun $f'(0)$ – mavjud emas, lekin funksiya $x = 0$ nuqtada minimumga erishadi.

$[a,b]$ kesmada uzluksiz bo'lgan $f(x)$ funksiya o'zining shu kesmadagi eng katta (eng kichik) qiymatiga kritik nuqtada yoki kesmaning chegaraviy nuqtasida erishadi.

d) Funksiyaning qavariqligi, egilish nuqtalari

3-ta'rif. *Agar (a,b) oraliqda berilgan $y = f(x)$ funksiya grafigi $\forall [x_1, x_2] \subset (a,b)$ kesmaning chetki nuqtalarini tutashtiruvchi vatardan **yuqorida (pastda)** yotsa, unda $y = f(x)$ funksiya $[a,b]$ oraliqda **qavariq (botiq)** deb ataladi.*

5-teorema. *$y = f(x)$ funksiya (a,b) intervalda aniqlangan va bu intervalda chekli $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning (a,b) da **qavariq** $\cap$ **(botiq** $\cup$**)** bo'lishi uchun $f'(x)$ning (a,b)da **kamayuvchi (o'suvchi)** bo'lishi zarur va yetarli.*

6-teorema. *$y = f(x)$ funksiya (a,b) intervalda aniqlangan va* *bu intervalda ikkinchi tartibli $f''(x)$ hosilaga ega bo'lsin. $f(x)$ ning (a,b) intervalda $\cap(\cup)$ bo'lishi uchun shu intervalda $f''(x)\le 0$ $(f''(x)\ge 0)$ tengsizlikning bajarilishi zarur va yetarli.*

4-ta'rif. *Agar $x=a$ nuqtadan o'tishda $y=f(x)$ funksiyaning grafigi qovariqligi yoki botiqligini o'zgartirsa, u holda $x=a$ nuqta funksiya grafigining* ***egilish nuqtasi*** *deyiladi.*

e) Funksiya grafigining asimptotalari

5-ta'rif. *Agar $\lim\limits_{x\to a} f(x)=\infty$ bo'lsa, $x=a$ to'g'ri chiziq $y=f(x)$ funksiya grafigining* ***vertikal asimptotasi*** *deyiladi.*

6-ta'rif. *Agar $\lim\limits_{x\to\infty} f(x)=b$ bo'lsa, $y=b$ to'g'ri chiziq $y=f(x)$ funksiya grafigining* ***gorizontal asimptotasi*** *deyiladi.*

7-ta'rif. *Agar $\lim\limits_{x\to\infty}\left[f(x)-(ax+b)\right]=0$ bo'lsa, $y=ax+b$ to'g'ri chiziq $y=f(x)$ funksiya grafigining* ***og'ma asimptotasi*** *deyiladi.*

7-teorema. *$y=f(x)$ funksiya grafigi $x\to+\infty$ da $y=ax+b$ og'ma asimptotaga ega bo'lishi uchun*

$$\lim_{x\to+\infty}\frac{f(x)}{x}=a,\quad \lim_{x\to+\infty}\left[f(x)-ax\right]=b$$

bo'lishi zarur va yetarlidir.

Bu teorema $x\to-\infty$ da ham o'rinlidir.

9⁰. Funksiyalarni to'liq tekshirish va grafiklarini chizish

Funksiyani to'la tekshirish va grafigini yasash quyidagilarni aniqlash yordamida amalga oshiriladi.

1) Funksiyani aniqlanish sohasini topish.
2) Aniqlanish sohasining chegaraviy nuqtalaridagi xarakterini aniqlash.
3) Funksiyaning juft yoki toqligini va, agar imkon bo'lsa, boshqa markaz va simmetriya o'qlarini aniqlash.
4) Davriylikka tekshirish.
5) Uzilish nuqtalarini topish va ularning turini aniqlash (2-punktni to'ldiradi).
6) Koordinata o'qlari bilan kesishish nuqtalarini topish.
7) Funksiyaning ishorasi o'zgarmaydigan oraliqlarni aniqlash.
8) Monotonlik va ekstremumga tekshirish.
9) Egilish nuqtalari, qavariqlik va botiqlik oraliqlarini topish.

10) Asimptotalarni aniqlash

11) Tekshirish natijalarini yo'llari $x, y, f(x), f'(x), f''(x)$ larga mos bo'lgan jadval ko'rinishida ifodalash (oxirgi yo'lda faqat ishora aniqlanadi).

12) Jadvaldagi nuqtalarni tekislikda ifodalash.

13) Asimptotalarni yasash.

14) Yuqoridagi tekshirish natijalarini hisobga olgan holda tekislikdagi nuqtalarni chiziq yordamida tutashtirish.

Izoh: Agar funksiya parametrik ko'rinishda yoki qutb koordinatalar sistemasida berilgan bo'lsa ham u yuqoridagi sxema yordamida tekshiriladi.

Nazorat savollari

1. Funksiya hosilasining ta'rifi.
2. Bir tomonli hosilalar.
3. Hosilaning geometrik ma'nosi.
4. Urinma tenglamasi.
5. Normal tenglamasi.
6. Hosilaning mexanik ma'nosi.
7. Funksiya differensialining ta'rifi.
8. Differensiallanuvchi va uzluksiz funksiyalar orasidagi bog'lanish.
9. Murakkab funksiyaning hosilasi.
10. Teskari funksiyaning hosilasi.
11. Parametrik ko'rinishda berilgan funksiyaning hosilasi.
12. Oshkormas ko'rinishda berilgan funksiyaning hosilasi.
13. Differensial yordamida taqribiy hisoblash.
14. Yuqori tartibli hosila va differensiallar.
15. Leybnis formulasi.
16. Ferma teoremasi.
17. Roll teoremasi.
18. Lagranj teoremasi.
19. Lagranj teoremasining natijalari.
20. Koshi teoremasi.
21. Lopitalning birinchi qoidasi.
22. Lopitalning ikkinchi qoidasi.
23. O-simvolika.
24. Teylor formulasi.
25. Funksiyaning monotonligi.
26. Birinchi tartibli hosila yordamida funksiyaning ekstremumini topish.
27. Yuqori tartibli hosilalar yordamida funksiyaning ekstremumini topish.
28. Funksiyaning qavariqligi va egilish nuqtalari.
29. Funksiya grafigining asimptotalari.
30. Funksiyani to'la tekshirish va grafigini yasash.

–B–

Mustaqil yechish uchun misol va masalalar

1-masala. Hosila ta'rifidan foydalanib $f'(0)$ topilsin (agar u mavjud bo'lsa).

1.1 $f(x)=\begin{cases} tg\left(x^3+x^2\sin\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0. \end{cases}$

1.2 $f(x)=\begin{cases} \sin\left(x\sin\frac{3}{x}\right), & x\neq 0, \\ 0, & x=0. \end{cases}$

1.3 $f(x)=\begin{cases} \arcsin\left(x^2\cos\frac{1}{9x}\right)+\frac{2}{3}x, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.4 $f(x)=\begin{cases} 0, & x=0, \\ \sqrt{1+\ln\left(1+x^2\sin\frac{1}{x}\right)^2}-1, & x\neq 0. \end{cases}$

1.5 $f(x)=\begin{cases} arctg\left(x\cos\frac{1}{5x}\right), & x\neq 0, \\ 0, & x=0. \end{cases}$

1.6 $f(x)=\begin{cases} \sin\left(e^{x^{2\sin\frac{5}{x}}}-1\right)+x, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.7 $f(x)=\begin{cases} \ln\left[1-\sin\left(x^2\sin\frac{1}{x}\right)\right], & x\neq 0, \\ 0, & x=0. \end{cases}$

1.8 $f(x)=\begin{cases} 0, & x=0, \\ x^2\cdot\cos\frac{4}{3x}+\frac{x^2}{2}, & x\neq 0. \end{cases}$

1.9 $f(x)=\begin{cases} arctg\left(x^3-x^{\frac{3}{2}}\sin\frac{1}{3x}\right), & x\neq 0, \\ 0, & x=0. \end{cases}$

1.10 $f(x)=\begin{cases} x^2\cdot\cos^2\frac{11}{x}, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.11 $f(x)=\begin{cases} \sin x\cdot\cos\frac{5}{x}, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.12 $f(x)=\begin{cases} 2x^2+x^2\cos\frac{1}{x}, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.13 $f(x)=\begin{cases} x+\arcsin\left(x^2\sin\frac{6}{x}\right), & x\neq 0, \\ 0, & x=0. \end{cases}$

1.14 $f(x)=\begin{cases} \frac{\ln\cos x}{x}, & x\neq 0, \\ 0, & x=0. \end{cases}$

1.15 $f(x)=\begin{cases} tg\left(2^{x^{2}\cos\frac{1}{8x}}-1+x\right), x\neq 0, \\ 0, \ x=0. \end{cases}$

1.16 $f(x)=\begin{cases} 6x+x\sin\dfrac{1}{x}, x\neq 0 \\ 0, x=0 \end{cases}$

1.17 $f(x)=\begin{cases} arctgx\cdot\sin\dfrac{7}{x}, x\neq 0, \\ 0, \ x=0. \end{cases}$

1.18 $f(x)=\begin{cases} e^{x\cdot\sin 5x}-1, \ x\neq 0, \\ 0, \ x=0. \end{cases}$

1.19 $\begin{cases} f(x)=2x^2+x^2\cos\dfrac{1}{9x}, \ x\neq 0, \\ 0, \ x=0. \end{cases}$

1.20 $f(x)=\begin{cases} 3^{x^2\cdot\sin\frac{2}{x}}-1+2x, \ x\neq 0, \\ 0, \ x=0. \end{cases}$

1.21 $f(x)=\begin{cases} 0, \ x=0, \\ \dfrac{e^{x^2}-\cos x}{x}, x\neq 0. \end{cases}$

2-masala. Funksiya grafigining abssissasi x_0 bo'lgan nuqtasiga o'tkazilgan normal (2.1-2.12 variantlarda) yoki urinma (2.13-2.21 variantlarda) tenglamasi topilsin.

2.1 $y=\dfrac{4x-x^2}{4}, x_0=2$.

2.2 $y=2x^2+3x-1, x_0=-2$.

2.3 $y=x-x^3, x_0=-1$.

2.4 $y=x^2+8\sqrt{x}-32, x_0=4$.

2.5 $y=x+\sqrt{x^3}, x_0=1$.

2.6 $y=\sqrt[3]{x^2}-20, x_0=-8$.

2.7 $y=\dfrac{1+\sqrt{x}}{1-\sqrt{x}}, x_0=4$.

2.8 $y=8\sqrt[4]{x}-70, x_0=16$.

2.9 $y=2x^2-3x+1, x_0=1$.

2.10 $y=\dfrac{x^2-3x+6}{x^2}, x_0=3$.

2.11 $y=\sqrt{x}-3\sqrt[3]{x}, x_0=64$.

2.12 $y=\dfrac{x^3+3}{x^3-2}, x_0=2$.

2.13 $y=2x^2+3, x_0=-1$.

2.14 $y=\dfrac{x^{29}+6}{x^4+1}, x_0=1$.

2.15 $y=2x+\frac{1}{x}, x_0=1$.

2.16 $y=-\frac{2\left(x^8+2\right)}{3\cdot\left(x^4+1\right)}, x_0=1$.

2.17 $y=\frac{x^5+1}{x^4+1}, x_0=1$.

2.18 $y=\frac{x^{16}+9}{1-5x^2}, x_0=1$.

2.19 $y=3\left(\sqrt[3]{x}-2\sqrt{x}\right), x_0=1$.

2.20 $y=\frac{1}{3x+2}, x_0=2$.

2.21 $y=\frac{x}{x^2+1}, x_0=-2$.

3-masala. Differensial yordamida ifodaning taqribiy qiymati hisoblansin.

3.1 $y=\sqrt[3]{x}, x=7{,}76$.

3.2 $y=\sqrt[3]{x}, x=27{,}54$.

3.3 $y=\frac{x+\sqrt{5-x^2}}{2}, x=0{,}98$.

3.4 $y=\arcsin x, x=0{,}08$.

3.5 $y=\sqrt[3]{x^2+2x+5}, x=0{,}97$.

3.6 $y=\sqrt{x^2+x+3}, x=1{,}97$.

3.7 $y=x^{11}, x=1{,}021$.

3.8 $y=x^{21}, x=0{,}998$.

3.9 $y=\sqrt[3]{x^2}, x=1{,}03$.

3.10 $y=x^6, x=2{,}01$.

3.11 $y=\sqrt{4x-1}, x=2{,}56$.

3.12 $y=\sqrt[5]{x^2}, x=1{,}03$.

3.13 $y=\frac{1}{\sqrt{2x^2+x+1}}, x=1{,}016$.

3.14 $y=\sqrt{1+x+\sin x}, x=0{,}01$.

3.15 $y=\frac{1}{\sqrt{x}}, x=4{,}16$.

3.16 $y=\sqrt[3]{3x+\cos x}, x=0{,}01$.

3.17 $y=x^7, x=2{,}002$.

3.18 $y=\sqrt[4]{2x-\sin\frac{\pi x}{2}}, x=1{,}02$.

3.19 $y=\sqrt{4x-3}, x=1{,}78$.

3.20 $y=\sqrt{x^2+5}, x=1{,}97$.

3.21 $y=\sqrt[3]{x^3+7x}, x=1{,}012$.

4-masala. Hosila hisoblansin.

4.1 $y=\dfrac{3x+\sqrt{x}}{\sqrt{x^2+2}}$.

4.2 $y=2\cdot\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$.

4.3 $y=\dfrac{(x+3)\sqrt{2x-1}}{2x+7}$.

4.4 $y=\dfrac{(2x+1)\sqrt{x^2-x}}{x^2}$.

4.5 $y=\dfrac{x^2+2}{2\sqrt{1-x^4}}$.

4.6 $y=\dfrac{x-1}{(x^2+5)\sqrt{x^2+5}}$.

4.7 $y=\dfrac{x\sqrt{x+1}}{x^2+x+1}$.

4.8 $y=\dfrac{(2x^2+3)\cdot\sqrt{x^2-3}}{9x^3}$.

4.9 $y=\dfrac{x+7}{6\sqrt{x^2+2x+7}}$.

4.10 $y=(1-x^2)\cdot\sqrt[5]{x^3+\dfrac{1}{x}}$.

4.11 $y=3\sqrt[3]{\dfrac{x+1}{(x-1)^2}}$.

4.12 $y=\dfrac{\sqrt{2x+3}\cdot(x-2)}{x^2}$.

4.13 $y=3\cdot\dfrac{\sqrt[3]{x^2+x+1}}{x+1}$.

4.14 $y=\dfrac{x^6+8x^3-128}{\sqrt{8-x^3}}$.

4.15 $y=\dfrac{1}{(x+2)\cdot\sqrt{x^2+4x+5}}$.

4.16 $y=\dfrac{\sqrt{(1+x^2)^3}}{3x^3}$.

4.17 $y=\dfrac{\sqrt{x-1}\cdot(3x+2)}{4x^2}$.

4.18 $y=\dfrac{(x^2-2)\cdot\sqrt{4+x^2}}{24x^3}$.

4.19 $y=\dfrac{1+x^2}{2\cdot\sqrt{1+2x^2}}$.

4.20 $y=\dfrac{4+3x^3}{x\cdot\sqrt[3]{(2+x^3)^2}}$.

4.21 $y=\dfrac{x^6+x^3-2}{\sqrt{1-x^3}}$.

5-masala. Hosila hisoblansin.

5.1 $y=(\operatorname{arctg}x)^{\frac{1}{2}\ln\operatorname{arctg}x}$.

5.2 $y=\left(\sin\sqrt{x}\right)^{\ln\sin\sqrt{x}}$.

5.3 $y=(\sin x)^{5e^{x}}$.

5.4 $y=(\arcsin x)^{e^{x}}$.

5.5 $y=(\ln x)^{3^{x}}$.

5.6 $y=x^{\arcsin x}$.

5.7 $y=(\operatorname{ctg}3x)^{2e^{x}}$.

5.8 $y=x^{e^{tgx}}$.

5.9 $y=(tgx)^{4e^{x}}$.

5.10 $y=(\cos 5x)^{e^{x}}$.

5.11 $y=(x\sin x)^{\sin(x\sin x)}$.

5.12 $y=\left(x^{3}+4\right)^{tgx}$.

5.13 $y=x^{\sin x^{3}}$.

5.14 $y=\left(x^{4}+5\right)^{ctgx}$.

5.15 $y=(\sin x)^{\frac{5x}{2}}$.

5.16 $y=\left(x^{2}+1\right)^{\cos x}$.

5.17 $y=19^{x^{19}}\cdot x^{19}$

5.18 $y=x^{3x}\cdot 2^{x}$.

5.19 $y=\left(\sin\sqrt{x}\right)^{e^{\frac{1}{x}}}$.

5.20 $y=x^{e^{\sin x}}$.

5.21 $y=x^{2x}\cdot 5^{x}$

6-masala. Funksiya grafigining abssissasi $x_0=x(t_0)$ bo'lgan nuqtasiga o'tkazilgan urinma va normal tenglamalari topilsin.

6.1 $\begin{cases} x=\sqrt{3}\cos t \\ y=\sin t, t_0=\dfrac{\pi}{3} \end{cases}$

6.2 $\begin{cases} x=t(t\cos t-2\sin t) \\ y=t(t\sin t+2\cos t), t_0=\dfrac{\pi}{4} \end{cases}$

6.3 $\begin{cases} x=2t-t^2 \\ y=3t-t^3, t_0=1 \end{cases}$

6.4 $\begin{cases} x=\dfrac{3at}{1+t^2} \\ y=\dfrac{3at^2}{1+t^2}, t_0=2 \end{cases}$

6.5 $\begin{cases} x=2\sin^3 t \\ y=2\cos^3 t, t_0=\dfrac{\pi}{3} \end{cases}$

6.6 $\begin{cases} x=2\ln(\operatorname{ctg}t)+1 \\ y=\operatorname{tg}t+\operatorname{ctg}t, t_0=\dfrac{\pi}{4} \end{cases}$

6.7 $\begin{cases} x=3(t-\sin t) \\ y=3(1-\cos t), t_0=\dfrac{\pi}{3} \end{cases}$

6.8 $\begin{cases} x=at\cos t \\ y=at\sin t, t_0=\dfrac{\pi}{2} \end{cases}$

6.9 $\begin{cases} x=\sin^2 t \\ y=\cos^2 t, t_0=\dfrac{\pi}{6} \end{cases}$

6.10 $\begin{cases} x=\dfrac{t+1}{t} \\ y=\dfrac{t-1}{t}, t_0=-1 \end{cases}$

6.11 $\begin{cases} x=\arcsin\dfrac{t}{\sqrt{1+t^2}} \\ y=\arccos\dfrac{1}{\sqrt{1+t^2}}, t_0=1 \end{cases}$

6.12 $\begin{cases} x=\ln\left(1+t^2\right) \\ y=t-\operatorname{arctg} t, t_0=1 \end{cases}$

6.13 $\begin{cases} x=\dfrac{1+\ln t}{t^2} \\ y=\dfrac{3+2\ln t}{t}, t_0=1 \end{cases}$

6.14 $\begin{cases} x=t\cdot(1-\sin t) \\ y=t\cdot\cos t, t_0=0 \end{cases}$

6.15 $\begin{cases} x=\dfrac{1+t}{t^2} \\ y=\dfrac{3}{2t^2}+\dfrac{2}{t}, t_0=2 \end{cases}$

6.16 $\begin{cases} x=\dfrac{1+t^3}{t^2-1} \\ y=\dfrac{t}{t^2-1} t_0=2 \end{cases}$

6.17 $\begin{cases} x=a\sin^3 t \\ y=a\cos^3 t, t_0=\dfrac{\pi}{6} \end{cases}$

6.18 $\begin{cases} x=3\cos t \\ y=4\sin t, t_0=\dfrac{\pi}{4} \end{cases}$

6.19 $\begin{cases} x=a(t\sin t+\cos t) \\ y=a(\sin t-t\cos t), t_0=\dfrac{\pi}{4} \end{cases}$

6.20 $\begin{cases} x=t-t^4 \\ y=t^2-t^3, t_0=1 \end{cases}$

6.21 $\begin{cases} x=1-t^2 \\ y=t-t^3, t_0=2 \end{cases}$

7-masala. Parametrik ko‘rinishda berilgan funksiyaning ikkinchi tartibli hosilasi hisoblansin.

7.1 $\begin{cases} x=\cos 2t \\ y=2\sec^2 t \end{cases}$

7.2 $\begin{cases} x=\sqrt{t^3-1} \\ y=\ln t \end{cases}$

7.3 $\begin{cases} x=\sqrt{1-t^2} \\ y=\dfrac{1}{t} \end{cases}$

7.4 $\begin{cases} x=\sqrt{t}-1 \\ y=\dfrac{1}{\sqrt{t}} \end{cases}$ 7.5 $\begin{cases} x=e^t\cos t \\ y=e^t\sin t \end{cases}$ 7.6 $\begin{cases} x=\cos^2 t \\ y=tg^2 t \end{cases}$

7.7 $\begin{cases} x=t+\sin t \\ y=2-\cos t \end{cases}$ 7.8 $\begin{cases} x=\sqrt{t-3} \\ y=\ln(t-2) \end{cases}$ 7.9 $\begin{cases} x=\dfrac{1}{t} \\ y=\dfrac{1}{1+t^2} \end{cases}$

7.10 $\begin{cases} x=\sin t \\ y=\ln(\cos t) \end{cases}$ 7.11 $\begin{cases} x=\sqrt{t} \\ y=\dfrac{1}{\sqrt{1-t}} \end{cases}$ 7.12 $\begin{cases} x=t-\sin t \\ y=2-\cos t \end{cases}$

7.13 $\begin{cases} x=\sin t \\ y=\sec t \end{cases}$ 7.14 $\begin{cases} x=\cos t \\ y=\ln(\sin t) \end{cases}$ 7.15 $\begin{cases} x=tgt \\ y=\dfrac{1}{\sin 2t} \end{cases}$

7.16 $\begin{cases} x=\cos t+t\sin t \\ y=\sin t-t\cos t \end{cases}$ 7.17 $\begin{cases} x=\sqrt{t-1} \\ y=\dfrac{t}{\sqrt{t-1}} \end{cases}$ 7.18 $\begin{cases} x=e^t \\ y=\arcsin t \end{cases}$

7.19 $\begin{cases} x=\sqrt{t} \\ y=\sqrt[3]{t-1} \end{cases}$ 7.20 $\begin{cases} x=2(t-\sin t) \\ y=4(2+\cos t) \end{cases}$ 7.21 $\begin{cases} x=t+\sin t \\ y=2+\cos t \end{cases}$

8-masala. n-tartibli hosila hisoblansin.

8.1 $y=\sin 2x+\cos(x+1)$. 8.2 $y=\sqrt[5]{e^{7x-1}}$.

8.3 $y=\dfrac{4x+7}{2x+3}$. 8.4 $y=\lg(5x+2)$.

8.5 $y=2^{3x}$. 8.6 $y=\dfrac{x}{2(3x+2)}$.

8.7 $y=\dfrac{2x+5}{13(3x+1)}$. 8.8 $y=4^{3x+5}$.

8.9 $y=\sin(x+1)+\cos 2x$. 8.10 $y=\sqrt[3]{e^{2x+1}}$.

8.11 $y=\dfrac{4x+15}{5x+1}$.

8.12 $y=\lg(3x+1)$.

8.13 $y=7^{5x}$.

8.14 $y=\dfrac{x}{9(4x+9)}$.

8.15 $y=\dfrac{4}{x}$.

8.16 $y=\dfrac{5x+1}{13\cdot(2x+3)}$.

8.17 $y=5^{2x+3}$.

8.18 $y=\sin(3x+1)+\cos 5x$.

8.19 $y=\sqrt{e^{3x+1}}$.

8.20 $y=\dfrac{11+12x}{6x+5}$.

8.21 $y=\lg(2x+7)$.

9-masala. Quyidagi tengsizliklar isbotlansin.

9.1 $\ln(1+x)>\dfrac{x}{1+x}, x>0$.

9.2 $\ln(1+x)<x, x>0$.

9.3 $e^x>1+x, x\in R$.

9.4 $e^x>ex, x>1$.

9.5 $b^n-a^n>n(b-a)a^{n-1}, 0<a<b, n\in N$.

9.6 $(a+b)^p\le a^p+b^p, 0\le p\le 1$.

9.7 $\cos x>1-\dfrac{x^2}{2}, x>0$.

9.8 $2\sqrt{x}>3-\dfrac{1}{x}, x>1$.

9.9 $\sin x>x-\dfrac{x^3}{6}, x>0$.

9.10 $arctgx>x-\dfrac{x^3}{3}, 0<x\le 1$.

9.11 $e^x\ge 1+x+\dfrac{x^2}{2!}+...+\dfrac{x^n}{n!}, x\ge 0, n\in N$.

9.12 $arctgx<x-\dfrac{x^3}{6}, 0<x\le 1$.

9.13 $\ln(1+x)\le x-\dfrac{x^2}{2}+\dfrac{x^3}{3}, x\ge 0$.

9.14 $e^x\ge 1+x+\dfrac{x^2}{2!}, x\ge 0$.

9.15 $\ln\dfrac{a}{b}<\dfrac{a-b}{b}, 0<b<a$.

9.16 $e^x\le 1+x+\dfrac{x^2e^x}{2!}, x\ge 0$.

9.17 $x^p-y^p\le px^{p-1}\cdot(x-y),\ 0<y<x,\ p>1$.

9.18 $\ln(1+x)\ge x-\dfrac{x^2}{2}, x\ge 0$.

9.19 $|\mathrm{arctg}\, a - \mathrm{arctg}\, b| \le |a-b|$.

9.20 $\ln\frac{a}{b} > \frac{a-b}{a}, 0 < b < a$.

9.21 $b^n - a^n < n\cdot(b-a)b^{n-1}, 0 < a < b, n \in N$.

10-masala. Limit hisoblansin.

10.1 $\lim\limits_{x\to-\infty}\frac{e^{-x}}{x^{10}}$.

10.2 $\lim\limits_{x\to1-0}\ln(1-x)\cdot ctg\pi x$.

10.3 $\lim\limits_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x^{100}}$.

10.4 $\lim\limits_{x\to\frac{1}{2}}(2-2x)^{tg\pi x}$.

10.5 $\lim\limits_{x\to+0}\frac{e^{-\frac{1}{x}}}{x^{10}}$.

10.6 $\lim\limits_{x\to\frac{\pi}{2}}\left[tgx-(1-\sin x)^{-1}\right]$.

10.7 $\lim\limits_{x\to+\infty}x^{100}(0.01)^x$.

10.8 $\lim\limits_{x\to+0}\sqrt{x}\ln^2 x$.

10.9 $\lim\limits_{x\to+\infty}\frac{x^2+\sin x}{e^x+\cos x}$.

10.10 $\lim\limits_{x\to+0}x^{\sin x}$.

10.11 $\lim\limits_{x\to+\infty}\frac{x^3+\ln x}{x^3+\cos x}$.

10.12 $\lim\limits_{x\to1}\left[(x-1)^{-1}-\ln^{-1}x\right]$.

10.13 $\lim\limits_{x\to+\infty}\frac{x^2+e^x}{\sin x+e^{2x}}$.

10.14 $\lim\limits_{x\to0}\left(x^{-2}-\sin^{-2}x\right)$.

10.15 $\lim\limits_{x\to+\infty}\frac{x^4+\cos x}{e^x+\sin x}$.

10.16 $\lim\limits_{x\to+0}x^{-2tgx}$.

10.17 $\lim\limits_{x\to0}\left(x^{-2}-ctg^2x\right)$.

10.18 $\lim\limits_{x\to0}\left[(e^x-1)^{-1}-x^{-1}\right]$.

10.19 $\lim\limits_{x\to0}\left(\frac{tgx}{x}\right)^{\frac{1}{x}}$.

10.20 $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{tg2x}$.

10.21 $\lim\limits_{x\to0}(\cos x)^{\frac{1}{\sin x}}$.

11-masala. Quyidagi masalalar yechilsin.

11.1 Yig'indisi o'zgarmas a soniga teng bo'lgan 2 ta musbat sonning m va n darajalari (m>0,n>0) ko'paytmasining eng katta qiymati topilsin.

11.2 Ko'paytmasi o'zgarmas a soniga teng bo'lgan 2 ta musbat sonning m va n darajalari (m>0,n>0) yig'indisining eng kichik qiymati topilsin.

11.3 Yuzasi Sga teng bo'lgan barcha to'g'ri to'rtburchaklar ichidan perimetri eng kichik bo'lganini aniqlang.

11.4 Kateti va gipotenuzasi yig'indisi o'zgarmas bo'lgan to'g'ri burchakli uchburchaklar ichida yuzasi eng katta bo'lganini aniqlang.

11.5 V hajmli yopiq silindrik bankaning o'lchamlari qanday bo'lganda u eng kichik to'la sirtga ega bo'ladi?

11.6 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsga tomonlari ellipsning o'qlariga parallel bo'lgan shunday ichki to'g'ri to'rtburchak chizingki, uning yuzasi eng katta bo'lsin.

11.7 R radiusli yarim sharga asosi kvadratdan iborat bo'lgan shunday ichki to'g'ri parallelepipedni chizingki, uning hajmi eng katta bo'lsin.

11.8 R radiusli sharga shunday ichki silindr chizingki, uning hajmi eng katta bo'lsin.

11.9 R radiusli sharga shunday ichki silindr chizingki, uning to'la sirti eng katta bo'lsin.

11.10 R radiusli sharga shunday tashqi konus chizingki, uning hajmi eng kichik bo'lsin.

11.11 Yasovchisi l ga teng bo'lgan eng katta hajmli konusning hajmini toping.

11.12 M(p,p) nuqta va $y^2=2px$ parabola orasidagi eng qisqa masofani toping.

11.13 A(2,0) nuqta va $x^2+y^2=1$ aylana orasidagi eng qisqa va eng uzun masofalar topilsin.

11.14 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0<b<a)$ ellipsning B$(0;-b)$ nuqtasidan o'tuvchi eng katta vatarini toping.

11.15 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsda shunday $M(x,y)$ nuqtani topingki, shu nuqtadan ellipsga o'tkazilgan urinma va koordinata o'qlari yordamida hosil bo'lgan uchburchakning yuzasi eng kichik bo'lsin.

11.16 R radiusli doiraga shunday ichki to'g'ri to'rtburchak chizingki, uning perimetri eng katta bo'lsin.

11.17 $A(1;2)$ nuqtadan shunday to'g'ri chiziq o'tkazingki, shu to'g'ri chiziq va musbat yarim o'qlar yordamida hosil bo'lgan uchburchakning yuzasi eng kichik bo'lsin.

11.18 a musbat sonni shunday 2ta musbat qo'shiluvchiga ajratingki, ular kublarining yig'indisi eng kichik bo'lsin.

11.19 Uzunligi l ga teng bo'lgan setka bilan bir tomoni devor bilan to'silgan shunday to'g'ri to'rtburchak shaklidagi yer uchastkasini o'rash kerakki, uning yuzasi eng katta bo'lsin.

11.20 Teng yoqli uchburchakni 2 ta teng yuzali uchburchakka ajratuvchi eng kichik kesmaning uzunligi topilsin.

11.21 Derazaning perimetri P ga teng, yuqori qismi yarim doiradan iborat bo'lgan to'g'ri to'rtburchak shaklga ega. Derazaning o'lchamlari qanday bo'lganda undan eng ko'p yorug'lik o'tadi?

12-masala. Birinchi tartibli hosiladan foydalanib funksiyaning grafigini chizing.

12.1 $y=x^2(x-2)^2$.

12.2 $y=\frac{x^3-9x^2}{4}+6x-9$.

12.3 $y=2-3x^2-x^3$.

12.4 $y=(x+1)^2\cdot(x-1)^2$.

12.5 $y=2x^3-3x^2-4$.

12.6 $y=3x^2-2-x^3$.

12.7 $y=(x-1)^2\cdot(x-3)^2$.

12.8 $y=\frac{x^3+3x^2}{4}-5$.

12.9 $y=6x-8x^3$.

12.10 $y=16x^2\cdot(x-1)^2$.

12.11 $y=2x^3+3x^2-5$.

12.12 $y=2-12x^2-8x^3$.

12.13 $y=(2x+1)^2\cdot(2x-1)^2$.

12.14 $y=2x^3+9x^2+12x$.

12.15 $y=12x^2-8x^3-2$.

12.16 $y=(2x-1)^2\cdot(2x-3)^2$.

12.17 $y=\dfrac{27\left(x^3-x^2\right)}{4}-x$.

12.18 $y=\dfrac{x\cdot\left(12-x^2\right)}{8}$.

12.19 $y=-\dfrac{\left(x^2-4\right)^2}{16}$.

12.20 $y=16x^3-12x^2-4$.

12.21 $y=\dfrac{x^2\cdot(x-4)^2}{16}$.

13-masala. Funksiyaning asimptotalarini toping va grafigini yasang.

13.1 $y=\dfrac{x^3-2x^2-3x+2}{1-x^2}$.

13.2 $y=\dfrac{2x^2-9}{\sqrt{x^2-1}}$.

13.3 $y=\dfrac{x^2-11}{4x-3}$.

13.4 $y=\dfrac{2x^3-3x^2-2x+1}{1-3x^2}$.

13.5 $y=\dfrac{2x^2-1}{\sqrt{x^2-2}}$.

13.6 $y=\dfrac{21-x^2}{7x+9}$.

13.7 $y=\dfrac{x^3+3x^2-2x-2}{2-3x^2}$.

13.8 $y=\dfrac{x^2+16}{\sqrt{9x^2-8}}$.

13.9 $y=\dfrac{3x^2-7}{2x+1}$.

13.10 $y=\dfrac{x^2-6x+4}{3x-2}$.

13.11 $y=\dfrac{2-x^2}{\sqrt{9x^2-4}}$.

13.12 $y=\dfrac{x^2+1}{\sqrt{4x^2-3}}$.

13.13 $y=\dfrac{17-x^2}{4x-5}$.

13.14 $y=\dfrac{4x^2+9}{4x+8}$.

13.15 $y=\dfrac{x^3-4x}{3x^2-4}$.

13.16 $y=\dfrac{x^2-3}{\sqrt{3x^2-2}}$.

13.17 $y=\dfrac{4x^3+3x^2-8x-2}{2-3x^2}$.

13.18 $y=\dfrac{2x^3+2x^2-3x-1}{2-4x^2}$.

13.19 $y=\dfrac{2x^2-6}{x-2}$.

13.20 $y=\dfrac{x^3-5x}{5-3x^2}$.

13.21 $y=\dfrac{4x^3-3x}{4x^2-1}$.

14-masala. Funksiyani to'liq tekshiring va grafigini yasang.

14.1 $y=\dfrac{x^3+4}{x^2}$.

14.2 $y=\dfrac{4x^2}{3+x^2}$.

14.3 $y=\dfrac{2}{x^2+2x}$.

14.4 $y=\dfrac{12x}{9+x^2}$.

14.5 $y=\dfrac{x^2-3x+3}{x-1}$.

14.6 $y=\dfrac{4-x^3}{x^2}$.

14.7 $y=\dfrac{x^2-4x+1}{x-1}$.

14.8 $y=\dfrac{2x^3+1}{x^2}$.

14.9 $y=\dfrac{(x-1)^2}{x^2}$.

14.10 $y=\dfrac{x^2}{(x-1)^2}$.

14.11 $y=\left(1+\dfrac{1}{x}\right)^2$.

14.12 $y=\dfrac{12-3x^2}{x^2+12}$.

14.13 $\dfrac{9+6x-3x^2}{x^2-2x+13}$.

14.14 $y=-\dfrac{8x}{x^2+4}$.

14.15 $y=\left(\dfrac{x-1}{x+1}\right)^2$.

14.16 $y=\dfrac{3x^4+1}{x^3}$.

14.17 $y=\dfrac{4x}{(x+1)^2}$.

14.18 $y=\dfrac{8(x-1)}{(x+1)^2}$.

14.19 $y=\dfrac{4}{x^2+2x-3}$.

14.20 $y=\dfrac{x^2+2x-7}{x^2+2x-3}$.

14.21 $y=\dfrac{x^2-x+1}{x-1}$.

-D-

Namunaviy variant yechimi

1.21-masala. Hosila ta'rifidan foydalanib, $f'(0)$ topilsin.

$$f(x)=\begin{cases}\dfrac{e^{x^2}-\cos x}{x}, x\neq 0;\\ 0, x=0\end{cases}$$

◁

$$f'(0):=\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\dfrac{e^{\Delta x^2}-\cos\Delta x}{\Delta x}-0}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{\Delta x^2}-\cos\Delta x}{\Delta x^2}=\lim_{\Delta x\to 0}\frac{\left(e^{\Delta x^2}-1\right)+\left(1-\cos\Delta x\right)}{\Delta x^2}=$$

$$=\lim_{\Delta x\to 0}\frac{e^{\Delta x^2}-1}{\Delta x^2}+\lim_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x^2}=1+\lim_{\Delta x\to 0}\frac{2\sin^2\dfrac{\Delta x}{2}}{\Delta x^2}=1+\frac{1}{2}\lim_{\Delta x\to 0}\left(\frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}\right)^2=1+\frac{1}{2}=\frac{3}{2}.\triangleright$$

2.21-masala. Funksiya grafigining abssissasi x_0 bo‘lgan nuqtasiga o‘tkazilgan urinma tenglamasi topilsin.

$$y=\frac{x}{x^2+1}\ ,\quad x_0=-2\,;$$

$\triangleleft$ Ma’lumki, urinma tenglamasi

$$y=f(x_0)+f'(x_0)\cdot(x-x_0)$$

ko‘rinishga ega. $f(x_0)=f(-2)=\dfrac{-2}{(-2)^2+1}=-\dfrac{2}{5}$

$$f'(x)=\left(\frac{x}{x^2+1}\right)'=\frac{x'\cdot(x^2+1)-x\cdot(x^2+1)'}{(x^2+1)^2}=\frac{x^2+1-2x^2}{(x^2+1)^2}=$$

$$=\frac{1-x^2}{(x^2+1)^2}\Rightarrow f(x_0)=f'(-2)=\frac{1-4}{25}=-\frac{3}{25}\Rightarrow$$

$\Rightarrow y=-\dfrac{2}{5}-\dfrac{3}{25}\cdot(x+2)\Rightarrow$ Urinma tenglamasi: $3x+25y+16=0$ $\triangleright$

3.21-masala. Differensial yordamida ifodaning taqribiy qiymati hisoblansin.

$$y=\sqrt[3]{x^3+7x}\ ,\quad x=1{,}012$$

$\triangleleft$ Taqribiy qiymat

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\cdot\Delta x \qquad (1)$$

formula yordamida hisoblanadi.

Bizda

$$f(x)=\sqrt[3]{x^3+7x}\,, x_0=1\,,\quad \Delta x=0{,}012 \Rightarrow f'(x)=\left(\sqrt[3]{x^3+7x}\right)'=\left[\left(x^3+7x\right)^{\frac{1}{3}}\right]'=$$

$$=\frac{1}{3}\cdot\left(x^3+7x\right)^{-\frac{2}{3}}\cdot\left(x^3+7x\right)'=\frac{3x^2+7}{3\cdot\sqrt[3]{\left(x^3+7x\right)^2}}\Rightarrow f\left(x_0\right)=\sqrt[3]{1+7}=2\,,$$

$$f'\left(x_0\right)=\frac{10}{3\cdot 4}=\frac{10}{12}=\frac{5}{6}.$$

Topilgan ifodalarni (1) tenglikka olib borib qo‘yamiz:

$$\sqrt[3]{\left(1{,}012\right)^3+7\cdot 1{,}012}\approx 2+\frac{5}{6}\cdot 0{,}012=2+5\cdot 0{,}002=2{,}01 \triangleright$$

4.21-masala. Hosila hisoblansin.

$$y=\frac{x^6+x^3-2}{\sqrt{1-x^2}}.$$

$$\triangleleft\ y'=\left(\frac{x^6+x^3-2}{\sqrt{1-x^2}}\right)'=\frac{\left(x^6+x^3-2\right)'\cdot\sqrt{1-x^2}-\left(x^6+x^3-2\right)\cdot\left(\sqrt{1-x^2}\right)'}{\left(\sqrt{1-x^2}\right)^2}=$$

$$=\frac{\left(6x^5+3x^2\right)\cdot\sqrt{1-x^2}-\left(x^6+x^3-2\right)\cdot\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}=\frac{\left(6x^5+3x^2\right)\left(1-x^2\right)+x\left(x^6+x^3-2\right)}{\left(1-x^2\right)\cdot\sqrt{1-x^2}}=$$

$$=\frac{x\left(5x^6-6x^4+2x^3-3x+2\right)}{\left(x^2-1\right)\cdot\sqrt{1-x^2}} \triangleright$$

5.21-masala. Hosila hisoblansin.

$$y=x^{2x}\cdot 5^x.$$

$$\triangleleft\ y'=\left(x^{2x}\cdot 5^x\right)'=\left[e^{\ln\left(x^{2x}\cdot 5^x\right)}\right]'=e^{\ln\left(x^{2x}\cdot 5^x\right)}\cdot\left[\ln\left(x^{2x}\cdot 5^x\right)\right]'=x^{2x}\cdot 5^x\cdot\left[2x\ln x+x\cdot\ln 5\right]'=$$

$$=x^{2x}\cdot 5^x\cdot\left[2\ln x+2+\ln 5\right]=x^{2x}\cdot 5^x\cdot\left(2+\ln 5x^2\right). \triangleright$$

6.21-masala. Funksiya grafigining abssissasi $x_0 = x(t_0)$ bo'lgan nuqtasiga o'tkazilgan urinma va normal tenglamalari topilsin.

$$\begin{cases} x = 1 - t^2, \\ y = t - t^3, t_0 = 2. \end{cases}$$

◁ Biz $y = f(x_0) + f'(x_0)\cdot(x - x_0)$ -**(2)** (urinma tenglamasi),

$y = f(x_0) - \dfrac{1}{f'(x_0)}\cdot(x - x_0)$ -**(3)** (normal tenglamasi),

va $y'_x = \dfrac{y'_t}{x'_t}$ -**(4)** (parametrik ko'rinishda berilgan funksiyaning hosilasi) formulalardan foydalanamiz:

$$x_0 = 1 - 2^2 = -3;\ f(x_0) = 2 - 2^3 = -6;$$

$$y'_x = \frac{(t - t^3)'}{(1 - t^2)'} = \frac{1 - 3t^2}{-2t} \Rightarrow f'(x_0) = \frac{1 - 3\cdot 4}{-4} = \frac{11}{4}.$$

Topilgan qiymatlarni (2)va(3)-tengliklarga olib borib qo'yib urinma va normal tenglamalarni topamiz:

$$\begin{cases} y = -6 + \dfrac{11}{4}\cdot(x + 3) \\ y = -6 - \dfrac{4}{11}(x + 3) \end{cases} \Rightarrow \begin{cases} 4y - 11x - 9 = 0 - urinma \\ 4x + 11y + 78 = 0 - normal \end{cases}$$ ▷

7.21-masala. Parametrik ko'rinishda berilgan funksiyaning ikkinchi tartibli hosilasi hisoblansin.

$$\begin{cases} x = t + \sin t \\ y = 2 + \cos t \end{cases}$$

◁ Bu masalani (4)-formuladan ikki marta foydalanish yordamida yechamiz.

$$y'_x = \frac{y'_t}{x'_t} = \frac{(2 + \cos t)'}{(t + \sin t)'} = \frac{-\cos t}{1 + \cos t}$$

$$y''_{x^2} = \frac{(y'_x)'_t}{x'_t} = \frac{\left(\dfrac{-\cos t}{1 + \cos t}\right)'}{1 + \cos t} = \frac{\sin t\cdot(1 + \cos t) - \cos t \sin t}{(1 + \cos t)^3} = \frac{\sin t}{(1 + \cos t)^3}.$$ ▷

8.21-masala. n-tartibli hosila hisoblansin.

$$y=\ln(2x+7).$$

$$\triangleleft \quad y=\lg(2x+7)=\frac{\ln(2x+7)}{\ln 10}=\frac{1}{\ln 10}\cdot\ln(2x+7)$$

$$y'=\frac{1}{\ln 10}\cdot\frac{1}{2x+7}\cdot(2x+7)'=\frac{2}{\ln 10}\cdot\frac{1}{2x+7}$$

$$y''=(y')'=\frac{2}{\ln 10}\cdot\left(-\frac{1}{(2x+7)^2}\right)\cdot(2x+7)'=-\frac{2^2}{\ln 10}\cdot\frac{1}{(2x+7)^2}$$

$$y'''=(y'')'=\frac{2^3}{\ln 10}\cdot\frac{2!}{(2x+7)^3}.$$

Bu jarayonni davom ettirish natijasida $\forall n\in N$ uchun

$y^{(n)}=(-1)^{n-1}\cdot\frac{2^n}{\ln 10}\cdot\frac{(n-1)!}{(2x+7)^n}$ tenglikni hosil qilamiz. $\triangleright$

9.21-masala. Quyidagi

$$b^n-a^n<n\cdot(b-a)\cdot b^{n-1},0<a<b,n\in N$$

tengsizlik isbotlansin.

$\triangleleft$ Bu tengsizlikni Lagranj teoremasidan foydalanib, isbotlaymiz. $f(x)=x^n$ funktsiya uchun $[a,b]$ kesmada Lagranj teoremasini qo'llaymiz:

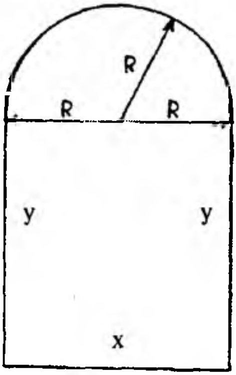
$$f(b)-f(a)=f'(x_0)\cdot(b-a),$$
$$x_0\in(a,b)\Rightarrow b^n-a^n=n\cdot x_0^{\,n-1}\cdot(b-a)<n\cdot(b-a)\cdot b^{n-1}.\triangleright$$

10.21-masala. Limit hisoblansin.

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin}}$$

$\triangleleft \lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}}=(1^\infty)=e^{\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}}=e^{\left(\frac{0}{0}\right)}=$ ((Lopital teoremasidan foydalanamiz)) $=e^{\lim_{x\to 0}\frac{(\ln(\cos x))'}{(\sin x)'}}=e^{\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{\cos x}}=e^{\lim_{x\to 0}\frac{\sin x}{\cos^2 x}}=e^0=1$ $\triangleright$

11.21-masala. Derazaning perimetri P ga teng, yuqori qismi yarim doiradan iborat bo‘lgan to‘g‘ri to‘rtburchak shaklga ega. Derazaning o‘lchamlari qanday bo‘lganda undan eng ko‘p yorug‘lik o‘tadi?



1-chizma.

◁ Masala shartiga ko‘ra deraza 1-chizmada ko‘rsatilgan shaklga ega. Chizmadan ko‘rinadiki,

$$R=\frac{x}{2}.\ \text{Unda}$$

$$P=x+2y+\pi R=x+2y+\frac{\pi x}{2}\Rightarrow$$

$$y=\frac{P}{2}-\frac{x}{2}-\frac{\pi x}{4}.$$

Endi derazaning yuzasini topamiz:

$$S=x\cdot y+\frac{\pi R^2}{2}=\frac{Px}{2}-\frac{x^2}{2}-\frac{\pi x^2}{4}+\frac{\pi x^2}{8}=\frac{Px}{2}-\frac{x^2}{2}-\frac{\pi x^2}{8} \qquad (5)$$

Derazadan eng ko‘p yorug‘lik o‘tishi uchun derazaning yuzasi eng katta bo‘lishi kerak. Buning uchun (5)-funksiyaga maksimum qiymatni beruvchi x ni topishimiz lozim.

$$S'(x)=\frac{P}{2}-x-\frac{\pi x}{4};\quad S'(x)=0\Rightarrow x\cdot\left(\frac{\pi}{4}+1\right)=\frac{P}{2}\Rightarrow x_0=\frac{\frac{P}{2}}{\frac{\pi}{4}+1}$$ – statsionar nuqta. Bu nuqtada $S''(x_0)=-1-\frac{\pi}{4}<0\Rightarrow \max$. Demak, derazadan yorug‘lik eng ko‘p o‘tishi uchun uning asosi $x=\frac{2P}{\pi+4}$ bo‘lishi kerak ekan. Balandligi esa

$$y=\frac{P}{2}-\frac{x}{2}-\frac{\pi x}{4}=\frac{P}{2}-\frac{P}{\pi+4}-\frac{\pi P}{2(\pi+4)}=\frac{P(\pi+4-2-\pi)}{2(\pi+4)}=\frac{P}{\pi+4}$$

bo‘lar ekan. ▷

12.21-masala. Birinchi tartibli hosiladan foydalanib

$$y=\frac{x^2\cdot(x-4)^2}{16}$$

funksiyaning grafigini chizing.

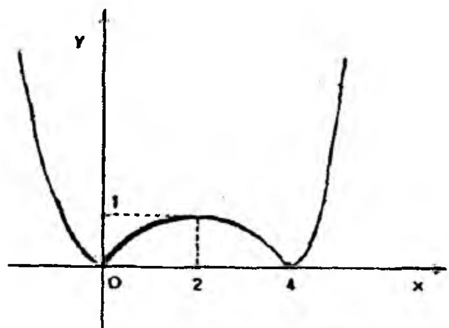
◁ Berilgan funksiyaning hosilasini hisoblaymiz:

$$y' = \frac{1}{16}\left[2x\cdot(x-4)^2 + x^2\cdot 2(x-4)\right] = \frac{2x\cdot(x-4)}{16}\cdot[x-4+x] = \frac{x(x-4)\cdot(2x-4)}{8} = \frac{x\cdot(x-2)(x-4)}{4}.$$

Intervallar usulidan foydalanib, bu ifodaning ishorasi saqlanadigan oraliqlarni topamiz va quyidagi jadvalni tuzamiz.

x	$(-\infty;0)$	0	$(0;2)$	2	$(2;4)$	4	$(4;+\infty)$
y'	-	0	+	0	-	0	+
y	↘	min 0	↗	max 1	↘	min 0	↗

Jadvaldagi ma'lumotlardan foydalanib, berilgan funksiyaning grafigini chizamiz (2-chizma). ▷



2-chizma.

13.21-masala $y = \dfrac{4x^3 - 3x}{4x^2 - 1}$ **funksiyaning asimptotalarini toping va grafigini yasang.**

$$◁ \quad y = \frac{4x^3 - 3x}{4x^2 - 1} = \frac{x\cdot(4x^2-3)}{(2x+1)(2x-1)} = \frac{x\left(x+\frac{\sqrt{3}}{2}\right)\left(x-\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)} \quad ▷$$

a) Vertikal asimptota: $x = -\frac{1}{2}$ va $x = \frac{1}{2}$ to'g'ri chiziqlar vertikal asimptota bo'ladi, chunki $\lim\limits_{x\to -\frac{1}{2}} f(x) = \infty$ va $\lim\limits_{x\to \frac{1}{2}} f(x) = \infty$.

Funksiyaning shu nuqtadagi o‘ng va chap limitlarini ham hisoblaymiz:

$$\lim_{x\to -\frac{1}{2}+0} f(x) = -\infty, \quad \lim_{x\to -\frac{1}{2}-0} f(x) = +\infty$$

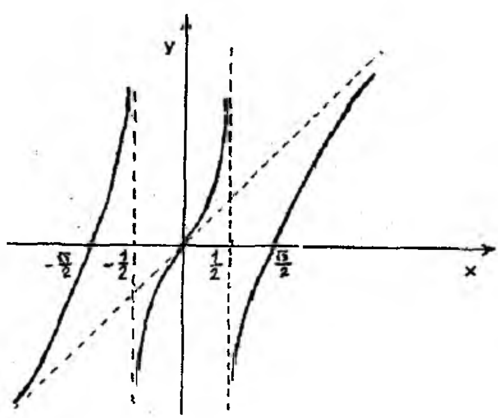
$$\lim_{x\to \frac{1}{2}+0} f(x) = -\infty \quad \lim_{x\to \frac{1}{2}-0} f(x) = +\infty$$

b) Gorizontal asimptota: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x^3-3x}{4x^2-1} = \infty \Rightarrow$ gorizontal asimptota yo‘q.

d) Og‘ma asimptota: $a = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{4x^3-3x}{x(4x^2-1)} = 1,$

$b = \lim\limits_{x\to\infty}[f(x)-ax] = \lim\limits_{x\to\infty}\left[\dfrac{4x^3-3x}{4x^2-1} - x\right] = \lim\limits_{x\to\infty}\dfrac{4x^3-3x-4x^3+x}{4x^2-1} = \lim\limits_{x\to\infty}\dfrac{-2x}{4x^2-1} = 0 \Rightarrow y = x$ — og‘ma asimptota.

Bu asimtotalardan foydalanib, funksiya grafigini chizamiz (3-chizma). ▷



3-chizma.

14.21-masala. $y = \dfrac{x^2-x+1}{x-1}$ **funksiyani to‘liq tekshiring va grafigini chizing.**

◅ Funksiyani paragrafning A bo‘limi 9-punktida taklif qilingan sxema asosida to‘liq tekshiramiz.

Funksiyaning aniqlanish sohasi: $Д(y)=\{x\neq 1\}$

Funksiya juft ham, toq ham, davriy ham emas.

$x=1$ nuqta funksiyaning 2-tur uzilish nuqtasi, chunki $\lim\limits_{x\to 1-0} f(x)=-\infty$ va $\lim\limits_{x\to 1+0} f(x)=+\infty$ OY o‘qi bilan kesishish nuqtasi: $y=f(0)=-1$.

OX o‘qi bilan kesishish nuqtasi: $y=0 \Rightarrow x^2-x+1=0 \Rightarrow x\in \text{O} \Rightarrow$ OX o‘qi bilan kesishishmaydi.

Funksiyaning ishorasi o‘zgarmaydigan oraliqlar:

X	$(-\infty;1)$	$(1;+\infty)$
Y	–	+

Endi funksiyani monotonlik va ekstremumga tekshiramiz:

$$y'=\left(\frac{x^2-x+1}{x-1}\right)'=\frac{(2x-1)\cdot(x-1)-(x^2-x+1)\cdot 1}{(x-1)^2}=\frac{2x^2-3x+1-x^2+x-1}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\frac{x\cdot(x-2)}{(x-1)^2}$$

Intervallar usulidan foydalanib, bu ifodaning ishorasi saqlanadigan oraliqlarni topamiz va quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;1)$	1	$(1;2)$	2	$(2;+\infty)$
y'	+	0	–	∄	–	0	+
y	↗	max -1	↘	∄	↘	min 3	↗

Qavariqlikka tekshirish uchun y'' ni hisoblaymiz:

$$y''=(y')'=\left[\frac{x^2-2x}{(x-1^2)}\right]'=\left[1-\frac{1}{(x-1)^2}\right]'=\frac{2}{(x-1)^3}\Rightarrow x<1\partial a\cap \text{ va } x>1\partial a\cup.$$

Funksiya asimptotalarini topamiz:

a) Vertikal asimptota: $x=1$ -vertikal asimptota.

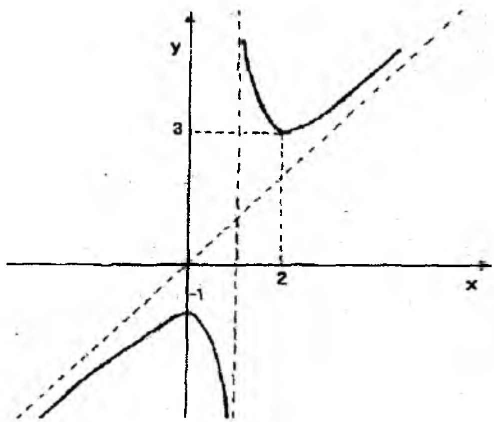
b) Gorizontal asimptota: $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{x^2-x+1}{x-1}=\infty \Rightarrow$ gorizontal asimptota yo'q.

d) Og'ma asimptota: $a=\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\frac{x^2-x+1}{x\cdot(x-1)}=1$

$$b=\lim_{x\to\infty}[f(x)-ax]=\lim_{x\to\infty}\left(\frac{x^2-x+1}{x-1}-x\right)=\lim_{x\to\infty}\frac{x^2-x+1-x^2+x}{x-1}=\lim_{x\to\infty}\frac{1}{x-1}=0\Rightarrow y=x$$

og'ma asimptota.

Endi topilgan ma'lumotlardan foydalanib, funktsiya grafigini chizamiz (4-chizma). ▷



4-chizma.

4-§. 3-MUSTAQIL ISH
Aniqmas va aniq integrallar, ularning tatbiqlari

Boshlang'ich funksiya.
Aniqmas integral.
Aniqmas integralni hisoblash usullari.
Ratsional funksiyalarni integrallash.
Ba'zi irratsional ko'rinishdagi funksiyalarni integrallash.
Binomial differensial va trigonometrik funksiyalarni integrallash.
Aniq integral va uning tatbiqlari.
Elliptik integrallar.

-A-

Asosiy tushuncha va teoremalar
1^0. Aniqmas integral va uni hisoblash usullari

$f(x)$ funksiya biror (a,b) intervalda aniqlangan bo'lsin. Quyidagi masalani qaraymiz: $\exists F(x)$ funksiyani topish kerakki $\forall x\in(a,b)$ uchun $F'(x)=f(x)$ bo'lsin.

1-ta'rif. *Agar* $\forall x\in(a,b)$ *uchun* $F'(x)=f(x)$ *bo'lsa u holda* $F(x)$ *funksiya* (a,b) *intervalda* $f(x)$ *funksiyaning* ***boshlang'ich funktsiyasi*** *deyiladi.*

Ma'lumki $F(x)$ funksiya boshlang'ich funksiya bo'lsa $F(x)+c$ ham boshlang'ich funksiya bo'ladi.

2-ta'rif. (a,b) intervalda berilgan $f(x)$ funksiya boshlang'ich funksiyalarining umumiy ifodasi $F(x)+c$ shu $f(x)$ funksiyaning **aniqmas integrali** deb ataladi va

$$\int f(x)dx$$

kabi belgilanadi.

Demak,

$$\int f(x)dx=F(x)+c \qquad (1)$$

Integrallashning umumiy qoidalari va aniqmas integrallar jadvali 1-§ ning 12^0 va 13^0 punktlarida keltirilgan. Biz ularga to'xtalmay aniqmas integralni hisoblash usullarini keltiramiz.

1-teorema. **(O'zgaruvchilarni almashtirish).** Agar

$$\int f(t)dt=F(t)+c \qquad (2)$$

bo'lsa unda

$$\int f[\varphi(x)]\varphi'(x)dx = F[\varphi(x)] + c \tag{3}$$

bo'ladi ((3)-tenglikda $f(t), \varphi(x), \varphi'(x)$ funksiyalar uzluksiz deb faraz qilinadi).

2-teorema. Agar $u = u(x)$ va $v = v(x)$ funksiyalar (a,b) intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalariga ega bo'lsa unda shu intervalda ushbu

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) \tag{4}$$

bo'laklab integrallash formulasi o'rinli bo'ladi.

Amaliyot shuni ko'rsatadiki bo'laklab integrallash usulini qo'llab hisoblanadigan integrallarni asosan, uch guruhga ajratish mumkin.

Birinchi guruhga ko'paytuvchining biri ma'lum funktsiyaning hosilasi bo'lgan ikkinchisi esa ushbu

$\ln(x), \arcsin x, \arccos x, \operatorname{arctg} x, (\operatorname{arctg} x)^2, (\arccos x)^2, \ln\varphi(x), \ldots$ funksiyalardan biriga teng bo'lgan funksiyalarning integrallari kiritiladi. Bu holda $u(x)$ deb shu funksiyalar belgilanadi.

Ikkinchi guruhga $\int(ax+b)^n\cos(cx)dx, \int(ax+b)^n\cdot\sin(cx)dx$ va $\int(ax+b)^n e^{cx}dx$ ko'rinishidagi integrallar kiritiladi. Bu holda $u(x) = (ax+b)^n$ deb olinib bo'laklab integrallash formulasi n marta qo'llaniladi.

Uchinchi guruhga $\int e^{ax}\cos bxdx, \int e^{ax}\sin bxdx, \int\sin(\ln x)dx, \int\cos(\ln x)dx, \ldots$ ko'rinishidagi integrallar kiritiladi. Bunda integralni I deb belgilab bo'laklab integrallash formulasini ikki marta qo'llasak, I ga nisbatan chiziqli tenglamaga kelamiz.

Bu uchta guruhga kirmagan ba'zi bir integrallarni ham bo'laklab integrallash usuli bilan hisoblash mumkin. Masalan

$$I_n = \int\frac{dx}{(x^2+a^2)^n}, (n \in N)$$

integral yuqoridagi uchta guruhga kirmaydi lekin bu integralni ham bo'laklab integrallash usuli bilan **rekkurent formulaga** keltirish yordamida hisoblash mumkin:

$$I_{n+1} = \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\cdot\frac{1}{a^2}I_n \tag{5}$$

$I_1 = \int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg \frac{x}{a} + c$ Agar (5)-tenglikda $n=1$ desak

$I_2 = \int \frac{dx}{\left(x^2 + a^2\right)^2} = \frac{1}{2a^2} \cdot \frac{x}{x^2 + a^2} + \frac{1}{2a^2} arctg \frac{x}{a} + c$ ekanini topamiz.

Izoh: Ma'lumki, elementar funksiyaning hosilasi yana elementar funksiya bo'lar edi, lekin integral olish uchun bu tasdiq o'rinli bo'lishi shart emas, ya'ni ba'zi bir elementar funksiyalarning integrallari elementar funksiya bo'lmay qolishi mumkin. Masalan, ushbu

1. $\int e^{-x^2} dx$; 2. $\int \cos x^2 dx$;

3. $\int \sin x^2 dx$; 4. $\int \frac{dx}{\ln x}$ $(x \geq 0, x \neq 1)$;

5. $\int \frac{\cos x}{x} dx$ $(x \neq 0)$; 6. $\int \frac{\sin x}{x} dx$.

integrallarning har biri elementar funksiyalar yordamida ifodalanmaydi. Bu funksiyalar amaliyotda ko'p uchraganligi sababli ularning qiymatlarini hisoblash uchun alohida jadvallar tuzilgan va ularning grafiklari yasalgan. Shu yo'l bilan elementar funksiyalarda integrallanmaydigan funksiyalar ham to'la o'rganilgan.

2^0. Ratsional funksiyalarni integrallash

3-ta'rif. *Agar $R(x)$ funksiyani ikkita ko'phadning nisbati ko'rinishida yozish mumkin bo'lsa, u holda $R(x)$* ***ratsional funksiyalar (yoki ratsional kasr)*** *deyiladi, ya'ni*

$$R(x) = \frac{P_n(x)}{Q_m(x)}. \tag{6}$$

$P_n(x) - n$-tartibli, $Q_m(x) - m$-tartibli ko'phad.

Agar $n \geq m$ bo'lsa kasr **noto'g'ri kasr**; $n < m$ bo'lsa **to'g'ri kasr** deyiladi.

Ixtiyoriy noto'g'ri kasr berilgan bo'lsa, ko'phadni ko'phadga bo'lish yordamida har doim uni ko'phad va to'g'ri kasrning yig'indisi shaklida ifodalash mumkin. Ixtiyoriy to'g'ri kasrni quyidagi 4ta ko'rinishdagi sodda kasrlarning yig'indisi kabi ifodalash mumkin.

I. $\frac{A}{x-a}$; II. $\frac{A}{(x-a)^k}\,(k=2,3,4,...)$;

III. $\frac{Mx+N}{x^2+px+q}$; IV $\frac{Mx+N}{(x^2+px+q)^m}\,(m=2,3,..)$.

III va IV da $x^2+px+q\neq 0$, ya'ni $q-\frac{p^2}{4}>0$.

I va II ko'rinishidagi sodda kasrlar to'g'ridan to'g'ri integrallanadi. III va IV ko'rinishidagi sodda kasrlarni integrallash uchun esa $x+\frac{p}{2}=t$ almashtirish bajarish lozim.

3^0. Ba'zi irratsional ko'rinishidagi funksiyalarni integrallash. Eyler almashtirishlari.

$R(x,y)$ deganda x va y o'zgaruvchiga nisbatan ratsional bo'lgan funksiyani tushunamiz.

a) $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ integralni hisoblashda $t=\sqrt[n]{\frac{ax+b}{cx+d}}$ almashtirish bajarilsa, ratsional funksiyani integrallashga kelinadi.

b) $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ integralni hisoblashda quyidagi 3ta hol qaraladi.

1-hol. ax^2+bx+c kvadrad uchhad har xil x_1 va x_2 haqiqiy ildizlarga ega bo'lsin. $\Rightarrow ax^2+bx+c=a(x-x_1)(x-x_2)$. Bunda

$$\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1) \qquad (7)$$

almashtirish bajaramiz.

2-hol. $a>0$ bo'lsin. Unda

$$\sqrt{ax^2+bx+c}=t-\sqrt{a}x\left(\text{yoki}\sqrt{ax^2+bx+c}=t+\sqrt{a}x\right) \qquad (8)$$

almashtirish bajaramiz.

3-hol. $c>0$ bo'lsin. U holda

$$\sqrt{ax^2+bx+c}=tx+\sqrt{c}\left(\text{yoki }\sqrt{ax^2+bx+c}=tx-\sqrt{c}\right) \qquad (9)$$

almashtirishni bajarish yordamida hisoblanishi kerak bo'lgan integral ratsional funksiyani integrallashga keltiriladi.

(7)-(9) almashtirishlarga **Eyler almashtirishlari** deb ataladi.

4^0. Binomial differensiallarni va trigonometrik funksiyalarni integrallash.

a) ***4-ta'rif.*** *Ushbu* $x^m\left(a+bx^n\right)^p dx$ *ko'rinishidagi ifodaga* ***binomial differensial*** *deb ataladi.* Bu yerda m, n, p, -lar ratsional sonlar.

$$I=\int x^m\left(a+bx^n\right)^p dx \qquad (10)$$

integral quyidagi 3 ta holda ratsional funksiyaning integraliga kelar ekan.

1-hol. p-butun son. $x=t^N$ almashtirish bajariladi. Bu yerda N soni m va n ratsional sonlar (ya'ni kasrlar) maxrajlarning eng kichik umumiy karralisi.

2-hol. $\frac{m+1}{n}$ **-butun son.** Bu holda $a+bx^n=Z^N, N-p$ ratsional sonning maxaraji, almashtirish bajarish kerak.

3-hol. $\frac{m+1}{n}+p$ **- butun son.** Bunda $\frac{a}{x^n}+b=Z^N, N-p$ ning maxraji, almashtirish bajarish yetarli.

b) $I=\int R(\sin x,\cos x)dx;$

integral berilgan bo'lsin. Bu integralni ushbu

$$tg\frac{x}{2}=t \quad -\pi<x<\pi;$$

universal almashtirish yordamida har doim ratsional funksiyani integrallashga keltirish mumkin:

$$\sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2}\quad dx=\frac{2dt}{1+t^2}.$$

d) Aytaylik,

$$I=\int \sin^n x\cdot\cos^m x dx,\ (n,m\in Z)$$

integral berilgan bo'lsin. Bu integralni hisoblash uchun quyidagi hollar qaraladi.

1-hol. n-toq, m-juft $\Rightarrow \cos x=t$ almashtirish bajariladi.

2-hol. n-juft, m-toq $\Rightarrow \sin x = t$ almashtirish bajariladi

3-hol. n va m-toq. Bunda $\cos x = t, \sin x = t$ yoki $tg\, x = t$ almashtirishlardan biri bajariladi.

4-hol. n va m-juft. Bu holda

$$\sin 2x = 2\sin x \cdot \cos x \quad \text{va} \quad \cos 2x = \cos^2 x - \sin^2 x$$

formulalardan foydalanib, tartib pasaytiriladi va yuqoridagi hollardan biriga keltiriladi.

5^0. Aniq integral va uning tatbiqlari.

Aniq integral tushunchasi va uni hisoblash usullari maktab kursida qisman va ma'ruzalarda batafsil o'tilishini hisobga olib, biz aniq integralni hisoblash usullariga qisman to'xtalamiz hamda asosiy e'tiborimizni uning tatbiqlariga qaratamiz.

1. Nyuton-Leybnis formulsi. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa va $F'(x) = f(x)$ tenglik bajarilsa, u holda

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

formula o'rinli bo'ladi.

Formulaning isbotida uzluksiz $f(x)$ funksiya uchun ham bajariladigan

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

tenglikdan foydalaniladi.

2. Bo'laklab integrallash formulasi. Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada uzluksiz differensiallanuvchi bo'lsa, u holda

$$\int_a^b f(x)g'(x)dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx;$$

bo'ladi.

3. O'zgaruvchini almashtirish. Agar $\varphi(t)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz differensiallanuvchi va $\varphi(t) \in [\alpha,\beta]\, F\ a = \varphi(\alpha)\, F\ b = \varphi(\beta)$ bo'lib F $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, unda

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt;$$

bo'ladi.

4. O'rta qiymat haqidagi birinchi teorema.

Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada chegaralangan va integrallanuvchi bo'lib, $g(x)$ funksiya (a,b) da ishorasini o'zgartirmasa, shunday $\mu \in [m,\ M]$ $\left(m = \inf\limits_{[a,b]}\{f(x)\},\ M = \sup\limits_{[a,b]}\{f(x)\}\right)$ nuqta topiladiki,

$$\int_a^b f(x)g(x)dx = \mu \cdot \int_a^b g(x)dx;$$

tenglik bajariladi.

a) Aniq integral yordamida tekis shaklning yuzasini hisoblash.

1) Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

$f(x) \in C[a,b]$ bo'lib $\forall x \in [a,b]$ uchun $f(x) \geq 0$ tengsizlik bajarilsin va D soha quyidagicha aniqlansin:

$$D = \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases}$$ -egri chiziqli trapetsiya.

Unda

$$S = \int_a^b f(x)dx \qquad (11)$$

tenglik o'rinli.

Agar $f_1(x) \in C[a,b]$, $f_2(x) \in C[a,b]$ bo'lib,

$$D = \begin{cases} a \leq x \leq b \\ f_1(x) \leq y \leq f_2(x); \end{cases}$$

bo'lsa, u holda

$$S = \int_a^b [f_2(x) - f_1(x)]dx \qquad (12)$$

bo'ladi.

2) Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

Agar D soha qutb koordinatalar sistemasida

$$D = \begin{cases} \alpha \leq \varphi \leq \beta \\ 0 \leq r \leq r(\varphi) \end{cases}$$

ko'rinishida berilgan bo'lib, $r(\varphi) \in C[\alpha, \beta]$ bo'lsa,

$$S=\frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)d\varphi \qquad (13)$$

formula o'rinli bo'ladi.

b) Aniq integral yordamida yoy uzunligini hisoblash.

1) Dekart koordinatalar sistemasida berilgan yoy uzunligini hisoblash.

$f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo'lsin. Uning grafigi quyidagi

$$\{(x,f(x)):x\in[a,b]\}$$

nuqtalar to'plamidagi iborat. Shu grafikdagi $A(a,f(a))$ va $B(b,f(b))$ nuqtalar orasidagi $\overset{\smile}{AB}$ egri chiziq yoyi uzunligi l ni topish talab qilinsin. Agar $f'(x)\in C[a,b]$ bo'lsa, unda

$$l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx \qquad (14)$$

bo'ladi.

Agar (14) da $b=x$ desak, $l(x)=\int_a^x\sqrt{1+[f'(x)]^2}\,dx$ bo'lib,

$$\frac{dl}{dx}=\sqrt{1+[f'(x)]^2} \Rightarrow dl=\sqrt{1+[f'(x)]^2}\,dx\,.$$

Bu ifodaga **yoy differensiali** deb ataladi.

2) Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligi hisoblash.

Agar

$$\overset{\smile}{AB}:\begin{cases}x=\varphi(t)\\ y=\psi(t),\end{cases}\quad \alpha\le t\le\beta;$$

bo'lib, $\varphi'(t)\in C[\alpha,\beta]$ va $\psi'(t)\in C[\alpha,\beta]$ bo'lsa,

$$l=\int_{\alpha}^{\beta}\sqrt{\varphi'(t)^2+[\psi'(t)]^2}\,dt \qquad (15)$$

bo'ladi.

3) Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligi hisoblash.

Agar

$$\breve{AB}:\begin{cases}\alpha\le\varphi\le\beta,\\ r=r(\varphi)\end{cases}$$

bo‘lib, $r'(\varphi)\in C[\alpha,\beta]$ bo‘lsa, unda

$$l=\int_{\alpha}^{\beta}\sqrt{r^2(\varphi)+[r'(\varphi)]^2}\,d\varphi \tag{16}$$

formula o‘rinli bo‘ladi.

d) Aylanma sirtning yuzasi.

Aytaylik, $f(x)\in C[a,b]$ bo‘lib, $f(x)\ge 0$ bo‘lsin. $\breve{AB}$ yoyni OX o‘qi atrofida aylantiramiz va **aylanma sirtni** hosil qilamiz. Agar $f'(x)\in C[a,b]$ bo‘lsa, unda shu aylanma sirtning yuzasi ushbu

$$S=2\pi\int_a^b f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx \tag{17}$$

formula yordamida hisoblanadi.

e) Aniq integral yordamida hajm hisoblash.

Faraz qilaylik, bizga biror T jism berilgan bo‘lib, uning OY o‘qiga parallel bo‘lgan kesimlarining yuzasi ma’lum bo‘lsin. Bu yuza x o‘zgaruvchining funksiyasi bo‘ladi, uni $S=S(x)$ deb belgilaylik. Agar $S(x)\in C[a,b]$ bo‘lsa, unda T jismning hajmi V ushbu

$$V=\int_a^b S(x)\,dx \tag{18}$$

formula yordamida hisoblanadi.

Natija. (Aylanma jismning hajmi). Ushbu

$$D=\begin{cases}a\le x\le b\\ 0\le y\le f(x)\end{cases}$$

egri chiziqli trapetsiyani OX o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma jismning hajmi

$$V=\pi\int_a^b [f(x)]^2\,dx \tag{19}$$

formula yordamida hisoblanadi.

f) O‘zgaruvchi kuchning bajargan ishi.

OX o‘qida shu o‘q bo‘ylab biror jism $F=F(x)$ kuch ta’sirida harakat qilayotgan bo‘lsin. Agar $F(x)\in C[a,b]$ bo‘lsa, $F=F(x)$ kuch ta'sirida jismni a nuqtadan b nuqtaga o'tkazishda bajarilgan ish ushbu

$$A=\int_a^b F(x)dx \tag{20}$$

formula yordamida hisoblanadi.

g) Statik moment. Og'irlik markazi.

Aytaylik, m massaga ega bo'lgan M(x,y)-material nuqta berilgan bo'lsin. my va mx ko'paytmalarga mos ravishda berilgan nuqtaning ***OX* va *OY* o'qlarga nisbatan statik momentlari** deb ataladi.

Egri chiziqning OX va OY o'qlarga nisbatan **statik momentlari** M_x va M_y lar ham shu kabi aniqlanadi hamda

$$M_x=\int_0^l ydl,\quad M_y=\int_0^l xdl \tag{21}$$

formulalar yordamida hisoblanadi. Bu yerda $dl=\sqrt{(dx)^2+(dy)^2}$ -yoy differensiali, l esa berilgan egri chiziq uzunligi.

Berilgan egri chiziq og'irlik markazining koordinatalari esa ushbu

$$\bar{x}=\frac{M_y}{l},\quad \bar{y}=\frac{M_x}{l} \tag{22}$$

formulalar yordamida hisoblanadi.

h) Geometrik figuralarning statik momentlari va og'irlik markazi.

Agar geometrik figura

$$D=\begin{cases} a\le x\le b \\ 0\le y\le f(x) \end{cases}$$

egri chiziqli trapetsiyadan iborat bo'lsa, unda

$$M_x=\frac{1}{2}\int_a^b y^2dx,\quad M_y=\frac{1}{2}\int_a^b xydx \tag{23}$$

va

$$\left(\bar{x},\bar{y}\right)=\left(\frac{M_y}{S},\frac{M_x}{S}\right) \tag{24}$$

bo'ladi. Bu yerda $S=\int_a^b y(x)dx$ -trapetsiyaning yuzi.

6^0. Elliptik integrallar.

5-ta'rif. Ushbu

$$F(k,\varphi)=\int_0^{\varphi}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{25}$$

$$E(k,\varphi)=\int_0^{\varphi}\sqrt{1-k^2\sin^2 x}dx \tag{26}$$

ko'rinishdagi integrallar I va II tipdagi **elliptik integrallarning Lejandr formasi** deb ataladi.

(25) va (26)-integral ostidagi funksiyalarning boshlang'ich funksiyalari elementar funksiyalar yordamida ifodalanmaydi. Shuning uchun ham ularning qiymatlarini hisoblash uchun maxsus jadvallar yaratilgan.

Agar (25) va (26)-integrallarda $\varphi=\frac{\pi}{2}$ bo'lsa, u holda bunday integrallar **to'liq elliptik integrallar** deb ataladi va ular $F(k), E(k)$ kabi belgilanadi.

Demak,

$$F(k)=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{27}$$

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2}dx \tag{28}$$

To'liq elliptik integrallarning qiymatlari ham maxsus jadvallar yordamida hisoblanadi.

Misol. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **ellips yoyining uzunligi hisoblansin.**

Ellipsni parametrik ko'rinishida $\begin{cases} x=a\sin t \\ y=b\sin t, \ 0\le t\le 2\pi \end{cases}$ kabi ifodalab olamiz.

Unda $l=4l_1=4\cdot\int_0^{\frac{\pi}{2}}\sqrt{[x'(t)]^2+[y'(t)]^2}dt=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2 t+b^2\sin^2 t}dt=$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\left(1-\sin^2 t\right)+b^2\sin^2 t}\,dt=4a\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2 t}\,dt=4aE\left(\frac{\sqrt{a^2-b^2}}{a}\right)=4aE(\varepsilon)$$

bu yerda $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$-ellipsning ekssentrisiteti.

Nazorat savollari

1. Boshlang'ich funksiya tushunchasi.
2. Aniqmas integral va uning xossalari.
3. Aniqmas integralda o'zgaruvchini almashtirish.
4. Aniqmas integralda bo'laklab integrallash formulasi.
5. Ratsional funksiyalarni integrallash.
6. Ba'zi irratsional ko'rinishdagi funktsiyalarni integrallash.
7. Eyler almashtirishlari.
8. Binomial differensiallarni integrallash.
9. Trigonometrik funksiyalarni integrallash.
10. Aniq integral tushunchasi.
11. Nyuton-Leybnis formulasi.
12. Aniq integralda bo'laklab integrallash formulasi.
13. Aniq integralda o'zgaruvchini almashtirish.
14. O'rta qiymat haqidagi birinchi teorema.
15. Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.
16. Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.
17. Dekart koordinatalar sistemasida berilgan yoy uzunligini hisoblash.
18. Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligini hisoblash.
19. Qutb koordinatalar sistemasida berilgan yoy uzunligini hisoblash.
20. Aylanma sirtning yuzasini hisoblash.
21. Aniq integral yordamida hajm hisoblash.
22. O'zgaruvchi kuchning bajargan ishi.
23. Egri chiziqning koordinata o'qlariga nisbatan statik momentlarini topish.
24. Egri chiziq og'irlik markazining koordinatalarini topish.
25. Geometrik figuralarning statik momentlari.
26. Geometrik figura og'irlik markazining koordinatalarini topish.
27. Elliptik integrallar.

-B-

Mustaqil yechish uchun misol va masalalar

1-masala. Aniqmas integral topilsin.

1.1 $\int\frac{xdx}{\sin^2 x}$.

1.2 $\int x\sin^2 xdx$.

1.3 $\int\left(\sqrt{2}-8x\right)\sin 3xdx$.

1.4 $\int\frac{xdx}{\cos^2 x}$.

1.5 $\int(4x+3)\sin 5xdx$.

1.6 $\int(7x-10)\sin 4xdx$.

1.7 $\int(x+5)\sin 3xdx$.

1.8 $\int(2-3x)\sin 2xdx$.

1.9 $\int(2x-5)\cos 4xdx$.

1.10 $\int(8-3x)\cos 5xdx$.

1.11 $\int\left(x\sqrt{2}-3\right)\cos 2xdx$.

1.12 $\int(4x+7)\cos 3xdx$.

1.13 $\int(5x+6)\cos 2xdx$.

1.14 $\int(3x-2)\cos 5xdx$.

1.15 $\int arctg\sqrt{3x-1}dx$.

1.16 $\int arctg\sqrt{5x-1}dx$.

1.17 $\int e^{-3x}(2-9x)dx$.

1.18 $\int e^{-2x}(4x-3)dx$.

1.19 $\int\ln\left(4x^2+1\right)dx$.

1.20 $\int(2-4x)\sin 2xdx$.

1.21 $\int(4x-2)\cos 2xdx$.

2-masala. Aniqmas integral hisoblansin.

2.1 $\int\frac{dx}{x\sqrt{x^2+1}}$.

2.2 $\int\frac{1+\ln x}{x}dx$.

2.3 $\int\frac{dx}{x\sqrt{x^2-1}}$.

2.4 $\int\frac{x^2+\ln x^2}{x}dx$.

2.5 $\int\frac{xdx}{\sqrt{x^4+x^2+1}}$.

2.6 $\int\frac{(\arccos x)^3-1}{\sqrt{1-x^2}}dx$.

2.7 $\int tgx\cdot\ln(\cos x)dx$.

2.8. $\int\frac{tg(x+1)}{\cos^2(x+1)}dx$.

2.9. $\int\frac{x^3}{\left(x^2+1\right)^2}dx$.

2.10 $\int\frac{1-\cos x}{(x-\sin x)^2}dx$.

2.11. $\int \frac{\sin x - \cos x}{(\cos x + \sin x)^5} dx$.

2.12 $\int \frac{x\cos x + \sin x}{(x\sin x)^2} dx$.

2.13 $\int \frac{x^3 + x}{x^4 + 1} dx$.

2.14 $\int \frac{xdx}{\sqrt{x^4 - x^2 + 1}}$.

2.15. $\int \frac{xdx}{\sqrt[3]{x-1}}$.

2.16 $\int \frac{1 + \ln(x-1)}{x-1} dx$.

2.17 $\int \frac{(x^2+1)dx}{(x^3 + 3x + 1)^5}$.

2.18 $\int \frac{4arctgx - x}{1 + x^2} dx$.

2.19 $\int \frac{x^3 dx}{x^2 + 4}$.

2.20 $\int \frac{x + \cos x}{x^2 + 2\sin x} dx$.

2.21 $\int \frac{2\cos x + 3\sin x}{(2\sin x - 3\cos x)^3} dx$.

3-masala. Aniqmas integral hisoblansin.

3.1 $\int \frac{x^3 + 1}{x^2 - x} dx$.

3.2 $\int \frac{x^3 - 3x^2 - 12}{(x-4)(x-3)(x-2)} dx$.

3.3 $\int \frac{3x^3 + 1}{x^2 - 1} dx$.

3.4 $\int \frac{x^3 - 3x^2 - 12}{(x-4)(x-3)x} dx$.

3.5 $\int \frac{x^3 - 17}{x^2 - 4x + 3} dx$.

3.6 $\int \frac{4x^3 + x^2 + 2}{x(x-1)(x-2)} dx$.

3.7 $\int \frac{2x^3 + 5}{x^2 - x - 2} dx$.

3.8 $\int \frac{3x^3 - 2}{x^3 - x} dx$.

3.9 $\int \frac{2x^3 - 1}{x^2 + x - 6} dx$.

3.10 $\int \frac{x^3 - 3x^2 - 12}{(x-4)(x-2)x} dx$.

3.11 $\int \frac{3x^3 + 25}{x^2 + 3x + 2} dx$.

3.12 $\int \frac{x^5 - x^3 + 1}{x^2 - x} dx$.

3.13 $\int \frac{x^3 + 2x^2 + 3}{(x-1)(x-2)(x-3)} dx$.

3.14 $\int \frac{x^5 + 3x^3 - 1}{x^2 + x} dx$.

3.15 $\int\frac{3x^3+2x^2+1}{(x+2)(x-2)(x-1)}dx$.

3.16 $\int\frac{3x^5-12x^3-7}{x^2+2x}dx$.

3.17 $\int\frac{x^3}{(x-1)(x+1)(x+2)}dx$.

3.18 $\int\frac{x^3-5x^2+5x+23}{(x-1)(x+1)(x-5)}dx$.

3.19 $\int\frac{-x^5+25x^3+1}{x^2+5x}dx$.

3.20 $\int\frac{-x^5+9x^3+4}{x^2+3x}dx$.

3.21 $\int\frac{2x^5-8x^3+2}{x^2-2x}dx$.

4-masala. Aniqmas integral hisoblansin.

4.1 $\int\frac{x^3+4x^2+3x+2}{(x+1)^2\cdot(x^2+1)}dx$.

4.2 $\int\frac{-3x^3+13x^2-13x+1}{(x-2)^2\cdot(x^2-x+1)}dx$.

4.3 $\int\frac{2x^3+7x^2+7x-1}{(x+2)^2\cdot(x^2+x+1)}dx$.

4.4 $\int\frac{x^3+2x^2+10x}{(x+1)^2\cdot(x^2-x+1)}dx$.

4.5 $\int\frac{2x^3+4x^2+2x-1}{(x+1)^2\cdot(x^2+2x+2)}dx$.

4.6 $\int\frac{4x^3+24x^2+20x-28}{(x+3)^2\cdot(x^2+2x+2)}dx$.

4.7 $\int\frac{x^3+6x^2+9x+6}{(x+1)^2(x^2+2x+2)}dx$.

4.8 $\int\frac{x^3+x+1}{(x^2+x+1)(x^2+1)}dx$.

4.9 $\int\frac{2x^3+11x^2+16x+10}{(x+2)^2\cdot(x^2+2x+3)}dx$.

4.10 $\int\frac{2x^3+4x^2+2x+2}{(x^2+x+1)(x^2+x+2)}dx$.

4.11 $\int\frac{3x^3+6x^2+5x-1}{(x+1)^2\cdot(x^2+2)}dx$.

4.12 $\int\frac{4x^2+3x+4}{(x^2+1)(x^2+x+1)}dx$.

4.13 $\int\frac{x^3+9x^2+21x+21}{(x+3)^2\cdot(x^2+3)}dx$.

4.14 $\int\frac{2x^2-x+1}{(x^2-x+1)(x^2+1)}dx$.

4.15 $\int\frac{x^3+6x^2+8x+8}{(x+2)^2\cdot(x^2+4)}dx$.

4.16 $\int\frac{x^3+x+1}{(x^2-x+1)\cdot(x^2+1)}dx$.

4.17 $\int \frac{x^3+5x^2+12x+4}{(x+2)^2\cdot(x^2+4)}dx$.

4.18 $\int \frac{x^3+2x^2+x+1}{(x^2+x+1)\cdot(x^2+1)}dx$.

4.19 $\int \frac{2x^3-4x^2-16x-12}{(x-1)^2\cdot(x^2+4x+5)}dx$.

4.20 $\int \frac{2x^3+2x^2+2x+1}{(x^2+x+1)(x^2+1)}dx$.

4.21 $\int \frac{x^3+4x^2+4x+2}{(x+1)^2\cdot(x^2+x+1)}dx$.

5-masala. Aniqmas integral hisoblansin.

5.1 $\int \frac{\sqrt{1+\sqrt{x}}}{x\cdot\sqrt[4]{x^3}}dx$.

5.2 $\int \frac{\sqrt[4]{\left(1+\sqrt{x}\right)^3}}{x\cdot\sqrt[8]{x^7}}dx$.

5.3 $\int \frac{\sqrt[3]{1+\sqrt{x}}}{x\cdot\sqrt[3]{x^2}}dx$.

5.4 $\int \frac{\sqrt[4]{\left(1+\sqrt[3]{x^2}\right)^3}}{x^2\cdot\sqrt[6]{x}}dx$.

5.5 $\int \frac{\sqrt{1+\sqrt[3]{x}}}{x\sqrt{x}}dx$.

5.6 $\int \frac{\sqrt{1+\sqrt[4]{x^3}}}{x^2\cdot\sqrt[8]{x}}dx$.

5.7 $\int \frac{\sqrt[3]{1+\sqrt[3]{x}}}{x\cdot\sqrt[9]{x^4}}dx$.

5.8 $\int \frac{\sqrt[3]{1+\sqrt[4]{x^3}}}{x^2}dx$.

5.9 $\int \frac{\sqrt[3]{1+\sqrt[3]{x^2}}}{x\cdot\sqrt[9]{x^8}}dx$.

5.10 $\int \frac{\sqrt[3]{\left(1+\sqrt[4]{x^3}\right)^2}}{x^2\cdot\sqrt[4]{x}}dx$.

5.11 $\int \frac{\sqrt[3]{\left(1+\sqrt[3]{x}\right)^2}}{x\cdot\sqrt[9]{x^5}}dx$.

5.12 $\int \frac{\sqrt[5]{\left(1+\sqrt{x}\right)^4}}{x\cdot\sqrt[10]{x^9}}dx$.

5.13 $\int \frac{\sqrt[3]{\left(1+\sqrt[3]{x^2}\right)^2}}{x^2\cdot\sqrt[9]{x}}dx$.

5.14 $\int \frac{\sqrt[5]{\left(1+\sqrt[3]{x^2}\right)^4}}{x^2\cdot\sqrt[5]{x}}dx$.

5.15 $\int \frac{\sqrt[3]{\left(1+\sqrt{x}\right)^2}}{x\cdot\sqrt[6]{x^5}}dx$.

5.16 $\int \frac{\sqrt{1+\sqrt[5]{x^4}}}{x^2\cdot\sqrt[5]{x}}dx$.

5.17 $\int\frac{\sqrt{1+\sqrt[3]{x^2}}}{x^2}dx$.

5.18 $\int\frac{\sqrt[3]{1+\sqrt[5]{x^4}}}{x^2\cdot\sqrt[15]{x}}dx$.

5.19 $\int\frac{\sqrt{1+x}}{x^2\cdot\sqrt{x}}dx$.

5.20 $\int\frac{\sqrt[3]{\left(1+\sqrt[5]{x^4}\right)^2}}{x^2\cdot\sqrt[3]{x}}dx$.

5.21 $\int\frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^3}}{x^2\cdot\sqrt[5]{x^2}}dx$.

6-masala. Aniq integral hisoblansin.

6.1 $\int\limits_{e+1}^{e^2+1}\frac{1+\ln(x-1)}{x-1}dx$.

6.2 $\int\limits_0^1\frac{(x^2+1)dx}{(x^3+3x+1)^2}$.

6.3 $\int\limits_0^1\frac{4arctgx-x}{1+x^2}dx$.

6.4 $\int\limits_0^2\frac{x^3dx}{x^2+4}$.

6.5 $\int\limits_{\pi}^{2\pi}\frac{x+\cos x}{x^2+2\sin x}dx$.

6.6 $\int\limits_0^{\frac{\pi}{4}}\frac{2\cos x+3\sin x}{(2\sin x-3\cos x)^3}dx$.

6.7 $\int\limits_0^{\frac{1}{2}}\frac{8x-arctg2x}{1+4x^2}$.

6.8 $\int\limits_0^1\frac{xdx}{x^4+1}$.

6.9 $\int\limits_{\sqrt{3}}^{\sqrt{8}}\frac{x+\frac{1}{x}}{\sqrt{x^2+1}}dx$.

6.10 $\int\limits_0^{\sqrt{3}}\frac{arctgx+x}{1+x^2}dx$.

6.11 $\int\limits_0^{\sqrt{3}}\frac{x-(arctgx)^4}{1+x^2}dx$.

6.12 $\int\limits_0^1\frac{x^3}{x^2+1}dx$.

6.13 $\int\limits_0^{\sin 1}\frac{(\arcsin x)^2+1}{\sqrt{1-x^2}}dx$.

6.14 $\int\limits_1^3\frac{1-\sqrt{x}}{\sqrt{x}\cdot(x+1)}dx$.

6.15 $\int\limits_{\sqrt{3}}^{\sqrt{8}}\frac{dx}{x\cdot\sqrt{x^2+1}}$.

6.16 $\int\limits_1^e\frac{1+\ln x}{x}dx$.

6.17 $\int_1^e \frac{x^2 + \ln x^2}{x} dx$.

6.18 $\int_0^{\pi/4} tgx \cdot \ln(\cos x) dx$.

6.19 $\int_0^1 \frac{x^3 \cdot dx}{(x^2+1)^2}$.

6.20 $\int_\pi^{2\pi} \frac{1-\cos x}{(x-\sin x)^2} dx$.

6.21 $\int_0^1 \frac{xdx}{\sqrt{x^4+x^2+1}}$.

7-masala. Aniq integral hisoblansin.

7.1 $\int_{-2}^0 (x^2+5x+6)\cos 2xdx$.

7.2 $\int_{-2}^0 (x^2-4)\cos 3xdx$.

7.3 $\int_{-1}^0 (x^2+4x+3)\cos xdx$.

7.4 $\int_{-2}^0 (x+2)^2 \cos 3xdx$.

7.5 $\int_{-4}^0 (x^2+7x+12)\cos xdx$.

7.6 $\int_0^\pi (2x^2+4x+7)\cos 2xdx$.

7.7 $\int_0^\pi (9x^2+9x+11)\cos 3xdx$.

7.8 $\int_0^\pi (8x^2+16x+17)\cos 4xdx$.

7.9 $\int_0^{2\pi} (3x^2+5)\cos 2xdx$.

7.10 $\int_0^{2\pi} (2x^2-15)\cos 3xdx$.

7.11 $\int_0^{2\pi} (3-7x^2)\cos 2xdx$.

7.12 $\int_0^{2\pi} (1-8x^2)\cos 4xdx$.

7.13 $\int_{-1}^0 (x^2+2x+1)\sin 3xdx$.

7.14 $\int_0^3 (x^2-3x)\sin 2xdx$.

7.15 $\int_0^\pi (x^2-3x+2)\sin xdx$.

7.16 $\int_0^{\pi/2} (x^2-5x+6)\sin 3xdx$.

7.17 $\int_{-3}^0 (x^2+6x+9)\sin 2xdx$.

7.18 $\int_0^{\pi/2} (1-5x^2)\sin xdx$.

7.19 $\int_{\pi/4}^3 (3x-x^2)\sin 2xdx$.

7.20 $\int_1^2 x\ln^2 xdx$.

7.21 $\int_2^3 (x-1)^3 \cdot \ln^2(x-1)dx$.

8-masala. Aniq integral hisoblansin.

8.1 $\int_0^{\pi/2} \frac{\cos x dx}{(1+\cos x+\sin x)^2}$.

8.2 $\int_0^{2arctg\frac{1}{2}} \frac{(1-\sin x)dx}{\cos x(1+\cos x)}$.

8.3 $\int_{-\frac{2\pi}{3}}^{0} \frac{\cos x dx}{1+\cos x-\sin x}$.

8.4 $\int_{-\frac{\pi}{2}}^{0} \frac{\cos x dx}{(1+\cos x-\sin x)^2}$.

8.5 $\int_0^{2\pi} \frac{\cos x dx}{1+\cos x+\sin x}$.

8.6 $\int_0^{2arctg 1/3} \frac{\cos x dx}{(1+\cos x)(1-\sin x)}$.

8.7 $\int_0^{\pi/2} \frac{\sin x dx}{1+\cos x+\sin x}$.

8.8 $\int_0^{2arctg 1/2} \frac{1+\sin x}{(1-\sin x)^2} dx$.

8.9 $\int_{\pi/3}^{\pi/2} \frac{\cos x dx}{1+\sin x-\cos x}$.

8.10 $\int_0^{\pi/2} \frac{(1+\cos x)dx}{1+\cos x+\sin x}$.

8.11 $\int_0^{\pi/2} \frac{\cos x dx}{5+4\cos x}$.

8.12 $\int_0^{2\pi/3} \frac{1+\sin x}{1+\cos x+\sin x} dx$.

8.13 $\int_{2arctg 1/3}^{2arctg 1/2} \frac{dx}{\sin x \cdot (1-\sin x)}$.

8.14 $\int_{2arctg 1/2}^{\pi/2} \frac{dx}{(1+\sin x-\cos x)^2}$.

8.15 $\int_0^{\pi/2} \frac{\cos x-\sin x}{(1+\sin x)^2} dx$.

8.16 $\int_{2arctg 2}^{2arctg 3} \frac{dx}{\cos x \cdot (1-\cos x)}$.

8.17 $\int_{\frac{\pi}{2}}^{2arctg 2} \frac{dx}{\sin^2 x \cdot (1+\cos x)}$.

8.18 $\int_{2arctg\frac{1}{2}}^{\pi/2} \frac{\cos x dx}{(1-\cos x)^3}$.

8.19 $\int_{\pi/2}^{2arctg 2} \frac{dx}{\sin^2 x \cdot (1-\cos x)}$.

8.20 $\int_0^{\pi/2} \frac{\cos x dx}{2+\cos x}$.

8.21 $\int_0^{\pi/2} \frac{\sin x dx}{(1+\sin x)^2}$.

9-masala. Aniq integral hisoblansin.

9.1 $\int\limits_0^{2\pi} \sin^8 \frac{x}{4} dx$.

9.2 $\int\limits_0^{2\pi} \sin^6 \frac{x}{4} \cdot \cos^2 \frac{x}{4} dx$.

9.3 $\int\limits_{\frac{\pi}{2}}^{\pi} 2^8 \sin^8 x dx$.

9.4 $\int\limits_0^{\pi} 2^4 \sin^4 x \cos^4 x dx$.

9.5 $\int\limits_0^{\pi} 2^4 \sin^6 \frac{x}{2} \cdot \cos^2 \frac{x}{2} dx$.

9.6 $\int\limits_0^{\pi} 2^4 \sin^4 \frac{x}{2} \cdot \cos^4 \frac{x}{2} dx$.

9.7 $\int\limits_0^{2\pi} \sin^4 x \cdot \cos^4 x dx$.

9.8 $\int\limits_0^{2\pi} \sin^2 x \cos^6 x dx$.

9.9 $\int\limits_{-\frac{\pi}{2}}^{0} 2^8 \sin^4 x \cdot \cos^4 x dx$.

9.10 $\int\limits_{-\frac{\pi}{2}}^{0} 2^8 \sin^2 x \cdot \cos^6 x dx$.

9.11 $\int\limits_0^{2\pi} \sin^2 \frac{x}{4} \cdot \cos^6 \frac{x}{4} dx$.

9.12 $\int\limits_0^{2\pi} \cos^8 \frac{x}{4} dx$.

9.13 $\int\limits_0^{\pi} 2^4 \cdot \cos^8 \frac{x}{2} dx$.

9.14 $\int\limits_{-\pi}^{0} 2^8 \sin^6 x \cos^2 x dx$.

9.15 $\int\limits_{-\frac{\pi}{2}}^{0} 2^8 \sin^8 x dx$.

9.16 $\int\limits_0^{2\pi} \sin^6 x \cos^2 x dx$.

9.17 $\int\limits_0^{2\pi} \sin^8 x dx$.

9.18 $\int\limits_0^{\pi} 2^4 \sin^2 x \cos^6 x dx$.

9.19 $\int\limits_{\pi/2}^{\pi} 2^8 \sin^6 x \cos^2 x dx$.

9.20 $\int\limits_0^{2\pi} \sin^4 3x \cos^4 3x dx$.

9.21 $\int\limits_{\frac{\pi}{2}}^{\pi} 2^8 \cdot \sin^2 x \cdot \cos^6 x dx$.

10-masala. Aniq integral hisoblansin.

10.1 $\int\limits_2^4 \frac{\sqrt{x^2 - 4}}{x^4} dx$.

10.2 $\int\limits_3^6 \frac{\sqrt{x^2 - 9}}{x^4} dx$.

10.3 $\int\limits_0^2 \frac{x^4dx}{\sqrt{(8-x^2)^3}}$.

10.4 $\int\limits_0^2 \frac{dx}{\sqrt{(16-x^2)^3}}$.

10.5 $\int\limits_1^{\sqrt{3}} \frac{dx}{\sqrt{(1+x^2)^3}}$.

10.6 $\int\limits_{-3}^{3} x^2 \cdot \sqrt{9-x^2}\,dx$.

10.7 $\int\limits_0^{2\sqrt{2}} \frac{x^4dx}{(16-x^2)\cdot\sqrt{16-x^2}}$.

10.8 $\int\limits_{\sqrt{2}}^{2\sqrt{2}} \frac{\sqrt{x^2-2}}{x^4}dx$.

10.9 $\int\limits_0^{4\sqrt{3}} \frac{dx}{\sqrt{(64-x^2)^3}}$.

10.10 $\int\limits_0^5 x^2\sqrt{25-x^2}\,dx$.

10.11 $\int\limits_0^{5/2} \frac{x^2dx}{\sqrt{25-x^2}}$.

10.12 $\int\limits_0^4 x^2\sqrt{16-x^2}\,dx$.

10.13 $\int\limits_0^4 \frac{dx}{(16+x^2)^{3/2}}$.

10.14 $\int\limits_0^2 \frac{x^2dx}{\sqrt{16-x^2}}$.

10.15 $\int\limits_0^1 \frac{x^4dx}{(2-x^2)^{3/2}}$.

10.16 $\int\limits_0^{\sqrt{3}} \frac{dx}{\sqrt{(4-x^2)^3}}$.

10.17 $\int\limits_0^{\sqrt{2}/2} \frac{x^4dx}{\sqrt{(1-x^2)^3}}$.

10.18 $\int\limits_1^2 \frac{\sqrt{x^2-1}}{x^4}dx$.

10.19 $\int\limits_0^3 \frac{dx}{(9+x^2)^{3/2}}$.

10.20 $\int\limits_0^5 \frac{dx}{(25+x^2)\sqrt{25+x^2}}$.

10.21 $\int\limits_0^2 \frac{dx}{(4+x^2)\sqrt{4+x^2}}$.

11-masala. Quyidagi chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.

11.1 $y=(x-2)^3; y=4x-8.$

11.2 $y=x\sqrt{9-x^2}; y=0, (0\le x\le 3)$.

11.3 $y=4-x^2; y=x^2-2x.$

11.4 $y=\sqrt{4-x^2}; y=0; x=0; x=1.$

11.5 $y=x^2\sqrt{4-x^2}; y=0; (0\le x\le 2)$.

11.6 $y=\sqrt{e^x-1}; y=0; x=\ln 2$.

11.7 $y=\dfrac{1}{x\sqrt{1+\ln x}}; y=0; x=1; x=e^3.$

11.8 $y=(x+1)^2; y^2=x+1.$

11.9 $y=\arccos x; y=0; x=0.$

11.10 $y=2x-x^2+3; y=x^2-4x+3$.

11.11 $y=x\sqrt{36-x^2}; y=0; (0\le x\le 6)$.

11.12 $x=\arccos y; x=0; y=0.$

11.13 $y=x\,arctgx; y=0; x=\sqrt{3}$.

11.14 $y=x^2\sqrt{8-x^2}; y=0; (0\le x\le 2\sqrt{2}).$

11.15 $x=\sqrt{e^y-1}; x=0; y=\ln 2$.

11.16 $y=x\sqrt{4-x^2}; y=0; (0\le x\le 2)$.

11.17 $y=\dfrac{x}{1+\sqrt{x}}; y=0; x=1$.

11.18 $x=(y-2)^3; x=4y-8$.

11.19 $y=\dfrac{x}{(x^2+1)^2}\ y=0;\ x=1$.

11.20 $x=4-y^2; x=y^2-2y.$

11.21 $y=(x-1)^2; y^2=x-1.$

12-masala. Tenglamalari qutb koordinatalar sistemasida berilgan chiziqlar bilan chegarlangan shaklning yuzasi hisoblansin.

12.1 $r=3\sin\varphi; r=5\sin\varphi.$

12.2 $r=2\sin\varphi;\quad r=4\sin\varphi$.

12.3 $r=2\cos 6\varphi$.

12.4 $r=\cos\varphi-\sin\varphi$.

12.5 $r=\cos\varphi+\sin\varphi$.

12.6 $r=2\sin 4\varphi$.

12.7 $r=\sin 6\varphi$.

12.8 $r=2\cos\varphi; r=3\cos\varphi$.

12.9 $r=\dfrac{3}{2}\cos\varphi; r=\dfrac{5}{2}\cos\varphi$.

12.10 $r=4\cos 4\varphi$.

12.11 $r=1+\sqrt{2}\sin\varphi$.

12.12 $r=\dfrac{5}{2}\sin\varphi; r=\dfrac{3}{2}\sin\varphi$.

12.13 $r=\dfrac{1}{2}+\cos\varphi$.

12.14 $r=\cos\varphi; r=2\cos\varphi$.

12.15 $r=\sin\varphi; r=2\sin\varphi$.

12.16 $r=6\cos3\varphi; r=3(r\geq3)$.

12.17 $r=\frac{1}{2}+\sin\varphi$.

12.18 $r=6\sin3\varphi; r=3(r\geq3)$.

12.19 $r=\cos3\varphi$.

12.20. $r=3\sin3\varphi$.

12.21 $r=4\cos3\varphi; r=2.(r\geq2)$.

13-masala. Parametrik ko'rinishda berilgan egri chiziq yoyining uzunligi hisoblansin.

13.1 $x=5(t-\sin t); y=5(1-\cos t); 0\leq t\leq\pi$.

13.2 $x=3(2\cos t-\cos2t); y=3(2\sin t-\sin2t); 0\leq t\leq2\pi$.

13.3 $x=4(\cos t+t\sin t); y=4(\cos t-t\sin t);\ 0\leq t\leq2$.

13.4 $x=(t^2-2)\sin t+2t\cos t; y=(2-t^2)\cos t+2t\sin t; 0\leq t\leq\pi$.

13.5 $x=10\cos^3 t; y=10\sin^3 t; 0\leq t\leq\frac{\pi}{2}$.

13.6 $x=e^t(\cos t+\sin t); y=e^t(\cos t-\sin t); 0\leq t\leq\pi$.

13.7 $x=3(t-\sin t); y=3(1-\cos t); \pi\leq t\leq2\pi$.

13.8 $x=\frac{1}{2}\cos t-\frac{1}{4}\cos2t; y=\frac{1}{2}\sin t-\frac{1}{4}\sin2t; \frac{\pi}{2}\leq t\leq\frac{2\pi}{3}$.

13.9 $x=3(\cos t+t\sin t); y=3(\sin t-t\cos t); 0\leq t\leq\frac{\pi}{3}$.

13.10 $x=(t^2-2)\sin t+2t\cos t; y=(2-t^2)\cos t+2t\sin t; 0\leq t\leq\frac{\pi}{3}$.

13.11 $x=6\cos^3 t; y=6\sin^3 t; 0\leq t\leq\frac{\pi}{3}$.

13.12 $x=e^t(\cos t+\sin t); y=e^t(\cos t-\sin t); \frac{\pi}{2}\leq t\leq\pi$.

13.13 $x=2{,}5(t-\sin t); y=2{,}5(1-\cos t); \frac{\pi}{2}\leq t\leq\pi$.

13.14 $x=3{,}5(2\cos t-\cos2t); y=3{,}5(2\sin t-\sin2t); 0\leq t\leq\frac{\pi}{2}$.

13.15 $x=6(\cos t+t\sin t); y=6(\sin t-t\sin t); 0\leq t\leq\pi$.

13.16 $x=(t^2-2)\sin t+2t\cos t; y=(2-t^2)\cos t+2t\sin t; 0\leq t\leq\frac{\pi}{2}$.

13.17 $x = 8\cos^3 t; y = 8\sin^3 t; 0 \le t \le \frac{\pi}{6}$.

13.18 $x = e^t(\cos t + \sin t); y = e^t(\cos t - \sin t); 0 \le t \le 2\pi$.

13.19 $x = 4(t - \sin t); y = 4(1 - \cos t); \frac{\pi}{2} \le t \le \frac{2\pi}{3}$.

13.20 $x = 2(2\cos t - \cos 2t); y = 2(2\sin t - \sin 2t); 0 \le t \le \frac{\pi}{3}$.

13.21 $x = 8(\cos t + t\sin t); y = 8(\sin t - t\cos t); 0 \le t \le \frac{\pi}{4}$.

14-masala. Quyidagi sirtlar bilan chegaralangan jismning hajmi topilsin.

14.1 $\frac{x^2}{9} + y^2 = 1; z = y; z = 0.(y \ge 0)$.

14.2 $z = x^2 + 4y; z = 2$.

14.3 $\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1; z = 0; z = 3$.

14.4 $\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{36} = -1; z = 12$.

14.5 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1; z = 1; z = 0$.

14.6 $x^2 + y^2 = 9; z = y; z = 0.(y \ge 0)$.

14.7 $z = x^2 + 9y; z = 3$.

14.8 $\frac{x^2}{4} + y^2 - z^2 = 1; z = 0; z = 3$.

14.9 $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{64} = -1; z = 16$.

14.10 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{16} = 1; z = 2; z = 0$.

14.11 $z = 2x^2 + 8y^2; z = 4$.

14.12 $\frac{x}{81} + \frac{y^2}{25} - z^2 = 1; z = 0; z = 2$.

14.13 $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{36} = -1; z = 12$.

14.14 $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} = 1; z = 0; z = 3$.

14.15 $z = x^2 + 5y^2, z = 5$.

14.16 $\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1; z = 0; z = 4$.

14.17 $\frac{x^2}{9} + \frac{y^2}{25} - \frac{z^2}{100} = -1; z = 20$.

14.18 $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{64} = 1; z = 4; z = 0$.

14.19 $z = 4x^2 + 9y^2, z = 6$.

14.20 $z = 2x^2 + 18y^2, z = 6$.

14.21 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$

15-masala. **Quyidagi chiziqlar bilan chegaralangan shaklni 1-16 variantlarda Ox o'qi atrofida, 17-21 variantlarda esa OY o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi topilsin.**

15.1 $y=-x^2+5x-6; y=0$.

15.2 $2x-x^2-y=0; 2x^2-4x+y=0$.

15.3 $y=3\sin x; y=\sin x; 0\le x\le \pi$.

15.4 $y=5\cos x; y=\cos x; x=0; x\ge 0$.

15.5 $y=\sin^2 x, x=\dfrac{\pi}{2}, y=0$.

15.6 $x=\sqrt[3]{y-2}, x=1, y=1$.

15.7 $y=xe^x, y=0, x=1$.

15.8 $y=2x-x^2; y=-x+2; x=0$.

15.9 $y=2x-x^2; y=-x+2$.

15.10 $y=e^{2-x}, y=0, x=0, x=1$.

15.11 $y=x^2, y^2-x=0$.

15.12 $x^2+(y-2)^2=1$.

15.13 $y=1-x^2, x=\sqrt{y-2}, x=1$.

15.14 $y=x^2, y=1, x=2$.

15.15 $y=x^2, y=\sqrt{x}$.

15.16 $y=\sin\dfrac{\pi x}{2}, y=x^2$.

15.17 $y=x^2, x=2, y=0$.

15.18 $y=(x-1)^2, y=1$.

15.19 $y=\ln x, x=2, y=0$.

15.20 $y=x^2-2x+1, x=2, y=0$.

15.21 $y=x^2+1, y=x, x=0, x=1$.

16-masala.

Quyidagi chiziqlarni aylantirishdan hosil bo'lgan aylanma sirtlarning yuzalari hisoblansin.

16.1 $y=\dfrac{x^2}{2}$ $(0\le y\le 1,5)$ OY o'qi atrofida.

16.2 $y^2=4+x$ $(x\le 2)$ OX o'qi atrofida.

16.3 $3x^2+4y^2=12$ ellipisni OY o'qi atrofida.

16.4 $x=\dfrac{1}{4}y^2-\dfrac{1}{2}\ln y$ $(1\le y\le e)$ OX o'qi atrofida.

16.5 $y^2+4x=2\ln y$ $(1\le y\le 2)$ OY o'qi atrofida.

16.6 $y=x^3$ $(0\le x\le 1)$ OX o'qi atrofida.

16.7 $x=e^t\sin t; y=e^t\cos t$ $\left(0\le t\le \dfrac{\pi}{2}\right)$ OX o'qi atrofida.

16.8 $x=2\cos t-\cos 2t, y=2\sin t-\sin 2t$ ni OX o'qi atrofida.

16.9 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning OX o'qidan yuqorida joylashgan bo'lagigining koordinata o'qlariga nisbatan statik momentlari topilsin.

16.10 $x+y=1, x=0, y=0$ chiziqlar bilan chegaralangan uchburchakning OX va OY o'qlarga nisbatan statik momentlari topilsin.

16.11 $y^2=2x,(y>0,0\le x\le 2)$ parabola yoyining OX va OY o'qlarga nisbatan statik momentlari topilsin.

16.12 $y=\cos x\left(-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right)$ egri chiziq yoyining OX o'qiga nisbatan statik momenti topilsin.

16.13 $\frac{x}{a}+\frac{y}{b}=1$ to'g'ri chiziqning koordinata o'qlari orasida joylashgan kesmasining koordinata o'qlariga nisbatan statik momentlari topilsin.

16.14 $y=\frac{2}{1+x^2}$ va $y=x^2$ chiziqlari bilan chegaralangan shaklning OX o'qiga nisbatan statik momenti topilsin.

16.15 $x^2+y^2=a^2, y\ge 0$ -yarim aylananing og'irlik markazi topilsin.

16.16 $x^{2/3}+y^{2/3}=a^{2/3}, x\ge 0, y\ge 0$ – astroida yoyining og'irlik markazi topilsin.

16.17 $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1(x\ge 0, y\ge 0)$ ning og'irlik markazi topilsin.

16.18 $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$ $(1\le y\le 2)$ chiziq yoyining og'irlik markazi topilsin.

Quyidagi chiziqlar bilan chegaralangan tekis shaklning og'irlik markazi topilsin.

16.19 $ax=y^2, \quad ay=x^2 \quad (x\succ 0)$.

16.20 $y=\frac{2}{\pi}x, \quad y=\sin x \quad (x\ge 0)$.

16.21 $x^2+4y-16=0; y=0$.

- D -

Namunaviy variant yechimi.

1.21-masala. $\int(4x-2)\cos 2x dx$ **aniqmas integral hisoblansin.**

◁ Bu integralni bo'laklab integrallash usulidan foydalanib, hisoblaymiz:

$$\int(4x-2)\cos 2x dx = \left(\begin{cases} u=4x-2 \\ dv=\cos 2x dx \end{cases} \Rightarrow, \begin{vmatrix} du=4dx \\ v=\frac{1}{2}\sin 2x \end{vmatrix} \right) = \frac{4x-2}{2}\sin 2x - 2\int \sin 2x dx =$$

$$=(2x-1)\sin 2x + \cos 2x + c \triangleright$$

2.21-masala. $\int \frac{2\cos x + 3\sin x}{(2\sin x - 3\cos x)^3}dx$ **aniqmas integral hisoblansin.**

◁ Bu integralni o'zgaruvchilarni almashtirish usulidan foydalanib, hisoblaymiz:

$$\int \frac{2\cos x + 3\sin x}{(2\sin x - 3\cos x)^3}dx = \Big(\big(2\sin x - 3\cos x = t \text{ deb belgilaymiz} \Rightarrow$$

$$\Rightarrow (2\sin x - 3\cos x)' dx = t' \cdot dt \text{ yoki } (2\cos x + 3\sin x)dx = dt \big) \Big) =$$

$$= \int \frac{dt}{t^3} = \int t^{-3}dt = \frac{t^{-2}}{-2} + c = -\frac{1}{2t^2} + c = -\frac{1}{2(2\sin x - 3\cos x)^2} + c \triangleright$$

3.21-masala. $\int \frac{2x^5 - 8x^3 + 2}{x^2 - 2x}dx$ aniqmas integral hisoblansin.

◁ Biz bu integralni ratsional funksiyani integrallash usulidan foydalanib hisoblaymiz. Avval noto'g'ri kasrni to'g'ri kasrga keltiramiz, so'ngra uni sodda kasrlarga yoyamiz:

$$R(x) = \frac{2x^5 - 8x^3 + 2}{x^2 - 2x} = 2x^3 + 4x^2 + \frac{2}{x(x-2)} = 2x^3 + 4x^2 + \frac{1}{x-2} - \frac{1}{x} \Rightarrow \int \frac{2x^5 - 8x^3 + 3}{x^2 - 2x} =$$

$$= \int \left[2x^3 + 4x^2 + \frac{1}{x-2} - \frac{1}{x} \right] dx = \frac{x^4}{2} + \frac{4x^3}{3} + \ln|x-2| - \ln|x| + c = \frac{x^4}{2} + \frac{4x^3}{3} + \ln\left|\frac{x-2}{x}\right| + c \triangleright$$

4.21-masala. $\int \frac{x^3 + 4x^2 + 4x + 2}{(x+1)^2 \cdot (x^2 + x + 1)}dx$ **aniqmas integral hisoblansin.**

◁ Bu integral ostida ham ratsional funksiya turibdi. Bu funksiyani sodda kasrlarga yoyamiz.

$$R(x)=\frac{x^3+4x^2+4x+2}{(x+1)^2\cdot(x^2+x+1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}.$$

Bu tenglikning o‘ng tomonidagi noma’lum A, B, C va D larni noma’lum koeffitsientlar usulidan foydalanib topamiz. Buning uchun tenglikning o‘ng tomonini umumiy maxrajga keltiramiz va berilgan kasr hamda hosil bo‘lgan kasrlarning suratlarini bir-biriga tenglaymiz:

$$A(x+1)+B(x^2+x+1)+(Cx+D)\cdot(x+1)^2=x^3+4x^2+4x+2$$

$$(A+C)x^3+(2A+B+2C+D)x^2+(2A+B+C+2D)x+(A+B+D)=x^3+4x^2+4x+2$$

$$\Downarrow$$

$$\begin{cases} A+C=1 \\ 2A+B+2C+D=4 \\ 2A+B+C+2D=4 \\ A+B+D=2 \end{cases}$$

Bu sistemani yechib $A=0$ va $B=C=D=1$ ekanligini topamiz. Demak, $R(x)=\frac{1}{(x+1)^2}+\frac{x+1}{x^2+x+1}$ ekan.

$$\Rightarrow \int R(x)dx=\int\frac{dx}{(x+1)^2}+\int\frac{x+1}{x^2+x+1}dx=I_1+I_2 \quad (*)$$

$$I_1=\int\frac{dx}{(x+1)^2}=\int(x+1)^{-2}d(x+1)=-\frac{1}{x+1}+c_1;$$

$$I_2=\int\frac{x+1}{x^2+x+1}dx\int\frac{x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\left(\left(\begin{array}{l} x+\frac{1}{2}=t \\ x=t-\frac{1}{2}\Rightarrow dx=dt \end{array}\right)\right)=\int\frac{t+\frac{1}{2}}{t^2+\frac{3}{4}}dt=\int\frac{tdt}{t^2+\frac{3}{4}}+\frac{1}{2}\int\frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}=$$

$$=\frac{1}{2}\int\frac{d\left(t^2+\frac{3}{4}\right)}{t^2+\frac{3}{4}}+\frac{1}{2}\int\frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}\ln\left|t^2+\frac{3}{4}\right|+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}+c_2=$$

$$=\frac{1}{2}\ln(x^2+x+1)+\frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+c_2.$$

I_1 va I_2 ning qiymatlarini (*) tenglikka olib borib qo'yib, ushbu natijaga kelamiz.

$$\int\frac{x^3+4x^2+4x+2}{(x+1)^2\cdot(x^2+x+1)}dx=-\frac{1}{x+1}+\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+c \quad \triangleright$$

5.21-masala. $\int\frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^3}}{x^2\cdot\sqrt[5]{x^2}}dx$ **aniqmas integral hisoblansin.**

$\triangleleft$ Integralni differensial binomni integrallashdan foydalanib hisoblaymiz:

$$I=\int\frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^3}}{x^2\cdot\sqrt[5]{x^2}}dx=\int x^{-\frac{12}{5}}\cdot\left(1+x^{\frac{4}{5}}\right)^{\frac{3}{4}}dx\Rightarrow a=b=1,m=-\frac{12}{5},n=\frac{4}{5},p=\frac{3}{4}\Rightarrow$$

$$\Rightarrow\frac{m+1}{n}+p=\frac{-\frac{12}{5}+1}{\frac{4}{5}}+\frac{3}{4}=-\frac{7}{4}+\frac{3}{4}=-1\Rightarrow\frac{1}{x^{4/5}}+1=z^4-\text{almashtirish}$$

bajarish lozim $\Rightarrow x=\left(z^4-1\right)^{-5/4}\Rightarrow dx=-5z^3\cdot\left(z^4-1\right)^{-\frac{9}{4}}dz$ Bularni I ga olib borib qo'yib topamiz:

$$I=-5\int z^6dz=-\frac{5z^7}{7}+c=-\frac{5}{7}\cdot\frac{\sqrt[4]{\left(1+\sqrt[5]{x^4}\right)^7}}{x\cdot\sqrt[5]{x^2}}+c\ \triangleright$$

6.21-masala. $\int\limits_0^1\frac{xdx}{\sqrt{x^4+x^2+1}}$ **aniq integral hisoblansin.**

$$\triangleleft\int\limits_0^1\frac{xdx}{\sqrt{x^4+x^2+1}}=\int\limits_0^1\frac{xdx}{\sqrt{\left(x^2+\frac{1}{2}\right)^2+\frac{3}{4}}}=\left(\left(\begin{matrix}x^2+\frac{1}{2}=t,x=0\Rightarrow t=\frac{1}{2}\\ xdx=\frac{1}{2}dt,x=1\Rightarrow t=\frac{3}{2}\end{matrix}\right)\right)=\frac{1}{2}\int\limits_{\frac{1}{2}}^{3/2}\frac{dt}{\sqrt{t^2+\frac{3}{4}}}=$$

((1-§ ning 16^0-punktidagi 24-formuladan foydalanamiz))

$$=\frac{1}{2}\ln\left|t+\sqrt{t^2+\frac{3}{4}}\right|_{\frac{1}{2}}^{3/2}=\frac{1}{2}\left[\ln\left(\frac{3}{2}+\sqrt{\frac{9}{4}+\frac{3}{4}}\right)-\ln\left(\frac{1}{2}+1\right)\right]=\frac{1}{2}\ln\frac{3+\sqrt{12}}{3}=\frac{1}{2}\ln\frac{3+2\sqrt{3}}{3}.\triangleright$$

7.21-masala. $\int_2^3 (x-1)^3 \cdot \ln^2(x-1)dx$ **aniq integral hisoblansin.**

◁Bu integralni bo'laklab integrallash usulidan foydalanib, hisoblaymiz:

$$\int_2^3 (x-1)^3 \cdot \ln^2(x-1)dx = \left(\begin{cases} u=\ln^2(x-1) \\ dv=(x-1)^3dx \end{cases} \Rightarrow \begin{matrix} du=\dfrac{2\ln(x-1)}{x-1}dx \\ v=\dfrac{(x-1)^4}{4} \end{matrix}\right) =$$

$$= \frac{(x-1)^4}{4}\ln^2(x-1)\Big|_2^3 - \frac{1}{2}\int_2^3 (x-1)^3 \cdot \ln(x-1)dx =$$

$$= 4\ln^2 2 - \frac{1}{2}\int_2^3 (x-1)^3 \cdot \ln^2(x-1)dx = \left(\begin{cases} u=\ln^2(x-1) \\ dv=(x-1)^3dx \end{cases} \Rightarrow \begin{matrix} du=\dfrac{1}{x-1}dx \\ v=\dfrac{(x-1)^4}{4} \end{matrix}\right) =$$

$$=4\ln^2 2-\frac{1}{2}\left[\frac{(x-1)^4}{4}\cdot\ln(x-1)\Big|_2^3-\frac{1}{4}\int_2^3(x-1)^3dx\right]=4\ln^2 2-\frac{1}{2}\left[4\ln 2-\frac{1}{16}(x-1)^4\Big|_2^3\right]=$$

$$= 4\ln^2 2 - 2\ln 2 + \frac{1}{32}\cdot(16-1) = 4\ln^2 2 - 2\ln 2 + \frac{15}{32}. \triangleright$$

8.21-masala. $\int_0^{\pi/2} \frac{\sin x dx}{(1+\sin x)^2}$ **aniq integral hisoblansin.**

◁Bu integralni hisoblash uchun $tg\frac{x}{2}=t$ universal almashtirish bajaramiz. Unda $x=0 \Rightarrow t=0; x=\frac{\pi}{2} \Rightarrow t=1$. va $dx=\frac{2dt}{1+t^2}; \sin t=\frac{2t}{1+t^2}$.

Almashtirishdan so'ng berilgan integral quyidagi ko'rinishga keladi:

$$\int_0^{\pi/2}\frac{\sin x dx}{(1+\sin x)^2}=\int_0^1\frac{2tdt}{(t+1)^2}=2\int_0^1\frac{t+1-1}{(t+1)^2}dt=2\left[\int_0^1\frac{dt}{t+1}-\int_0^1\frac{dt}{(t+1)^2}\right]=$$

$$=2\left[\ln|t+1|\Big|_0^1+\frac{1}{t+1}\Big|_0^1\right]_0^1=2\left[\ln 2+\frac{1}{2}-1\right]=2\ln 2-1 \rhd$$

9.21-masala. $\int_{\frac{\pi}{2}}^{\pi}2^8\cdot\sin^2 x\cdot\cos^6 xdx$ **aniq integral hisoblansin.**

$\lhd$ Integralni $\sin 2\alpha=2\sin\alpha\cdot\cos\alpha,$ $\cos^2\alpha=\frac{1+\cos 2\alpha}{2}$ va $\sin^2\alpha=\frac{1-\cos 2\alpha}{2}$ formulalaridan foydalanib, hisoblaymiz:

$$I=\int_{\frac{\pi}{2}}^{\pi}2^8\sin^2 x\cdot\cos^6 xdx=2^6\int_{\frac{\pi}{2}}^{\pi}(2\sin x\cdot\cos x)^2\cdot\cos^4 xdx=2^4\int_{\frac{\pi}{2}}^{\pi}\sin^2 2x\cdot(1+\cos 2x)^2dx=$$

$$=2^4\int_{\frac{\pi}{2}}^{\pi}\left[\sin^2 2x+2\sin^2 2x\cdot\cos 2x+\sin^2 2x\cdot\cos^2 2x\right]dx=2^3\cdot\int_{\frac{\pi}{2}}^{\pi}(1-\cos 4x)dx+2^4\int_{\frac{\pi}{2}}^{\pi}\sin 2xd(\sin 2x)+$$

$$+2^2\int_{\frac{\pi}{2}}^{\pi}\sin^2 4xdx=2^3\left(x-\frac{1}{4}\sin 4x\right)\Big|_{\frac{\pi}{2}}^{\pi}+2^4\cdot\frac{\sin^3 2x}{3}\Big|_{\frac{\pi}{2}}^{\pi}+2\int_{\frac{\pi}{2}}^{\pi}(1-\cos 8x)dx=$$

$$=4\pi+2\left(x-\frac{1}{8}\sin 8x\right)\Big|_{\frac{\pi}{2}}^{\pi}=5\pi. \rhd$$

10.21-masala. $\int_0^2\frac{dx}{(4+x^2)\sqrt{4+x^2}}$ **aniq integral hisoblansin.**

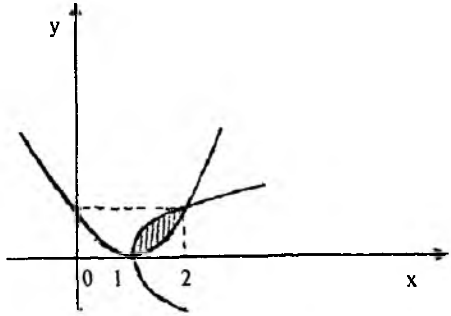
$\lhd$ Bu integralni hisoblashda uchun 2-hol uchun Eyler almashtirishini bajaramiz, ya'ni $\sqrt{x^2+4}=t+x$ almashtirish bajaramiz.

$$\Rightarrow x=\frac{4-t^2}{2t}\Rightarrow dx=-\frac{t^2+4}{2t^2}dt\Rightarrow\int_0^2\frac{dx}{(4+x^2)\sqrt{4+x^2}}=\int_{2(\sqrt{2}-1)}^{2}\frac{4tdt}{(t^2+4)^2}=2\int_{2(\sqrt{2}-1)}^{2}\frac{d(t^2+4)}{(t^2+4)^2}=$$

$$=-\frac{2}{t^2+4}\Big|_{2(\sqrt{2}-1)}^{2}=\frac{1}{4\sqrt{2}}. \rhd$$

11.21-masala. Quyidagi $y=(x-1)^2, y^2=x-1$ chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.

◁ $\begin{cases} y=(x-1)^2 \\ y^2=x-1 \end{cases}$ sistemani yechib, bu chiziqlarning kesishish nuqtalarini topamiz: $M_1(1;0)$ va $M_2(2;1)$ Bu chiziqlar bilan chegaralangan soha **5-chizmada tasvirlangan.**



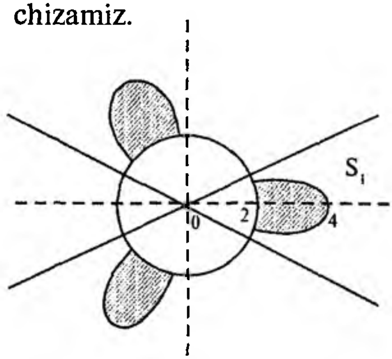
5-chizma.

$$S=\int_1^2\left[\sqrt{x-1}-(x-1)^2\right]dx=\left[\frac{2}{3}\sqrt{(x-1)^3}-\frac{(x-1)^3}{3}\right]\Big|_1^2=\frac{1}{3}.\ \triangleright$$

12.21-masala. Tenglamalari qutb koordinatalari sistemasida berilgan chiziqlar bilan chegaralangan shakilning yuzasi hisoblansin.

$$r=4\cos 3\varphi;\quad r=2,(r\geq 2)$$

◁ Birinchi navbatda bu funksiyaning aniqlanish sohasini topamiz va uning chizmasini chizamiz.



6-chizma.

$$D(r)=\left[0,\frac{\pi}{6}\right]\cup\left[\frac{\pi}{2},\frac{5\pi}{6}\right]\cup\left[\frac{7\pi}{6},\frac{3\pi}{2}\right]\cup\left[\frac{11\pi}{6},2\pi\right]$$

Yuzasini hisoblashimiz kerak bo‘lgan soha 6-chizmada shtrixlab ko‘rsatilgan. Integrallash chegarasini topishimiz uchun $\begin{cases} r=4\cos 3\varphi \\ r=2 \end{cases}$ sistemasidan φ ni topamiz

$$\Rightarrow \cos 3\varphi=\frac{1}{2}\Rightarrow 3\varphi=\frac{\pi}{3}\Rightarrow \varphi=\frac{\pi}{9}\Rightarrow$$

$$S=6S_1=6\cdot\frac{1}{2}\int_0^{\frac{\pi}{9}}\left[(4\cos 3\varphi)^2-2^2\right]d\varphi=3\cdot\left[16\int_0^{\pi/9}\cos^2 3\varphi d\varphi-4\int_0^{\pi/9}d\varphi\right]=$$

$$=3\left[8\cdot\int_0^{\pi/9}(1+\cos 6\varphi)d\varphi-4\varphi\Big|_0^{\frac{\pi}{9}}\right]=3\left[8\left(\varphi+\frac{1}{6}\sin 6\varphi\right)\Big|_0^{\frac{\pi}{9}}-\frac{4\pi}{9}\right]=$$

$$=3\left[\frac{8\pi}{9}+\frac{4}{3}\sin\frac{2\pi}{3}-\frac{4\pi}{9}\right]=3\left[\frac{4\pi}{9}+\frac{2\sqrt{3}}{3}\right]=\frac{4\pi}{3}+2\sqrt{3}.\triangleright$$

13.21-masala. Parametrik ko‘rinishda berilgan egri chiziq yoyining uzunligi hisoblansin.

$$\triangleleft\ x=8(\cos t+t\sin t);\quad y=8(\sin t-t\cos t);\quad 0\le t\le\frac{\pi}{4}.$$

$$\left.\begin{aligned} x'(t)&=8(-\sin t+\sin t+t\cos t=8t\cos t)\\ y'(t)&=8(\cos t-\cos t+t\sin t)=8t\sin t \end{aligned}\right\}\Rightarrow\sqrt{(x'(t))^2+(y'(t))^2}=8\cdot|t|\Rightarrow$$

$$l=\int_0^{\pi/4}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt=\int_0^{\pi/4}8tdt=4t^2\Big|_0^{\frac{\pi}{4}}=\frac{\pi^2}{4}.\triangleright$$

14.21-masala. Quyidagi $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **sirt bilan chegaralangan jismning hajmi topilsin.**

$\triangleleft$ Hajimni (18)-formulaga ko‘ra

$$V=\int_{x_1}^{x_2} S(x)dx$$

formula yordamida hisoblaymiz. Buning uchun $S(x)$ ni topish lozim.

O'zgaruvchi x ni fiksirlasak, ellipsoid kesimida.

$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$ yoki $\frac{y^2}{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2}+\frac{z^2}{\left(c\sqrt{1-\frac{x^2}{a^2}}\right)^2}=1$ ellips hosil

bo'ladi. Bizga ma'lumki, $\frac{y^2}{m^2}+\frac{z^2}{n^2}=1$ ellipsning yuzasi πmn ga teng edi. $\Rightarrow$

$$S(x)=\pi\cdot b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}=\pi bc\cdot\left(1-\frac{x^2}{a^2}\right)\Rightarrow V=\int_{-a}^{a}S(x)dx=$$

$$=\pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\pi bc\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}=\frac{4}{3}\pi abc.\triangleright$$

Natija. Agar $a=b=c=R$ bo'lsa ellipsoid sharga aylanadi va shar hajmini hisoblash ucun

$$V=\frac{4}{3}\pi R^3;$$

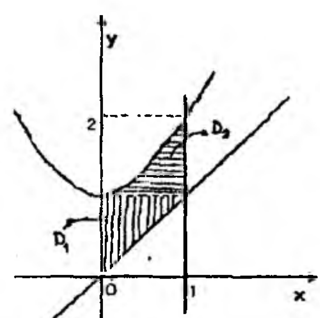
formulani hosil qilamiz.

15.21-masala. Quyidagi

$$y=x^2+1,\ y=x,\ x=0,\ x=1$$

chiziqlar bilan chegaralangan shaklni OY o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi topilsin.

$\triangleleft$Avval OY o'qi atrofida aylantirish kerak bo'lgan D sohani chizib olamiz (7-chizma).



7-chizma.

$$D = D_1 \cup D_2 \Rightarrow V = V_1 + V_2 = \pi \cdot \left[\int_0^1 y^2 dy + \int_1^2 (1-(y-1))dy \right] =$$

$$= \pi \left[\frac{y^3}{3} \Big|_0^1 + \left(2y - \frac{y^2}{2} \right) \Big|_1^2 \right] = \pi \left(\frac{1}{3} + 3 - \frac{3}{2} \right) = \frac{11\pi}{6}. \triangleright$$

16.21-masala. Quyidagi

$$x^2 + 4y - 16 = 0, y = 0$$

chiziqlar bilan chegaralangan shakilning og'irlik markazi topilsin.

$\triangleleft$ Masala shartidan ko'rinadiki, berilgan chiziqlar bilan chegaralangan D sohani ushbu

$$D = \begin{cases} -4 \le x \le 4 \\ 0 \le y \le 4 - \dfrac{x^2}{4} \end{cases}$$

ko'rinishda ifodalash mumkin. Bu shaklning og'irlik markazining koordinatalarini (23) va (24)-formulalardan foydalanib, topamiz.

$$S = \int_{-4}^{4} \left(4 - \frac{x^2}{4} \right) dx = \left(4x - \frac{x^3}{12} \right)\Big|_{-4}^{4} = \frac{64}{3}.$$

$$M_x = \frac{1}{2} \int_{-4}^{4} y^2 dx = \frac{1}{2} \int_{-4}^{4} \left(4 - \frac{x^2}{4} \right)^2 dx = \frac{1}{2} \int_{-4}^{4} \left(16 - 2x^2 + \frac{x^4}{16} \right) dx = \frac{1}{2} \left(16 - \frac{2x^3}{3} + \frac{x^5}{5 \cdot 16} \right)\Big|_{-4}^{4} = \frac{512}{15}.$$

$$M_y = \int_{-4}^{4} xy dx = \int_{-4}^{4} x \cdot \left(4 - \frac{x^2}{4} \right) dx = \int_{-4}^{4} \left(4x - \frac{x^3}{4} \right) dx = \left(2x^2 - \frac{x^4}{16} \right)\Big|_{-4}^{4} = 0.$$

Bu yerdan $\left(\bar{x}, \bar{y} \right) = \left(\dfrac{M_y}{S}, \dfrac{M_x}{S} \right) = \left(0, \dfrac{8}{5} \right)$ ekanligini topamiz. $\triangleright$

5-§. 4-MUSTAQIL ISH
Ko'p o'zgaruvchili funksiyalar
R^m fazo

R^m fazoda ketma-ketlik va uning limiti.
Ko'p o'zgaruvchili funksiyaning limiti va uzluksizligi.
Ko'p o'zgaruvchili funksiyaning hosila va differensiallari.
Yo'nalish bo'yicha hosila.
Ko'p o'zgaruvchili funksiyalarning ekstremumlari.
Shartli ekstremum.
O'zgaruvchilarni almashtirish.

-A-
Asosiy tushuncha va teoremalar

10. R^m fazoda ketma-ketlik va uning limiti

Ushbu

$$R^m = \underbrace{R \times R \times \ldots R}_{m-ta} = \left\{(x_1, x_2, \ldots, x_m) : x_k \in R, k = \overline{1,m}\right\}$$

to'plamga **m o'lchovli Yevklid fazosi** deyiladi.

Ixtiyoriy $x = (x_1, \ldots, x_m) \in R^m$ va $y = (y_1, \ldots y_m) \in R^m$ nuqtalarni olaylik. Quyidagi

$$\rho(x, y) = \sqrt{(y_1 - x_1)^2 + \ldots + (y_m - x_m)^2} = \sqrt{\sum_{k=1}^{m}(y_k - x_k)^2} \qquad (1)$$

miqdor x va y nuqtalar orasidagi **masofa** deb ataladi.

U quyidagi xossalarga ega:

1) $\rho(x, y) \geq 0$ va $(\rho(x, y) = 0 \Leftrightarrow x = y)$;

2) $\rho(x, y) = \rho(y, x)$;

3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$; (uchburchak tengsizligi).

Natural sonlar to'plami N va R^m fazo berilgan bo'lib, f har bir $n \in N$ ga R^m fazoning biror $x^{(n)} = \left(x_1^{(n)}, \ldots, x_m^{(n)}\right) \in R^m$ nuqtasini mos qo'yuvchi akslantirish bo'lsin:

$$f : N \to R^m \text{ yoki } n \to x^{(n)};$$

$f:N\to R^m$ akslantirish obrazlaridan tuzilgan

$$x^{(1)},x^{(2)},...x^{(n)}.... \qquad (2)$$

to‘plam **ketma-ketlik** deb ataladi va u $\{x^{(n)}\}$ kabi belgilanadi.

Demak, (2)-ketma-ketlikning hadlari R^m fazo nuqtalaridan iborat. $\{x^{(n)}\}$ ketma-ketlikning mos koordinatalaridan tuzilgan $\{x_1^{(n)}\},...,\{x_m^{(n)}\}$ lar sonli ketma-ketlik bo‘lib, $\{x^{(n)}\}$ ketma-ketlikni shu m ta ketma-ketlikning birgalikda qaralishi deb hisoblash mumkin.

Aytaylik R^m fazoda $\{x^{(n)}\}$ ketma-ketlik va $a=(a_1,...,a_m)\in R^m$ nuqta berilgan bo‘lsin.

1-ta’rif. $\forall\varepsilon>0, \exists n_0(\varepsilon)\in N:\forall n>n_0\Rightarrow \rho\left(x^{(n)},a\right)<\varepsilon$, unda a nuqta $\{x^{(n)}\}$ ketma-ketlikning **limiti** deb ataladi va $\lim\limits_{n\to\infty} x^{(n)}=a$ kabi belgilanadi.

Limitga quyidagicha ham ta’rif berish mumkin.

2-ta’rif. Agar a nuqtaning $\forall\bigcup_\delta(a)=\{x\in R^m:\rho(x,a)<\varepsilon\}$ atrofi olinganda ham $\exists n_0(\varepsilon)\in N:\forall n>n_0\Rightarrow x_n\in\bigcup_\delta(a)$, unda a nuqta $\{x^{(n)}\}$ ketma-ketlikning **limiti** deb ataladi.

Teorema. R^m fazoda $\{x^{(n)}\}=\left\{\left(x_1^{(n)},...,x_m^{(n)}\right)\right\}$ ketma-ketlikning $a=(a_1,...,a_m)\in R^m$ nuqtaga yaqinlashishi uchun $n\to\infty$ da bir yo‘la $x_1^{(n)}\to a_1,...,x_m^{(n)}\to a_m$ bo‘lishi zarur va yetarli.

R^m fazodagi ketma-ketlik uchun ham sonli ketma-ketlik uchun o‘rinli bo‘lgan xossalar o‘rinli. Biz ularga to‘xtalmaymiz.

2⁰. Ko‘p o‘zgaruvchili funksiya limiti va uning uzluksizligi

Ko‘p o‘zgaruvchili funksiya, funksiyaning aniqlanish sohasi va qiymatlar to‘plami, ko‘p o‘zgaruvchili murakkab funksiya ta’riflari bir o‘zgaruvchili funksiyadagi mos ta’riflar kabi kiritiladi.

$M\subset R^m$ to‘plam berilgan bo‘lib, $\underline{a}$ nuqta M to‘plamning limit nuqtasi va $y=f(x)=f(x_1,...,x_m)$ funksiya M to‘plamda aniqlangan bo‘lsin.

1-ta’rif. Agar $\forall\varepsilon>0$ uchun $\exists\delta=\delta(\varepsilon,a)>0$ ushbu $0<\rho(x,a)<\delta$ tengsizlikni qanoatlantiruvchi $\forall x\in M$ uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, unda b soni $f(x)$ **funksiyaning *a***

nuqtadagi limiti (yoki karrali limiti) deyiladi va

$$\lim_{x\to a} f(x)=b \quad \text{yoki} \quad \lim_{\substack{x_1\to a_1\\ \dots\dots\\ x_m\to a_m}} f(x)=b;$$

kabi belgilanadi.

Agar $a=\infty$ yoki $b=\infty$ bo'lsa , unda ham shu kabi ta'riflarni berish mumkin. Ko'p o'zgaruvchili funksiyalar uchun boshqa formadagi limit tushunchasini ham kiritish mumkin. Masalan, bunda avval bir o'zgaruvchi bo'yicha limitga o'tilib, qolgan $m-1$ ta o'zgaruvchini fiksirlangan deb faraz qilinadi. Keyin, qolgan $m-1$ ta o'zgaruvchining biri bo'yicha limitga o'tilib, $m-2$ ta o'zgaruvchini fiksirlangan deb faraz qilinadi. Bu jarayonni m marta qaytarish natijasida hosil qilingan limitga $f(x_1,...,x_m)$ funksiyaning **takroriy limiti deyiladi.** m o'zgaruvchili funksiyaning jami **m!** ta takroriy limitini qarash mumkin. Masalan, ikki o'zgaruvchili $f(x_1,x_2)$ funksiya uchun 2 ta $\lim\limits_{x_1\to a_1}\lim\limits_{x_2\to a_2} f(x_1,x_2)$ va $\lim\limits_{x_2\to a_2}\lim\limits_{x_1\to a_1} f(x_1,x_2)$ takroriy limitni ko'rish mumkin.

Misol. Ushbu

$$f(x,y)=\begin{cases} x+y\sin\frac{1}{x}, x\neq 0,\\ 0, x=0\end{cases}$$

funksiyaning (0,0) nuqtadagi takroriy va karrali limitlari hisoblansin.

$$\lhd \lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{x\to 0} x=0-\exists,$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0}\left(y\lim_{x\to 0}\sin\frac{1}{x}\right)-\nexists, \text{ lekin}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)-\exists \quad \text{va} \quad =0. \text{ Darhaqiqat,}$$

$$0\le |f(x,y)-0|=\left|x+y\sin\frac{1}{x}\right|\le|x|+|y|(x\neq 0)\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=0 \rhd$$

Tabiiy savol tug'iladi: **Qanday shartlar bajarilganda karrali va takroriy limitlar bir-biriga teng bo'ladi?**

Bu savolga quyidagi teoremalar javob beradi.

Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : |x-x_0|<a_1,\ |y-y_0|<a_2\}$$

to'plamda aniqlangan bo'lsin.

1-teorema. Agar

1) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=b-\exists,$

2) Har bir fiksirlangan x da $\lim\limits_{y\to y_0} f(x,y)=\varphi(x)-\exists$ bo'lsa, u holda $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ takroriy limit $\exists$ bo'lib, $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)=b$ bo'ladi.

2-teorema. Agar

1) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=b-\exists$

2) Har bir fiksirlangan y da $\lim\limits_{x\to x_0} f(x,y)=\varphi(y)-\exists$ bo'lsa, unda $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)-\exists$ va b ga teng bo'ladi.

Natija. Agar bir vaqtning o'zida 1 va 2-teoremalarning shartlari bajarilsa, unda

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) \text{ bo'ladi.}$$

Endi funksiyaning uzluksizligi ta'rifini beramiz.

$M\subset R^m$ to'plamda $f(x)=f(x_1,...,x_m)$ funksiya berilgan bo'lib, $a=(a_1,...,a_m)\in M$ nuqta M to'plamning limit nuqtasi bo'lsin. Quyidagi belgilashlarni kiritamiz:

$\Delta f(a)=f(a_1+\Delta x_1,...,a_m+\Delta x_m)-f(a_1,...,a_m)$ funksiyaning a nuqtadagi **to'liq orttirmasi;**

$\Delta x_k f(a)=f(a_1,...,a_{k-1},a_k+\Delta x_k,a_{k+1},...,a_m)-f(a_1,...,a_m)$ — funksiyaning a nuqtadagi x_k **o'zgaruvchi bo'yicha xususiy orttirmasi** $(k=\overline{1,m})$.

2-ta'rif. Agar $\lim\limits_{x\to a} f(x)=f(a)$ $\left(\text{yoki } \lim\limits_{\substack{\Delta x_1\to 0\\ \dots\dots\\ \Delta x_m\to 0}} \Delta f(a)=0\right)$ bo'lsa, unda $f(x)$ funksiya **a nuqtada uzluksiz** deb ataladi.

3-ta'rif. Agar $\lim\limits_{\Delta x_k \to 0} \Delta_{x_k} f(a) = 0$ bo'lsa, $f(x_1,...,x_m)$ funksiya $a=(a_1,...,a_m)$ nuqtada x_k **o'zgaruvchi bo'yicha uzluksiz** deb ataladi. Odatda funksiyaning bunday uzluksizligi uning **hususiy uzluksizligi** deb ataladi.

3-teorema. Agar $f(x_1,...,x_m)$ funksiya $a \in M$ nuqtada uzluksiz bo'lsa, funksiya shu nuqtada har bir o'zgaruvchisi bo'yicha ham hususiy uzluksiz bo'ladi.

Izoh: Teoremaning aksi har doim ham o'rinli bo'lavermaydi. Masalan,

$$f(x,y)=\begin{cases} \dfrac{2xy}{x^2+y^2}, x^2+y^2 \neq 0 \\ 0, x^2+y^2=0 \end{cases}$$

funksiya (0,0) nuqtada har bir o'zgaruvchi bo'yicha xususiy uzluksiz, lekin shu nuqtada bir yo'la uzluksiz emas, bu nuqtada hatto limitga ega emas.

4-ta'rif. *Agar* $\lim\limits_{x \to a} f(x) - \nexists$ *yoki* $= \infty$, *yoki* $\lim\limits_{x \to a} f(x) = b \neq f(a)$ *bo'lsa, u holda* $f(x)$ *funksiya **a nuqtada uzilishga ega** deyiladi.*

5-ta'rif. *Agar* $\forall\, \varepsilon > 0$ *uchun* $\exists \delta = \delta(\varepsilon) > 0$ M *to'plamning* $\rho(x', x'') < \delta$ *tengsizlikni qanoatlantiruvchi* $\forall x'$ *va* x'' *nuqtalarida* $|f(x'') - f(x')| < \varepsilon$ *tengsizlik bajarilsa,* $f(x)$ *funktsiya M **to'plamda tekis uzluksiz funksiya** deb ataladi.*

4-teorema. (Kantor teoremasi). *Agar* $f(x)$ *funksiya chegaralangan yopiq* $M \subset R^m$ *to'plamda uzluksiz bo'lsa, u holda funksiya shu to'plamda tekis uzluksiz bo'ladi.*

3⁰. Ko'p o'zgaruvchili funksiyaning hosila va differensiallari.

1-ta'rif. *Ushbu*

$$\lim_{\Delta x_k \to 0} \frac{\Delta_{x_k} f(x^0)}{\Delta x_k}, \ \left(k=\overline{1,m}\right);$$

limitga $f(x) = f(x_1,...,x_m)$ *funksiyaning* x^0 *nuqtadagi* x_k *o'zgaruvchi bo'yicha **xususiy hosilasi** deyiladi va u* $\dfrac{\partial f(x^0)}{\partial x_k}$ *kabi belgilanadi.*

Xususiy hosilaning geometrik ma'nosini bilish uchun $M \subset R^2$ to'plamda aniqlangan $z=f(x,y)$ funksiyani qaraymiz. Aytaylik $(x_0,y_0)\in M$ bo'lib, bu nuqtada $\frac{\partial f(x_0,y_0)}{\partial x}$ va $\frac{\partial f(x_0,y_0)}{\partial y}$ lar $\exists$ bo'lsin. $z=f(x,y)$ funksiya grafigi R^3 da biror sirtni aniqlaydi. $\Rightarrow$ $z=f(x,y_0)$ ning grafigi sirt bilan $y=y_0$ tekislikning kesishishida hosil bo'lgan Γ_1 chiziq bo'ladi. $z=f(x_0,y)$ ning grafigi Γ_2 chiziq bo'ladi. Agar Γ_1 va Γ_2 chiziqlarning $(x_0,y_0,f(x_0,y_0))$ nuqtasiga o'tkazilgan urinmaning Oxy tekisligi bilan hosil qilgan burchaklarini mos ravishda α va β deb belgilasak, unda

$$\frac{\partial f(x_0,y_0)}{\partial x}=tg\alpha \quad \text{va} \quad \frac{\partial f(x_0,y_0)}{\partial y}=tg\beta;$$

bo'ladi. Bundan $z=f(x,y)$ sirtning (x_0,y_0,z_0) nuqtasiga o'tkazilgan urinma tekislik tenglamasi ushbu

$$z-z_0=\frac{\partial f(x_0,y_0)}{\partial x}\cdot(x-x_0)+\frac{\partial f(x_0,y_0)}{\partial y}\cdot(y-y_0) \qquad (3)$$

ko'rinishda bo'lishi hosil qilamiz.

1-teorema. *Agar $f(x)$ funksiya x^0 nuqtada chekli $\frac{\partial f(x^0)}{\partial x_k},(k=\overline{1,m})$ xususiy hosilaga ega bo'lsa, unda $f(x)$ funksiya shu nuqtada mos x_k o'zgaruvchi bo'yicha xususiy uzluksiz bo'ladi.*

2-ta'rif. Agar $f(x)$ funksiya x^0 nuqtadagi $\Delta f(x^0)$ orttirmasini

$$\Delta f(x^0)=A_1\cdot\Delta x_1+\ldots+A_m\cdot\Delta x_m+\alpha_1\cdot\Delta x_1+\ldots+\alpha_m\cdot\Delta x_m \qquad (4)$$

ko'rinishda ifodalash mumkin bo'lsa, $f(x)$ funksiya x^0 **nuqtada differensiallanuvchi** deyiladi. Bu $A_1,\ldots,A_m$ lar $\Delta x_1,\ldots,\Delta x_m$ ga bog'liq bo'lmagan o'zgarmaslar va $\lim\limits_{\substack{\Delta x_1\to 0\\ \cdots\cdots \\ \Delta x_m\to 0}}\alpha_k=0,(k=\overline{1,m})$ tengliklar bajariladi.

(4)-tenglik ushbu

$$\Delta f(x_0)=A_1\cdot\Delta x_1+\ldots+A_m\cdot\Delta x_m+o(\rho) \qquad (5)$$

tenglikka ekvivalent. Bu yerda $\rho=\sqrt{(\Delta x_1)^2+...+(\Delta x_m)^2}$.

2-teorema. *Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsa, u holda bu funksiya shu nuqtada uzluksiz bo'ladi.*

3-teorema. *Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsa, unda bu funksiyaning shu nuqtadagi barcha hususiy hosilalari $\exists$ va $\frac{\partial f(x^0)}{\partial x_1}=A_1,..,\frac{\partial f(x^0)}{\partial x_m}=A_m$ tengliklar o'rinli bo'ladi.*

Izoh: Teoremaning aksi har doim ham o'rinli bo'lavermaydi, ya'ni barcha xususiy hosilalari $\exists$ bo'lgan funksiya differensiallanuvchi bo'lishi shart emas.

$$\text{Masalan,}\quad f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

funksiyaning (0,0) nuqtada hususiy hosilalari $\exists$, lekin u bu nuqtada differensiallanuvchi emas.

Demak, xususiy hosilalari $\exists$ bo'lishi funksiyaning differensiallanuvchi bo'lishi uchun zaruriy shart ekan.

4-teorema. (Yetarli shart). *Agar $f(x)$ funksiya x^0 nuqtaning biror atrofida barcha o'zgaruvchilari bo'yicha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar x^0 nuqtada uzluksiz bo'lsa, unda $f(x)$ funksiya shu x^0 nuqtada differensiallanuvchi bo'ladi.*

Ushbu

$$df(x^0)=\frac{\partial f(x^0)}{\partial x_1}dx_1+...+\frac{\partial f(x^0)}{\partial x_m}dx_m \quad \text{va}$$

$$d_{x_k}f(x_0)=\frac{\partial f(x^0)}{\partial x_k}dx_k,(k=\overline{1,m})$$

ifodalarga mos ravishda $f(x)$ funksiyaning x^0 nuqtadagi **differensiali (to'liq differensiali)** va x_k **o'zgaruvchi bo'yicha xususiy differensiali** deyiladi.

Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lib, $df(x^0)\neq 0$ bo'lsa

$$\Delta f(x^0)=df(x^0)+o(\rho) \text{ va } \lim_{\rho\to 0}\frac{\Delta f(x^0)}{df(x^0)}=1 \text{ bo'ladi } \Rightarrow \Delta f(x^0)\approx df(x^0)$$

yoki

$$f(x_1^0+\Delta x_1,\dots,x_m^0+\Delta x_m)\approx f(x_1^0,\dots,x_m^0)+\frac{df(x^0)}{dx_1}\Delta x_1+\dots+\frac{df(x^0)}{dx_m}\Delta x_m \quad (6)$$

bo'ladi (6)-formulaga **taqribiy hisoblash formulasi** deyiladi.

Endi yo'nalish bo'yicha hosila tushunchasini kiritamiz.

Ikki o'zgaruvchili $z=f(x,y)$ funksiya ochiq $M\subset R^2$ to'plamda berilgan bo'lsin. $\forall A_0(x_0,y_0)\in M$ nuqta olib, bu nuqtadan biror ℓ to'g'ri chiziq o'tkazaylik. Bu to'g'ri chiziqning OX va OY koordinata o'qlari bilan hosil qilgan burchaklari α va β bo'lsin.

3-ta'rif. *Agar A nuqta ℓ to'g'ri chiziq bo'ylab A_0 nuqtaga intilganda ushbu*

$$\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0,A)};$$

limit mavjud bo'lsa, uning qiymatiga $f(x,y)=f(A)$ funksiyaning $A_0=(x_0,y_0)$ nuqtadagi ℓ ***yo'nalish bo'yicha hosilasi*** *deyiladi va $\dfrac{\partial f(A_0)}{\partial \ell}$ yoki $\dfrac{\partial f(x_0,y_0)}{\partial \ell}$ kabi belgilanadi.*

Demak,

$$\frac{\partial f(A_0)}{\partial \ell}:=\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0,A)} \quad (7)$$

5-Teorema. *Agar $f(x,y)$ funksiya $A_0=(x_0,y_0)$ nuqtada differensiallanuvchi bo'lsa, u holda shu funksiya A_0 nuqtada $\forall\ell$ yo'nalish bo'yicha hosilaga ega va*

$$\frac{\partial f(A_0)}{\partial \ell}:=\frac{\partial f(x_0,y_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0)}{\partial y}\cos\beta \quad (8)$$

tenglik o'rinli.

Izoh: Funksiya biror nuqtada differensiallanuvchi bo'lmasa ham u shu nuqtada biror yo'nalish bo'yicha hosilaga ega bo'lishi mumkin.

Agar differensiallanuvchi $w=f(x,y,z)$ va $x=\varphi(u,v)$, $y=\psi(u,v)$, $z=\chi(u,v)$ funksiyalar berilgan bo'lib, ular yordamida $w=f\left[\varphi(u,v),\psi(u,v),\chi(u,v)\right]=F(u,v)$ murakkab funksiya aniqlangan bo'lsa, unda murakkab funksiya ham differensiallanuvchi bo'ladi va

$$\begin{cases}\dfrac{\partial w}{\partial u}=\dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial u}+\dfrac{\partial w}{\partial z}\cdot\dfrac{\partial z}{\partial u},\\[2mm] \dfrac{\partial w}{\partial v}=\dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial v}+\dfrac{\partial w}{\partial z}\cdot\dfrac{\partial z}{\partial v}.\end{cases} \tag{9}$$

tengliklar o'rinli bo'ladi.

Ko'p o'zgaruvchili funksiyaning ikkinchi tartibli xususiy hosilalari quyidagi tenglik yordamida aniqlanadi:

$$f''_{x_i x_k}=\frac{\partial^2 f}{\partial x_i \partial x_k}:=\frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_i}\right),\quad \left(i,k=\overline{1,m}\right) \tag{10}$$

Agar $i=k$ bo'lsa, $\dfrac{\partial^2 f}{\partial x_k \partial x_k}=\dfrac{\partial^2 f}{\partial x_k^2}=f''_{x_k^2}$ kabi yoziladi.

Agar $i\neq k$ bo'lsa $\dfrac{\partial^2 f}{\partial x_i \partial x_k}$ - **aralash hosila** deb ataladi.

Yuqori tartibli xususiy hosilalar ham shu kabi aniqlanadi.

M to'plamda 1,2,3,..., k-tartibli uzluksiz xususiy hosilalarga ega bo'lgan funksiyalar sinfi $C^{(k)}(M;R)$ yoki $C^{(k)}(M)$ kabi belgilanadi.

4-ta'rif. Agar $f(x)$ funksiyaning x nuqtadagi barcha ikkinchi tartibli xususiy hosilalari mavjud bo'lsa, unda **funksiyaning ikkinchi tartibli differensiali** quyidagi tenglik yordamida aniqlanadi:

$$d^2 f(x):=\sum_{i,k=1}^{m}\frac{\partial^2 f(x)}{\partial x_i \partial x_k}dx_i dx_k=\left(\frac{\partial}{\partial x_1}dx_1+\ldots+\frac{\partial}{\partial x_m}dx_m\right)^2 f(x)$$

Xuddi shunga o'xshash

$$d^n f:=d\left(d^{n-1}f\right)=\left(\frac{\partial}{\partial x_1}dx_1+\ldots+\frac{\partial}{\partial x_m}dx_m\right)^n f \tag{11}$$

bo'ladi.

***6-teorema*. (Teylor formulasi).** Agar x va $x+h$ nuqtalarning o'zi va ularni tutashtiruvchi kesma *M* to'plamga tegishli bo'lib, $f(x)\in C^{(n)}(M)$ bo'lsa, u holda **ushbu Peano ko'rinishidagi qoldiq hadli Teylor formulasi** o'rinli bo'ladi:

$$f(x_1+h_1,\dots,x_m+h_m)-f(x_1,\dots,x_m)=$$

$$=\sum_{k=1}^{m}\frac{1}{k!}\left(h_1\frac{\partial}{\partial x_1}+\dots+h_m\frac{\partial}{\partial x_m}\right)^k f(x)+o(h^n).$$

4⁰. Ko‘p o‘zgaruvchili funksiyaning ekstremumlari

$f(x)=f(x_1,\dots,x_m)$ funksiya ochiq $M\subset R^2$ to‘plamda berilgan bo‘lib, $x_0=(x_1^0,\dots,x_m^0)\in M$ bo‘lsin.

5-ta'rif. Agar x^0 nuqtaning $\exists\bigcup_\delta(x^0)\subset M$ atrofi topilsaki, $\forall x\in\bigcup_\delta(x^0)$ uchun $f(x)\le f(x^0)$ ($f(x)\ge f(x^0)$) bo‘lsa, $f(x)$ funksiya x^0 nuqtada **min (max)** ga ega deyiladi. $f(x^0)$ qiymat esa $f(x)$ funksiyaning lokal **(max) min** qiymati deyiladi va

$$f(x^0)=\max_{x\in\bigcup_\delta(x^0)}\{f(x)\}\quad\left(f(x^0)=\min_{x\in\bigcup_\delta(x^0)}\{f(x)\}\right);$$

kabi belgilanadi.

Funksiyaning max va min qiymatlari uning **ekstremumlari** deb ataladi.

x^0 nuqtaning $\bigcup_\delta(x^0)$ atrofida

$$\Delta=f(x)-f(x^0)\qquad(12)$$

ayirmani ko‘raylik.

Agar bu ayirma $\bigcup_\delta(x^0)$da o‘z ishorasini saqlasa ya’ni har doim $\Delta\ge0(\Delta\le0)$ bo‘lsa, $f(x)$ funksiya x^0 nuqtada **min (max)** ga erishadi. Agar Δ ayirma x^0 nuqtaning $\forall$ atrofida ham o‘z ishorasini saqlamasa, unda $f(x)$ funksiya x^0 nuqtada ekstremumga ega bo‘la olmaydi.

1-teorema. **(Zaruriy shart)** *$f(x)$ funksiya x^0 nuqtada ekstremumga erishsa va shu nuqtada $f'_{x_1}(x_0),\dots,f'_{x_m}(x^0)$ xususiy hosilalar $\exists$ bo‘lsa, unda*

$$f'_{x_1}(x_0)=\dots=f'_{x_m}(x^0)=0\qquad(13)$$

bo‘ladi.

1-izoh. Teoremaning aksi har doim ham o'rinli bo'lavermaydi. Masala, $f(x,y)=x\cdot y$ funksiya uchun $f'_x(0,0)=f'_y(0,0)=0$, lekin funksiya (0,0) nuqtada ekstremumga erishmaydi, chunki u (0,0) nuqtaning $\forall$ atrofida har hil ishorali qiymatlarni qabul qiladi.

2-izoh. Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsa, u holda funksiyaning ekstremumga erishishining zaruriy shartini $df(x^0)=0$ ko'rinishda yozish mumkin.

2-teorema. **(Yetarli shart.)** *$f(x)$ funksiya x^0 nuqtaning biror $\bigcup_\delta(x^0)$ atrofida berilgan bo'lib quyidagi shartlarni bajarsin:*

1) $f(x)$ funksiya $\bigcup_\delta(x^0)$ da uzluksiz birinchi va ikkinchi tartibli xususiy hosilalarga ega;

2) x^0 nuqta $f(x)$ funksiyaning statsionar nuqtasi;

3) koeffitsientlari $a_{ik}=f''_{x_i x_k}(x^0)$, $(i,k=\overline{1,m})$ bo'lgan.

$$Q(\xi_1,\dots,\xi_m)=\sum_{i,k=1}^{m} a_{ik}\xi_i\xi_k \qquad (14).$$

kvadratik forma **musbat (manfiy)** aniqlangan.

U holda $f(x)$ funksiya x^0 nuqtada **min (max)** ga erishadi. Agar kvadratik forma noaniq bo'lsa, unda $f(x)$ funksiya x^0 nuqtada ekstremumga erishmaydi.

Bu teoremani $m=2$ bo'lgan holda alohida ko'ramiz:

$$a_{11}=\frac{\partial^2 f(x^0)}{\partial x_1^2} \quad a_{12}=\frac{\partial^2 f(x^0)}{\partial x_1 dx_2} \quad a_{22}=\frac{\partial^2 f(x^0)}{\partial x_2^2}$$

$$\Delta=\begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}^2$$ bo'lsin. Unda

1) $\Delta>0, a_{11}>0$ bo'lsa, **min**;

2) $\Delta>0, a_{11}<0$ bo'lsa, **max**;

3) $\Delta<0$ bo'lsa, **ekstremum mavjud emas.**

4) $\Delta=0$ bo'lsa, **shubhali hol** bo'ladi.

Biz shu vaqtgacha hech qanday shart berilmaganda $y=f(x_1,x_2,\dots,x_m)$ funksiya ekstremumini topish masalasi bilan shug'ullandik. Lekin matematikaning ko'p tatbiqlarida funksiyaning argumentlari ba'zi bir shartlarni qanoatlantirgandagi ekstremumlarini topish talab qilinadi. Biz shunday masalani eng sodda hol uchun keltiramiz.

Aytaylik,

$$u = f(x, y) \tag{15}$$

funksiyaning

$$F(x, y) = 0 \tag{16}$$

shartni qanoatlantiruvchi ekstremumini topish talab qilinsin. Bunday ekstremumga **shartli ekstremum** deyiladi.

Agar (16)-tenglamadan $y = \varphi(x)$ funksiyani topish mumkin bo'lsa, u holda shartli ekstremumni topish masalasi

$$u = f[x, \varphi(x)] = \Phi(x) \tag{17}$$

funksiyaning oddiy ekstremumini topish masalasiga keladi. Lekin har doim ham $y = \varphi(x)$ funksiyani topish imkoni yo'q. Shuning uchun (16)-tenglamani yechmay turib shartli ekstremumni topishni o'rganamiz. Bunda Lagranj usuli yaxshi natijaga olib keladi.

Ushbu

$$\Phi(x, y) = f(x, y) + \lambda F(x, y) \tag{18}$$

Lagranj funksiyasini olamiz. (18) dagi λ hozircha noma'lum o'zgarmas ko'paytuvchi.

$\Phi(x, y)$ funksiyaning oddiy ekstremumi $f(x, y)$ funksiyaning $F(x, y) = 0$ tenglamani qanoatlantiruvchi shartli ekstremumi bilan ustma-ust tushadi. $\Phi(x, y)$ funksiyaning statsionar nuqtasi va noma'lum koeffitsient λ quyidagi

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = 0 \\ \dfrac{\partial \Phi}{\partial y} = 0 \\ F(x, y) = 0 \end{cases} \tag{19}$$

shartlardan topiladi. Faraz qilaylik, $M_0(x_0, y_0)$ nuqta $\Phi(x, y)$ funksiyaning statsionar nuqtasi bo'lsin. Agar $d^2\Phi \,|_{M_0} > 0$ bo'lsa **min** va $d^2\Phi \,|_{M_0} > 0$ bo'lsa **max** bo'ladi. Bu yerda $d^2\Phi = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 \Phi$

O‘zgaruvchilari soni ko‘p bo‘lgan funksiyalar qaralganda shartli ekstremum shu kabi aniqlanadi va Lagranj funksiyasi yordamida topiladi.

5⁰. O‘zgaruvchilarni almashtirish

a) Oddiy hosilani o‘zida saqlovchi ifodalarda o‘zgaruvchilarni almashtirish

Aytaylik, $y=y(x)$ funksiya va

$$A=\Phi\left(x,y,y'_x,y''_{xx},\ldots\right) \tag{20}$$

differensial ifoda berilgan bo‘lib,

$$x=f(t,u),\quad y=g(t,u) \tag{21}$$

va $u=u(t)$ bo‘lsin. Differensial ifodada yangi t o‘zgaruvchiga o‘tish talab qilinsin. Unda (21) ga ko‘ra

$$y'_x=\frac{y'_t}{x'_t}=\frac{\dfrac{\partial g}{\partial t}+\dfrac{\partial g}{\partial u}\cdot u'_t}{\dfrac{\partial f}{\partial t}+\dfrac{\partial f}{\partial u}\cdot u'_t}$$

ekanligini topamiz. Shunga o‘xshash yuqori tartibli $y''_{xx},\ldots$ hosilalar ham topiladi va (20) ga olib borib qo‘yib, yangi

$$A=\Phi_1\left(t,u,u'_t,u''_{tt},\ldots\right)$$

differensial ifoda hosil qilinadi.

b) Xususiy hosilani o‘zida saqlovchi ifodalarda o‘zgaruvchilarni almashtirish.

Faraz qilaylik, $z=z(x,y)$ funksiya va

$$B=F\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2},\ldots\right) \tag{22}$$

differensial ifoda berilgan bo‘lib,

$$x=f(u,v)\quad y=g(u,v) \tag{23}$$

bo‘lsin. Bu yerda u va v lar yangi erkli o‘zgaruvchilar. Unda $\frac{dz}{dx},\frac{dz}{dy},\ldots$ hususiy hosilalar ushbu

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial f}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial g}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial f}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial g}{\partial v},$$

....

engliklardan topiladi va ular (22) ga olib borib qo‘yib, yangi

$$B=F_1\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v},\frac{\partial^2 z}{\partial u^2},\frac{\partial^2 z}{\partial u\partial v},\frac{\partial^2 z}{\partial v^2},\ldots\right)$$

ifferensial ifoda hosil qilinadi.

Umumiy holda (22) ifodada ushbu

$$x=f(u,v,w),\quad y=g(u,v,w,),\quad z=h(u,v,w) \tag{24}$$

lmashtirish bajarilgan bo‘lib, u,v lar yangi erkli o‘zgaruvchilar va $v=w(u,v)$ yangi funksiya bo‘lsin. Unda $\frac{\partial z}{\partial x};\frac{\partial z}{\partial y},\ldots$ xususiy hosila-rni topish uchun

$$\frac{\partial z}{\partial x}\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial u}\right)+\frac{\partial z}{\partial}\left(\frac{\partial g}{\partial u}+\frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial u}\right)=\frac{\partial h}{\partial u}+\frac{\partial h}{\partial w}\cdot\frac{\partial w}{\partial u},$$

$$\frac{\partial z}{\partial x}\left(\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial v}\right)+\frac{\partial z}{\partial y}\left(\frac{\partial g}{\partial v}+\frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial v}\right)=\frac{\partial h}{\partial v}+\frac{\partial h}{\partial w}\cdot\frac{\partial w}{\partial v},$$

....

nglamalar hosil qilinadi. Bu tenglamalar yordamida xususiy hosila-r topiladi va (22) ga olib borib qo‘yib, yangi

$$B=F_2\left(u,v,w,\frac{\partial w}{\partial u},\frac{\partial w}{\partial v},\frac{\partial^2 w}{\partial u^2},\frac{\partial^2 w}{\partial u\partial v},\frac{\partial^2 w}{\partial v^2},\ldots\right)$$

ifferensial ifoda hosil qilinadi.

Nazorat savollari

1. R^m fazo.
2. R^m fazoda metrika.
3. R^m fazoda ketma-ketlik tushunchasi va uning limiti.
4. Ko‘p o‘zgaruvchili funksiya (k. o‘. f.) tushunchasi.
5. K.o‘.f. ning karrali limiti tushunchasi.
6. K.o‘.f. ning takroriy limiti tushunchasi.
7. Karrali va takroriy limitlar orasidagi bog‘lanish.

8. Karrali va takroriy limitlarning tengligi haqidagi teorema.
9. K.o'.f. ning uzluksizligi.
10. K.o'.f. ning tekis uzluksizligi va Kantor teoremasi.
11. K.o'.f. ning xususiy hosilasi ta'rifi.
12. Urinma tekislik tenglamasi.
13. K.o'.f. ning differensiallanuvchiligi.
14. K.o'.f. differensiallanuvchi va uzluksiz funksiyalar orasidagi bog'lanish.
15. Taqribiy hisoblash formulasi.
16. Yo'nalish bo'yicha hosila.
17. K.o'.f. uchun Teylor formulasi.
18. K.o'.f. ning ekstremumlari.
19. Shartli ekstremum, Lagranj usuli.
20. Oddiy hosilani o'zida saqlovchi ifodalarda o'zgaruvchilarni almashtirish.
21. Xususiy hosilani o'zida saqlovchi ifodalarda o'zgaruchilarni almashtirish.

-B-

Mustaqil yechish uchun misol va masalalar

1-masala.

R^2 fazoda quyidagi ketma-ketliklarning limiti $a\left(a\in R^2\right)$ ekanligi ta'rif yordamida isbotlansin.

1.1 $x^{(n)}=\left(\frac{13-n^2}{1+2n^2},\frac{2n-1}{2-3n}\right)$; $a\left(-\frac{1}{2};-\frac{2}{3}\right)$. **1.2** $x^{(n)}=\left(\frac{3n^2+2}{4n^2-1},\frac{2n^3}{n^3-2}\right)$; $a\left(\frac{3}{4};2\right)$.

1.3 $x^{(n)}=\left(\frac{1-2n^2}{n^2+3},\frac{3n^2}{2-n^2}\right)$; $a(-2;-3)$. **1.4** $x^{(n)}=\left(\frac{4+2n}{1-3n},\frac{5n+15}{6-n}\right)$; $a\left(-\frac{2}{3};-5\right)$.

1.5 $x^{(n)}=\left(\frac{4n^2+1}{3n^2+2},\frac{4-n^3}{3+2n^3}\right)$; $a\left(\frac{4}{3};-\frac{1}{2}\right)$. **1.6** $x^{(n)}=\left(\frac{1-2n^2}{2+4n^2};-\frac{5n}{n+1}\right)$; $a\left(-\frac{1}{2};-5\right)$.

1.7 $x^{(n)}=\left(\frac{1}{n};\frac{2}{n}\cos n\pi\right)$; $a(0,0)$. **1.8** $x^{(n)}=\left(\frac{3n-2}{2n-1};\frac{4n-1}{2n+1}\right)$; $a\left(\frac{3}{2};2\right)$.

1.9 $x^{(n)}=\left(\frac{2n}{3n+1},\frac{1+n}{1-2n}\right)$; $a\left(\frac{2}{3};-\frac{1}{2}\right)$. **1.10** $x^{(n)}=\left(\frac{\cos n}{n},\frac{n-1}{n^2+1}\right)$; $a(0,0)$.

1.11 $x^{(n)}=\left(\frac{1}{n^2};\frac{5}{n}\right)$; $a(0,0)$. **1.12** $x^{(n)}=\left(\frac{2}{n},\frac{n}{n+1}\right)$; $a(0,1)$.

R^2 fazoda quyidagi ketma-ketliklarning limiti topilsin.

1.13. $x^{(n)}=\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2};\left(\frac{2n^2+2}{2n^2+1}\right)^{n^2}\right).$

1.14 $x^{(n)}=\left(\frac{(2n+1)!+(2n+2)!}{(2n+3)!};\left(\frac{n-1}{n+3}\right)^{n+2}\right).$

1.15. $x^{(n)}=\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n};\left(\frac{n^2-1}{n^2}\right)^{n^4}\right).$

1.16. $x^{(n)}=\left(\frac{1+2+\ldots+n}{\sqrt{9n^4+1}};\left(\frac{2n+3}{2n+1}\right)^{n+1}\right).$

1.17. $x^{(n)}=\left(\frac{1+4+7+\ldots+(3n-2)}{\sqrt{5n^4+n+1}};\left(\frac{n+1}{n-1}\right)^{n}\right).$

1.18. $x^{(n)}=\left(\frac{(n+4)!-(n+2)!}{(n+3)!};\left(\frac{n+3}{n+5}\right)^{n+4}\right).$

1.19 $x^{(n)}=\left(\frac{\sqrt[3]{n^3+5}-\sqrt{3n^4+2}}{1+3+5+\ldots+(2n-1)};\left(\frac{n^3+1}{n^3-1}\right)^{2n-n^3}\right)$

1.20 $x^{(n)}=\left(\frac{3}{4}+\frac{5}{16}+\frac{9}{64}+\ldots+\frac{1+2^n}{4^n},n\left(\sqrt[3]{5+8n^3}-2n\right)\right).$

1.21 $x^{(n)}=\left(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots+\frac{1}{n(n+1)};\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot\ldots\cdot\sqrt[2^n]{2}\right).$

2-masala. Quyidagi funksiyalarning aniqlanish sohalari topilsin va chizmada ko'rsatilsin.

2.1. $u=\arccos\frac{x}{x+y}.$ **2.2.** $u=\ln(xyz).$

2.3. $u=\ln(-x-y).$ **2.4.** $u=\arcsin\frac{y}{x}.$

2.5. $u=\sqrt{\sin\left(x^2+y^2\right)}$. **2.6.** $u=\sqrt{1-\left(x^2+y\right)^2}$.

2.7. $u=\sqrt{\dfrac{x^2+y^2-x}{2x-x^2-y^2}}$. **2.8.** $u=\sqrt{\left(x^2+y^2-1\right)\cdot\left(4-x^2-y^2\right)}$.

2.9. $u=\sqrt{1-x^2}+\sqrt{y^2-1}$. **2.10.** $u=\sqrt{\dfrac{x^2+2x+y^2}{x^2-2x+y^2}}$.

2.11. $u=\arccos\dfrac{z}{\sqrt{x^2+y^2}}$. **2.12.** $u=xy+\sqrt{\ln\dfrac{9}{x^2+y^2}}+\sqrt{x^2+y^2-9}$.

2.13. $u=\dfrac{\sqrt{4x-y^2}}{\ln\left(1-x^2-y^2\right)}$. **2.14.** $u=\ln\left(\dfrac{x^2}{9}-\dfrac{y^2}{4}-1\right)$.

2.15. $u=arctg\dfrac{x-y}{1+x^2y^2}$. **2.16.** $u=1+\sqrt{-(x-y)^2}$.

2.17. $u=\sqrt{x-\sqrt{y}}$. **2.18.** $u=\sqrt{y\sin x}$.

2.19. $u=\sqrt{x\cos y}$. **2.20.** $u=\arccos\dfrac{x^2+y^2}{9}$.

2.21. $u=\arcsin\dfrac{x}{y^2}+\arcsin(1-y)$.

3-masala. Karrali limitlar hisoblansin.

3.1 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin xy}{\sqrt{x^2+y^2}}$. **3.2** $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{x+y}{x^2-xy+y^2}$.

3.3 $\lim\limits_{\substack{x\to\infty\\ y\to 3}}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}$. **3.4** $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}\left(\dfrac{xy}{x^2+y^2}\right)^{x^2}$.

3.5 $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{x^2+y^2}{x^4+y^4}$. **3.6** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{x^2y^2}$.

3.7 $\lim\limits_{\substack{x\to\infty\\ y\to 0}}\left(1+\frac{2}{x}\right)^{\frac{x^2}{x+y}}$.   **3.8** $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}}\left(x^2+y^2\right)e^{-(x+y)}$.

3.9 $\lim\limits_{\substack{x\to 1\\ y\to 0}}\frac{\ln\left(x+e^y\right)}{\sqrt{x^2+y^2}}$.   **3.10** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sin\left(x^4y^2\right)}{\left(x^2+y^2\right)^2}$.

3.11 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{e^{-\frac{1}{x^4+y^4}}}{x^4+y^4}$.   **3.12** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(1+x^2y^2\right)^{\frac{1}{x^2+y^2}}$.

3.13 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(1+x^2+y^2\right)^{\frac{1}{x^2+y^2}}$.   **3.14** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt[3]{x^4y^2}}{x^2+y^2}$.

3.15 $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\left(x^2+y^2\right)\sin\frac{1}{x^2+y^2}$.   **3.16** $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\left(x^2+y^2\right)\ln\left(1+\sin\frac{1}{x^2+y^2}\right)$.

3.17 $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}(x+y)e^{-\left(x^2+y^2\right)}$.   **3.18** $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{x^2+y^2}{|x|^3+|y|^3}$.

3.19 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{|x|}$.   **3.20** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos\left(x^2y^2\right)}{\left(x^2+y^2\right)^2}$.

3.21 $\lim\limits_{\substack{x\to 1\\ y\to 0}}\frac{\ln^2(x+y)}{\sqrt{x^2+y^2-2x+1}}$.

4-masala. $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0}f(x,y)$ **va** $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0}f(x,y)$ **takroriy limitlar hisoblansin.**

4.1 $f(x,y)=\sin\frac{\pi x}{2x+y};x_0=\infty,y_0=\infty$.   **4.2** $f(x,y)=\frac{x^2+xy+y^2}{x^2-xy+y^2};x_0=y_0=0$.

4.3 $f(x,y)=\log_x(x+y);x_0=1,y_0=0$.   **4.4** $f(x,y)=\frac{\sin(x+y)}{2x+3y};x_0=y_0=0$.

4.5 $f(x,y)=\frac{\cos x-\cos y}{x^2+y^2};\ x_0=y_0=0$.   **4.6** $f(x,y)=\frac{\sin|x|-\sin|y|}{\sqrt{x^2+y^2}};x_0=y_0=0$.

4.7 $f(x,y)=\dfrac{\sin 3x-tg2y}{6x+3y};x_0=y_0=0$. 4.8 $f(x,y)=\dfrac{x^2+y^2}{x^2+y^4}x_0=y_0=\infty$.

4.9 $f(x,y)=\dfrac{x^y}{1+x^y};x_0=+\infty,y_0=+0$. 4.10 $f(x,y)=\dfrac{1}{xy}tg\dfrac{xy}{1+xy};x_0=0,y_0=\infty$.

4.11 $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^4+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0).x_0=y_0=0\end{cases}$. 4.12 $f(x,y)=\begin{cases}\left(1+\dfrac{1}{x+y}\right)^{x+y},x+y\neq0\\ 1,x+y=0x_0=y_0=\infty\end{cases}$.

4.13 $f(x,y)=\begin{cases}\dfrac{x-y+x^2+y^2}{x+y},x\neq-y\\ 0,x=-yx_0=y_0=0\end{cases}$ 4.14 $f(x,y)=\dfrac{\sin x+\sin y}{x+y},x_0=y_0=0$.

4.15 $f(x,y)=\dfrac{x^2\sin\frac{1}{x}+y}{x+y},x_0=y_0=0$. 4.16 $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{|x|-|y|},|x|\neq|y|\\ 0,|x|=|y|,x_0=y_0=0\end{cases}$

4.17 $f(x,y)=\dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}};x_0=1,y_0=0$. 4.18 $f(x,y)=\dfrac{5-\sqrt{25-xy}}{xy};x_0=y_0=0$

4.19 $f(x,y)=\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2};x_0=y_0=0$. 4.20 $f(x,y)=\dfrac{\ln(x+y)}{y};x_0=1,y_0=0$.

4.21 $f(x,y)=\sin\dfrac{\pi(x+y)}{2x+3y};x_0=y_0=\infty$.

5-masala.

5.1 $f(x,y)=\dfrac{x^2}{|x|+|y|}$ funksiya $O(0,0)$ nuqtada cheksiz kichik bo'lishi isbotlansin.

5.2 $f(x,y)=\sin(x+y)\cdot\ln(x^2+y^2)$ funksiya $O(0,0)$ nuqtada cheksiz kichik bo'lishi isbotlansin.

5.3 $f(x,y)=\dfrac{x^2y}{x^4+y^2}$ funksiya quyidagi hossalarga ega ekanligi isbotlansin.

a) $M(x,y)$ nuqta $O(0,0)$ nuqtaga shu $O(0,0)$ nuqtadan o'tuvchi $\forall$ to'g'ri chiziq bo'ylab intilganda ham funksiya limiti 0 ga teng. b) $O(0,0)$ nuqtada funksiya limiti mavjud emas. (x_0,y_0) **nuqtada** $f(x,y)$ **funksiyaning karrali va takroriy limitlari mavjudmi?**

5.4 $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2};x_0=y_0=0$. **5.5** $f(x,y)=\log_x(x+y);x_0=1,y_0=0$.

5.6 $f(x,y)=\dfrac{\sin x+\sin y}{x+y},x_0=y_0=0$. **5.7** $f(x,y)=\dfrac{x-y}{x+y};x_0=y_0=0$.

5.8 $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2},x_0=y_0=0$. **5.9** $f(x,y)=(x+y)\sin\dfrac{1}{x}\cdot\sin\dfrac{1}{y};\ x_0=y_0=0$.

5.10 $f(x,y)=\dfrac{2xy}{x^2+y^2};x_0=y_0=0$.

5.11 $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^4+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0).\ x_0=y_0=0\end{cases}$

5.12 $f(x,y)=\begin{cases}\left(1+\dfrac{1}{x+y}\right)^{x+y},x+y\neq 0\\ 1,x+y=0.x_0=y_0=\infty\end{cases}$

Quyidagi funksiyalarni berilgan nuqtalarda har bir o'zgaruvchi bo'yicha xususiy va ikkala o'zgaruvchi bo'yicha birgalikda uzluksizlikka tekshiring.

5.13 $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^4+y^4},x^4+y^4\neq 0,\\ 0,x^4+y^4=0\end{cases}$ $0(0,0)$ va $A(1;2)$.

5.14 $f(x,y)=\begin{cases}\dfrac{x^3y^2}{x^4+y^4},x^4+y^4\neq 0,\\ 0,x^4+y^4=0\end{cases}$ $0(0,0)$ va $A(10^{-4};10^{-5})$.

5.15 $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{x+y}, x+y\neq 0,\\ 0, x+y=0\end{cases}$ $0(0,0)$ **va** $A(-1;-1)$.

5.16 $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, x^2+y^2\neq 0,\\ 1, x^2+y^2=0\end{cases}$ $0(0,0)$ **va** $A(0;1)$.

5.17 $f(x,y)=\begin{cases}\dfrac{\sin x+\sin y}{x+y}, x^2+y^2\neq 0,\\ 1, x+y=0\end{cases}$ $0(0,0)$ **va** $A\left(\dfrac{\pi}{3};-\dfrac{\pi}{3}\right)$.

5.18 $f(x,y)=\begin{cases}\dfrac{\cos x-\cos y}{x-y}, x-y\neq 0,\\ 0, x-y=0\end{cases}$ $0(0,0)$ **va** $A\left(\dfrac{\pi}{4};\dfrac{\pi}{4}\right)$.

5.19 $f(x,y)=\begin{cases}\dfrac{1}{x^2+y^2}, (x,y)\neq(0,0)\\ 0, (x,y)=(0,0).\end{cases}$ $0(0,0)$ **va** $A(1;0)$.

5.20 $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, x^2+y^2\neq 0,\\ 0, x^2+y^2=0\end{cases}$ $0(0,0)$ **va** $A(1;0)$.

5.21 $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, x^2+y^2\neq 0,\\ 0, x^2+y^2=0\end{cases}$ $0(0,0)$ va $A(1;1)$.

6-masala.

$f(x,y)$ **funksiyani** M **to'plamda chegaralanganlikka tekshiring.**

6.1 $f(x,y)=x^2-y^2$, $M=\{(x,y)\in R^2, x^2+y^2\leq 25\}$.

6.2 $f(x,y)=x^2-y^2$, $M=\{(x,y)\in R^2, x^2+y^2>25\}$.

6.3 $f(x,y)=\dfrac{2x^2+3y^2}{x^2+y^2}$, $M=\{(x,y)\in R^2, x^2+y^2\neq 0\}$.

6.4 $f(x,y)=\dfrac{\cos(x+y)-\cos(x-y)}{xy}$, $M=\{(x,y)\in R^2, xy\neq 0\}$.

6.5 $f(x,y)=\dfrac{\sin(x+y)-\sin(x-y)}{xy}$, $M=\{(x,y)\in R^2, xy\neq 0\}$.

6.6 $f(x,y)=\dfrac{\ln x-\ln y}{x-y}$, $M=\{(x,y)\in R^2, x\neq y\}$.

Quyidagi funksiyalarning ko'rsatilgan to'plamda chegaralanganligini isbotlang, uning aniq chegaralarini toping va funksiya shu qiymatlarga erishish-erishmasligini aniqlang.

6.7 $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$, $M=\{(x,y)\in R^2, x^2+y^2\neq 0\}$.

6.8 $f(x,y)=\dfrac{x^6+y^6}{x^2+y^2}$, $M=\{(x,y)\in R^2, 0<x^2+y^2\leq 9\}$.

6.9 $f(x,y)=\dfrac{x^2y^2}{x^4+y^4}$, $M=\{(x,y)\in R^2, x^4+y^4\neq 0\}$.

6.10 $f(x,y)=xye^{-xy}$, $M=\{(x,y)\in R^2, x\geq 0, y\geq 0\}$.

6.11 $f(x,y,z)=\dfrac{4(x^2+y^2)+2z^2}{x^2+y^2+z^2}$ $M=\{(x,y,z)\in R^3, x^2+y^2+z\neq 0\}$.

$f(x,y)$ funksiyaning M to'plamda tekis uzluksiz bo'lishi ta'rif yordamida isbotlansin($\delta=\delta(\varepsilon)-?$).

6.12 $f(x,y)=2x+3y+5$, $M=R^2$.
6.13 $f(x,y)=x^2+y^2$ $M=\{(x,y)\in R^2, x^2+y^2<4\}$.
6.14 $f(x,y)=\sqrt{x^2+y^2}$ $M=R^2$.
6.15 $f(x,y)=x-2y+3$, $M=R^2$.

Quyidagi funksiyalarni ko'rsatilgan to'plamda tekis uzluksizlikka tekshiring.

6.16 $f(x,y)=\dfrac{x^2+y^2}{x^4+y^4}$, $M=\{(x,y)\in R^2, 0<x^2+y^2<1\}$.

6.17 $f(x,y)=\dfrac{\sqrt{x^4+y^4}}{x^2+y^2}$, $M=\{(x,y)\in R^2, 0<x^2\le 1\}$.

6.18 $f(x,y)=x\cdot\sin\dfrac{1}{y}$ $M=\{(x,y)\in R^2,\ 0<x<1, 0<y<1\}$.

6.19 $f(x,y)=xy\sin\dfrac{1}{y}$, $M=\{(x,y)\in R^2,\ 0<x<1,\ 0<y<1\}$.

6.20 $f(x,y)=y\cdot\cos\dfrac{1}{x}$, $M=\{(x,y)\in R^2,\ 0<x<1,\ 0<y<1\}$

6.21 $f(x,y)=x^3-y^3$ funksiyaning $M=\{(x,y)\in R^2, 1\le x\le 2, 0\le y\le 1\}$ to'plamda tekis uzluksiz ekanligi ta'rif yordamida isbotlansin.

7-masala. Quyidagi funksiya $O(0,0)$ nuqtada xususiy hosilalarga egami va bu nuqtada differensiallanuvchimi?

7.1 $u(x,y)=\sqrt{x^2+y^2}$.

7.2 $u(x,y)=\begin{cases}\dfrac{x^3+y^3}{|x|+|y|}, |x|+|y|\ne 0\\ 0,\ |x|+|y|=0\end{cases}$

7.3 $u(x,y)=\sqrt[3]{xy}$.

7.4 $u(x,y)=\sqrt{xy}\cdot\sin x$.

7.5 $u(x,y)=\sqrt[3]{x^4+y^4}$.

7.6 $u(x,y)=\sqrt[3]{x^2y}\cdot tgx$.

7.7 $u(x,y)=\sqrt[3]{x}\sin y$.

7.8 $u(x,y)=\sqrt[4]{x^3+y^3}$.

7.9 $u(x,y)=\sqrt[4]{x^4+y^4}$.

7.10 $u(x,y)=\sqrt{2x^2-3y^2}$.

7.11 $u(x,y)=\sqrt{x^4+y^4}$.

7.12 $u(x,y)=\begin{cases}e^{-\frac{1}{x^4+y^4}}\ x^4+y^4\ne 0\\ 0, x^4+y^4=0\end{cases}$

7.13 $u(x,y)=\sqrt[3]{x^2y^2}$. **7.14** $u(x,y)=\sqrt{x^3+y^4}$.

7.15 $u(x,y)=\sqrt[3]{x^3+y^3}$. **7.16** $u(x,y)=\begin{cases}\dfrac{x^4+y^4}{|x|+|y|}, & |x|+|y|\neq 0\\ 0, & |x|+|y|=0\end{cases}$

7.17 $u(x,y)=\begin{cases}e^{\frac{1}{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ **7.18** $u(x,y)=\sqrt[3]{x\,y^2}\cdot\sin x$.

7.19 $u(x,y)=\sqrt[3]{y}\cdot tgx$. **7.20** $u(x,y)=\sqrt[3]{x\,y\,\sin x}$.

7.21 $u(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$

8-masala.
Sirtga ko'rsatilgan nuqtada o'tkazilgan urinma tekislik tenglamasi topilsin.

8.1 $z=xy;\quad A(1,\ 0,\ 0)$. **8.2** $z=\sin(xy)\quad A\left(1,\frac{\pi}{3},\frac{\sqrt{3}}{2}\right)$.

8.3 $z=x+y^2;\ A(0,\ 1,\ 1)$. **8.4** $z=e^{x+y},\ A(1,\ 1,\ 1)$.

8.5 $z=x^3+y^3;\ A(1,-1,\ 0)$. **8.6** $z=x^2+y^2\ A(1,\ 2,\ 5)$.

8.7 $x^2+y^2+z^2=169;\ A(3,\ 4,\ 12)$. **8.8** $z=arctg\dfrac{y}{x},\ A\left(1,\ 1,\ \dfrac{\pi}{4}\right)$.

8.9 $z=y+\ln\dfrac{x}{z};\ A(1,1,1)$.

$z=0$ **tekislik** $O(0,0,0,)$ **nuqtada quyidagi sirtga urinma tekislik bo'ladimi?**

8.10 $z=x^2+y^2$; **-aylanma paraboloid.**

8.11 $z=\sqrt{x^2+y^2}$ ***-konus.***

8.12 $z=xy$; -giperbolik paraboloid.

Quyidagi miqdorlarning taqribiy qiymatlarini hisoblang.

8.13. $1{,}002\cdot 2{,}003^2\cdot 3{,}004^3$. **8.14** $\sin 29^0\cdot tg46^0$.

8.15. $\frac{1,03^2}{\sqrt[3]{0,98}\cdot\sqrt[4]{1,05^3}}$.

8.16 $2,67^{\sin 0,07}$.

8.17 $\sqrt{1,02^3+1,97^3}$.

8.18 $\sin 1,59\cdot tg3,09$.

8.19 $z=x^3-3x^2y+3xy^2+1$ funksiya M(3; 1) nuqtada shu nuqtadan (6; 5) nuqtaga qarab yo'nalgan yo'nalish bo'yicha hosilasi topilsin.

8.20 $z=arctg(xy)$ funksiyaning M(1;1) nuqtada birinchi chorakning bissektrissasi yo'nalishi bo'yicha hosilasi hisoblansin.

8.21 $z=x^2y^2-xy^3-3y-1$ funksiyaning M(2;1) nuqtada shu nuqtadan koordinata boshiga qarab yo'nalgan yo'nalish bo'yicha hosilasi hisoblansin.

9-masala.
Quyidagi murakkab funksiyalarning xususiy hosilalarini toping (f va g-differensiallanuvchi).

9.1 $u=f\left(\sqrt{x^2+y^2},\sqrt{y^2+z^2},\sqrt{z^2+x^2}\right)$.

9.2 $u=f\left(x-y^2,y-x^2,xy\right)$.

9.3 $u=\left[f(x-y)\right]^{g(x,y)}$.

9.4 $u=f(x-y,xy)$.

9.5 $u=f(xy)\cdot g(yz)$.

9.6 $f\left(x+y,x^2+y^2\right)$.

9.7 $u=f\left(\frac{x}{y},\frac{y}{x}\right)$.

Agar f-ixtiyoriy differensiallanuvchi funksiya bo'lsa, $u(x,y)$ funksiya mos tenglamani qanoatlantirishini tekshiring.

9.8 $u=f\left(x^2+y^2\right)$; $y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0$.

9.9 $u=x^n\cdot f\left(\frac{y}{x}\right)$; $x\frac{\partial u}{\partial y}-2y\frac{\partial u}{\partial y}=nu$.

9.10 $u=yf\left(x^2-y^2\right)$ $y^2\frac{\partial u}{\partial x}+xy\frac{\partial u}{\partial y}=xu$.

9.11 $u=\frac{y^2}{3x}+f(xy)$, $x^2\frac{\partial u}{\partial x}-xy\frac{\partial u}{\partial y}+y^2=0$.

9.12 $u=x^n f\left(\frac{y}{x^\alpha},\frac{z}{x^\beta}\right)$, $x\frac{\partial u}{\partial x}+\alpha y\frac{\partial u}{\partial y}+\beta z\frac{\partial u}{\partial z}=nu$.

9.13 $u=\frac{xy}{z}\ln x+xf\left(\frac{y}{z},\frac{z}{x}\right)$, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=u+\frac{xy}{z}$.

Funksiya differensialini ko'rsatilgan nuqtalarda toping.

9.14 $u=\frac{yz}{x}$, $M(x,y,z)$ va $M_0(1,2,3)$.

9.15 $u=\cos(xy+xz)$, $M(x,y,z)$ va $M_0\left(1,\frac{\pi}{6},\frac{\pi}{6}\right)$.

9.16 $u=x^y$, $M(x,y,)$ va $M_0(2,3)$.

9.17 $u=x\ln(xy)$, $M(x,y,)$ va $M_0(-1,-1)$.

$\frac{\partial u}{\partial x}$ **va** $\frac{\partial u}{\partial y}$ **xususiy hosilalarni hisoblash va f va g fuksiyalarning hosilalarini (f va g-differensiallanuvchi funksiyalar) yo'qotish yo'li bilan shunday tenglama tuzingki,** $u(x,y)$ **funksiya uni qanoatlantirsin.**

9.18 $u=f\left(\frac{x}{y},\frac{y}{z}\right)$.

9.19 $u=f(x-y,y-z)$.

9.20 $u=xf\left(\frac{x}{y^2}\right)$.

9.21 $u=x+f(xy)$.

10-masala. Ko'rsatilgan tartibdagi xususiy hosilalar va differensiallar hisoblansin.

10.1 $u=\frac{x+y}{x-y}$; $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

10.2 $u=x^m y^n$; $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

10.3 $u=e^{2x}\sin y+e^x\cos\frac{y}{2}$; $\frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

10.4 $u=e^{xyz}$; $\frac{\partial^3 u}{\partial x\,\partial y\partial z}$.

10.5 $u=\sin x\cdot\cos 2y$; $\dfrac{\partial^{10}u}{\partial x^4\partial y^6}$. **10.6** $u=x^4\cos y+y^4\sin x$; $\dfrac{\partial^8 u}{\partial x^4\partial y^4}$.

10.7 $u=\left(x^2+y\right)^{10}tgx$; $\dfrac{\partial^{10}u}{\partial x\,\partial y^9}$. **10.8** $u=\sin xy$; $\dfrac{\partial^3 u}{\partial x^2\partial y}$ **va** $\dfrac{\partial^3 u}{\partial x\,\partial y^2}$.

10.9 $u=\sqrt{x^2+y^2}\cdot e^{xy}$; d^2u. **10.10** $u=\left(\dfrac{x}{y}\right)^z$; d^2u.

10.11 $u=x^{yz}$; d^2u. **10.12** $u=f\left(x+y,x^2+y^2\right)$; d^2u.

10.13 $u=f(xy)\cdot g(xz)$; d^2u. **10.14** $u=f(\sin x+\cos y)$; d^2u.

10.15 $u=f\left(x+y,z^2\right)$; d^2u. **10.16** $u=f\left(xy,x^2+y^2\right)$; d^2u.

10.17 $u=f(2x-3y+4z)$; d^nu. **10.18** $u=f(2x,3y,2z)$; d^nu.

$z=z(x,y)$ **bo'lsa,** $\dfrac{\partial z}{\partial x}$ **va** $\dfrac{\partial z}{\partial y}$ **lar topilsin.**

10.19 $F(xy,yz,zx)=0$. **10.20** $F(xyz,x+y)=0$.

10.21 $F(y-zx,x-zy,z-xy)=0$.

11-masala. Quyidagi funksiyalar ekstremumga tekshirilsin.

11.1 $u=x^2+xy+y^2+\dfrac{1}{x}+\dfrac{1}{y}$. **11.2** $u=-x^2-xy-y^2+x+y$.

1.3 $u=x^3+y^3-3axy$. **11.4** $u=x^4+y^4-36xy$.

11.5 $u=x^4+y^4-2x^2+4xy-2y^2$. **11.6** $u=x^2-2xy^2+y^4-y^5$.

11.7 $u=e^{2x}\cdot\left(x+y^2+2y\right)$. **11.8** $u=3x^2y+y^3-18x-30y$.

11.9 $u=xy+yz+zx$. **11.10** $u=\left(x^2+y^2\right)e^{-\left(x^2+y^2\right)}$.

11.11 $u=4-\left(x^2+y^2\right)^{2/3}$. **11.12** $u=x^2+2y^2+z^2-2x+4y-6z+1$.

11.13 $u=2x^2+y^2+z^2-2xy+4z-x$. **11.14** $u=x^3+xy+y^2-2zx+2z^2+3y-1$.

11.15 $u=1-\sqrt{x^2+y^2}$. **11.16** $u=(x-y+1)^2$.

11.17 $u=2x^4+y^4-x^2-2y^2$. **11.18** $u=x^2y^3\cdot(6-x-y)$.

11.19 $u=x^4+y^4-x^2-2xy-y^2$. **11.20** $u=x^2-(y-1)^2$.

11.21 $u=x^2-2xy+4y^2+6z^2+6yz-6z$.

12-masala.

Berilgan funksiyaning ko'rsatilgan to'plamdagi eng katta va eng kichik qiymatlari topilsin.

12.1 $u=xy-x^2y-\frac{xy^2}{2}$, $0\le x\le 1, 0\le y\le 2$.

12.2 $u=x^2+3y^2-x+18y-4$ $0\le x\le 1, 0\le y\le 1$.

12.3 $u=x^3+3y^2-3xy$, $0\le x\le 2, 0\le y\le 1$.

12.4 $u=\frac{xy}{2}-\frac{x^2y}{6}-\frac{xy^2}{8}$, $x\ge 0, y\ge 0, \frac{x}{3}+\frac{y}{4}\le 1$.

12.5 $u=x^6+y^6-3x^2+6xy-3y^2$, $0\le y\le x\le 2$.

12.6 $u=\cos x\cdot\cos y\cdot\cos(x+y)$, $0\le x\le\pi, 0\le y\le\pi$.

12.7 $u=(x-y^2)\cdot\sqrt[3]{(1-x)^2}$, $y^2\le x\le 2$.

12.8 $u=x^3+y^3-9xy+27$, $0\le x\le 6, 0\le y\le 6$.

12.9 $u=x^4+y^4-2x^2+4xy-2y^2$,, $0\le x\le 2, 0\le y\le 2$.

12.10 $u=xy+yz+zx$, $x^2+y^2+z^2\le 9$.

12.11 $u=x+y+z$, $x^2+y^2\le z\le 1$.

12.12 $u=2\sin x+2\sin y+\sin(x+y)$, $0\le x\le\frac{\pi}{2}, 0\le y\le\frac{\pi}{2}$.

Oshkormas ko'rinishda berilgan $y=y(x)$ funksiyaning ekstremumlari topilsin.

12.13 $y^2-2y-\sin x=0, 0\le x\le 2\pi$. **12.14** $(y-x)^3+x+6=0$.

12.15 $(y-x^2)^2=x^5, x^2+y^2\ne 0$. **12.16** $x^2+y^2+xy=27$.

Oshkormas ko'rinishda berilgan $z = z(x, y)$ funksiyaning ekstremumlari topilsin.

12.17 $2x^2 + 2y^2 + z^2 + 8yz - z + 8 = 0$.

12.18 $x^4 + y^4 + z^4 = 2(x^2 + y^2 + z^2)$.

12.19 $5x^2 + 5y^2 + 5z^2 - 2xy - 2xz - 2yz - 72 = 0$.

12.20 $z^2 + xyz - xy^2 - x^3 = 0$.

12.21 $5z^2 + 4zy + y^2 - 2y + 3x^2 - 6x + 4 = 0$.

13-masala.

13.1 a tomoni va uning qarshisidagi A burchagiga ko'ra berilgan uchburchakning eng katta yuzini toping.

13.2 Uchburchakning a,v tomonlri va ular orasidagi S burchak ma'lum. Bu uchburchakning a va v tomonlaridan shunday kesma bilan teng ikkiga (yuzaga nisbatan) bo'lingki, natijada kesma uzunligi eng kichik bo'lsin.

13.3 $y = x^2$ parabola va $x - y - 2 = 0$ to'g'ri chiziq orasidagi eng kichik masofani toping.

13.4 (x_0, y_0, z_0) nuqta bilan $Ax + By + Cz + D$ tekislik orasidagi eng kichik masofani toping.

13.5 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidga ichki chizilgan eng katta hajmli to'g'ri burchakli paralepepipedning o'lchamlarini toping.

13.6 O'lchamlari qanday bo'lganda ko'ndalang kesimi yarim doira, sirtining yuzasi $3\pi m^2$ bo'lgan ochiq silindrik vanna eng katta hajmga ega bo'ladi?

13.7 O'lchamlari qanday bo'lganda usti ochiq, hajmi 32 sm^3 bo'lgan to'g'ri burchakli banka eng kichik sirtga ega bo'ladi?

13.8 Hajmi 54π bo'lgan silindrik banka, asos diametri d va balandligi h ning qanday qiymatlarida eng kichik sirtga ega bo'ladi.

13.9 Musbat a sonini 5 ta shunday musbat sonlarning yig'indisi ko'rinishida ifodalangki ularning ko'paytmasi eng katta bo'lsin.

13.10 Qirralari uzunliklarining yig'indisi a ga teng bo'lgan to'g'ri burchakli parallelepipedning o'lchamlari qanday bo'lganda uning hajmi eng katta bo'ladi?

13.11 Hajmi V ga teng bo‘lgan to‘g‘ri burchakli parallelepipedning o‘lchamlari qanday bo‘lganda uning to‘la sirti eng kichik bo‘ladi?

Lagranj usulidan foydalanib $u=u(x,y)$ (yoki $u=u(x,y,z)$) funksiyaning berilgan shartni qanoatlantiruvchi ekstremumlari topilsin.

13.12 $u=xyz, \quad x^2+y^2+z^2=3$.

13.13 $u=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}, \quad x^2+y^2+z^2=1 \quad (a>b>c>0)$.

13.14 $u=x-2y+z \quad x+y^2-z^2=1$.

13.15 $u=xy^2z^3, \quad x+2y+3z=6 \quad (x>0,\ y>0,\ z>0)$.

13.16 $u=x^3+y^3-z^3+5, \quad x+y-z=0$.

13.17 $u=x^2+y^2+2z^2 \quad x-y+z=1$.

13.18 $u=xy \quad x^2+y^2=1$.

13.19 $u=x+y, \quad \frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}$.

13.20 $u=x^2+y^2, \quad \frac{x}{a}+\frac{y}{b}=1$.

13.21 $u=x-2y+2z, \ x^2+y^2+z^2=1$.

14-masala. u va v larni yangi erkli o‘zgaruvchi sifatida qabul qilib, quyidagi tenglamalarda o‘zgaruvchilarni almashtiring.

14.1 $x^2\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=2, \quad u=xy, \quad v=\frac{y}{x}$.

14.2 $2y\frac{\partial z}{\partial x}+e^x\frac{\partial z}{\partial y}=4ye^x, \quad u=y^2+e^x, \quad v=y^2-e^x$.

14.3 $y\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial x}+xy=0, \quad u=\frac{y}{x}, \quad v=yx^3$.

14.4 $2y\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}+2y=0$, $y=v$, $x=\dfrac{4+v^2}{2}$.

14.5 $y\dfrac{\partial z}{\partial y}+4\dfrac{\partial z}{\partial x}=\sqrt{xy}$, $x=v,^2$ $y=(u-v)^2$.

14.6 $(x+z)\dfrac{\partial z}{\partial x}+(y+z)\dfrac{\partial z}{\partial y}=0$, $u=x$ $v=\dfrac{y+z}{x+z}$.

14.7 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$, $u=x$ $v=\dfrac{y}{x}$..

14.8 $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=y\cdot e^{x^2+y^2}$, $u=x^2+y^2$, $v=y$.

14.9 $(x+y)\dfrac{\partial z}{\partial x}-(x-y)\dfrac{\partial z}{\partial y}=0$, $u=\ln\sqrt{x^2+y^2}$, $v=arctg\dfrac{y}{x}$.

14.10 $\left(x\dfrac{\partial z}{\partial x}\right)^2+yz\dfrac{\partial z}{\partial y}=2z^2$, $u=\ln x$, $v=\ln y$.

14.11 $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=0$, $u=x$ $v=x^2+y^2$,

14.12 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$, $u=4x-7y$, $v=8\cdot\dfrac{y}{x}$.

14.13 $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=0$, $u=x-y$, $v=x+y$.

14.14 $\dfrac{\partial^2 z}{\partial x^2}+y\dfrac{\partial^2 z}{\partial y^2}+\dfrac{1}{2}\dfrac{\partial z}{\partial y}=0$, $(y>0)$, $u=x$, $v=2\sqrt{y}$.

14.15 $2\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x\partial y}+5\dfrac{\partial^2 z}{\partial y^2}-\dfrac{\partial z}{\partial x}=0$, $u=\dfrac{1}{3}(x-y)$, $v=\dfrac{1}{3}(2x+y)$.

14.16 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$, $x=u\cos v$, $y=u\sin v$.

14.17 $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$, $x=u\cos v$, $y=u\sin v$.

14.18 $\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=0$, $x=u\cos v$, $y=u\sin v$.

14.19 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$, $x=u\cos v$, $y=u\sin v$.

14.20 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+4z=0$, $x=e^4\cdot\cos v$, $y=e^4\cdot\sin v$.

14.21 $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0$, $u=xy$, $v=\frac{x}{y}$.

–D–

Namunaviy variant yechimi

1.21-masala. R^2 **fazoda ushbu**

$$x^{(n)}=\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)};\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot...\cdot\sqrt[2^n]{2}\right)$$

ketma-ketlikning limiti topilsin.

◁ $y_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}$ va $z_n=\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot...\cdot\sqrt[2^n]{2}$ deb belgilasak.

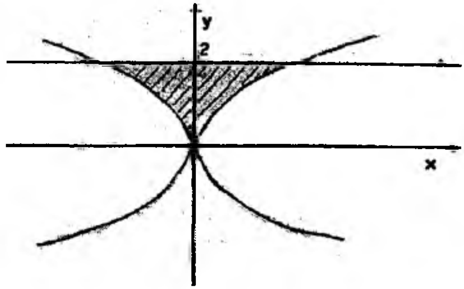
$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+...+\frac{1}{n}-\frac{1}{n+1}\right)=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1 \text{ va}$$

$\lim\limits_{n\to\infty}z_n=\lim\limits_{n\to\infty}2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}}=2^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}=2$ bo'lib, $\lim\limits_{n\to\infty}x^{(n)}=(1;2)$ ekanligini hosil qilamiz. ▷

2.21-masala. Quyidagi $u=\arcsin\frac{x}{y^2}+\arcsin(1-y)$ **funksiyaning aniqlanish sohasi topilsin va chizmada ko'rsatilsin.**

$$\triangleleft \quad D(u)=\begin{cases}\left|\dfrac{x}{y^2}\right|\le 1\\ |y-1|\le 1\end{cases}=\begin{cases}-y^2\le x\le y^2\\ 0<y\le 2\end{cases}=\left\{(x,y)\in R^2: 0<y\le 2, -y^2\le x\le y^2\right\}$$

Bu soha 8-chizmada tasvirlangan. $\triangleright$



8-chizma.

3.21-masala. Ushbu

$$\lim_{\substack{x\to 1\\ y\to 0}}\frac{\ln^2(x+y)}{\sqrt{x^2+y^2-2x+1}}$$ **karrali limit hisoblansin.**

$$\triangleleft \lim_{\substack{x\to 1\\ y\to 0}}\frac{\ln^2(x+y)}{\sqrt{x^2+y^2-2x+1}}=\left(\left(\ln(x+y)=\ln\left[1+(x-1+y)\right]\sim x-1+y\right)\right)=$$

$$=\lim_{\substack{x\to 1\\ y\to 0}}\frac{(x-1+y)^2}{\sqrt{(x-1)^2+y^2}}=\left(\left(\left.\begin{cases}x-1=r\cos\varphi & x\to 1\\ y=r\sin\varphi & y\to 0\end{cases}\right\}\Leftrightarrow r\to 0\right)\right)=$$

$$=\lim_{r\to 0}\frac{r^2(\cos\varphi+\sin\varphi)^2}{r}=(\cos\varphi+\sin\varphi)\lim_{r\to 0}r=0 \triangleright$$

4.21-masala. $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0}f(x,y)$ **va** $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0}f(x,y)$ **takroriy limitlar hisoblansin.**

$$f(x,y)=\sin\frac{\pi(x+y)}{2x+3y};\, x_0=y_0=\infty$$

$$\triangleleft \lim_{x\to x_0}\lim_{y\to y_0}f(x,y)=\lim_{x\to\infty}\lim_{y\to\infty}\sin\frac{\pi(x+y)}{2x+3y}=\lim_{x\to\infty}\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2} \quad \text{va}$$

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)=\lim_{y\to\infty}\lim_{x\to\infty}\sin\frac{\pi(x+y)}{2x+3y}=\lim_{y\to\infty}\sin\frac{\pi}{2}=1 \ \triangleright$$

5.21-masala. Quyidagi funksiyani berilgan nuqtalarda har bir o'zgaruvchi bo'yicha xususiy va ikkala o'zgaruvchi bo'yicha birgalikda uzluksizlikka tekshiring.

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, x^2+y^2\neq 0\\ 0, x^2+y^2=0\end{cases},\ O(0,0) \text{ va } A(1,1)$$

◁ **Ma'lumki, agar**

1) $\lim\limits_{x\to x_0} f(x,y_0)=f(x_0,y_0)$,

2) $\lim\limits_{y\to y_0} f(x_0,y)=f(x_0,y_0)$,

3) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0)$,

bo'lsa, unda $f(x,y)$ funksiya (x_0,y_0) nuqtada

1) x o'zgaruvchi bo'yicha xususiy,

2) y o'zgaruvchi bo'yicha xususiy

3) x va y o'zgaruvchilar bo'yicha birgalikda uzluksiz bo'ladi.

Shular asosida masalani yechamiz.

a) O(0,0) nuqtada tekshiramiz. Shartga ko'ra $f(0,0)=0$

$f(x,0)=f(0,y)=0 \Rightarrow \lim\limits_{x\to 0} f(x,0)=\lim\limits_{y\to 0} f(0,y)=f(0,0)\Rightarrow$ ikkala o'zgaruvchi bo'yicha xususiy uzluksiz.

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y_0)=\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{2xy}{x^2+y^2}=\left(\begin{pmatrix}x=r\cos\varphi\\ y=r\sin\varphi\end{pmatrix}\right)=\lim_{r\to 0}\frac{r^2\cdot\sin 2\varphi}{r^2}=\sin 2\varphi-\nexists\Rightarrow$$

birgalikda uzluksiz emas.

b) A(1,1) nuqtada tekshiramiz. Shartga ko'ra $f(1,\ 1)=1$ funksiya bu nuqtada ham xususiy, ham birgalikda uzluksiz ekanligini ko'rish qiyin emas. ▷

6.21-masala. $f(x,y)=x^3-y^3$ **funksiya** $M=\{(x,y)\in R^2: 1\le x\le 2, 0\le y\le 1\}$ **to'plamda tekis uzluksiz ekanligi ta'rif yordamida isbotlansin.**

◁ $\forall\varepsilon>0$ olib, quyidagi ayirmani baholaymiz:

$$|f(x_2,y_2)-f(x_1,y_1)|=|x_2^3-y_2^3-(x_1^3-y_1^3)|=|(x_2^3-x_1^3)-(y_2^3-y_1^3)|\le|x_2^3-x_1^3|+|y_2^3-y_1^3|=$$

$$=|x_2-x_1|\cdot\left|x_2^2+x_2\cdot x_1+x_1^2\right|+|y_2-y_1|\cdot\left|y_2^2+y_2y_1+y_1^2\right|<\delta\cdot\left(|x_2|^2+|x_2|\cdot|x_1|+|x_1|^2\right)+$$

$$+\delta\left(|y_2|^2+|y_2|\cdot|y_1|+|y_1|^2\right)\le\delta\cdot(4+2\cdot2+4)+\delta(1+1+1)=15\delta=\varepsilon\Rightarrow\delta=\frac{\varepsilon}{15}.$$

Demak, $\forall\varepsilon>0$ uchun $\delta=\frac{\varepsilon}{15}$ deb olsak , M to'plamning ushbu $|x_2-x_1|<\delta$ va $|y_2-y_1|<\delta$ tengsizliklarni qanoatlantiruvchi $\forall(x_1,y_1)$ va (x_2,y_2) nuqtalari uchun $|f(x_2,y_2)-f(x_1,y_1)|<\varepsilon$ tengsizlik bajariladi $\Rightarrow$ $f(x,y)$ funksiya M to'plamda tekis uzluksiz. ▷

7.21-masala. Quyidagi

$$u(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$$

funksiya O(0,0) nuqtada xususiy hosilalarga egami va bu nuqtada differensiallanuvchimi?

◁ $$\frac{\partial u(0,0)}{\partial x}:=\lim_{\Delta x\to0}\frac{u(\Delta x,0)-u(0,0)}{\Delta x}=\lim_{\Delta x\to0}\frac{\Delta x^4+0}{\left(\Delta x^2+0\right)\Delta x}=0,$$

$$\frac{\partial u(0,0)}{\partial x}:=\lim_{\Delta y\to0}\frac{u(0,\Delta y)-u(0,0)}{\Delta y}=\lim_{\Delta y\to0}\Delta y=0,$$ Demak, xususiy hosilalar $\exists$. Endi differensiallanuvchilikka tekshiramiz. Differensiallanuvchi bo'lishi uchun

$$\Delta u(0,0)=\frac{\partial u(0,0)}{\partial x}\cdot\Delta x+\frac{\partial u(0,0)}{\partial y}\cdot\Delta y+O(\rho)$$

yoki

$$\frac{\Delta x^4+\Delta y^4}{\Delta x^2+\Delta y^2}=0\left(\sqrt{\Delta x^2+\Delta y^2}\right)$$ bo'lishi kerak. $\Rightarrow$

$$\Rightarrow 0=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta u(0,0)}{\rho}=\lim_{\substack{\Delta x\to0\\ \Delta y\to0}}\frac{\Delta x^4+\Delta y^4}{\left(\Delta x^2+\Delta y^2\right)\cdot\sqrt{\Delta x^2+\Delta y^2}}=\left(\begin{pmatrix}\Delta x=r\cos\varphi\\ \Delta y=r\sin\varphi\end{pmatrix}\right)=$$

$$=\left(\cos^4\varphi+\sin^4\varphi\right)\lim_{r\to0}r=0\Rightarrow$$

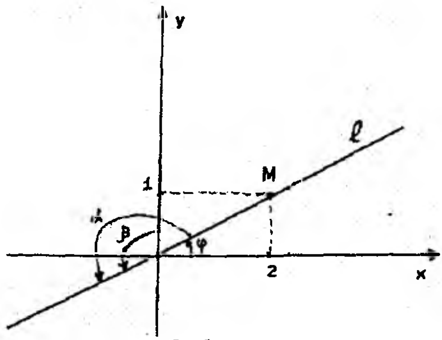
differensiallanuvchi. ▷

8.21-masala. $z = x^2y^2 - xy^3 - 3y - 1$ **funksiyaning M(2;1) nuqtada shu nuqtadan koordinata boshiga qarab yo'nalgan yo'nalish bo'yicha hosilasi hisoblansin.**

◁ Yo'nalish bo'yicha hosilani

$$\frac{\partial f(M)}{\partial \ell} = \frac{\partial f(M)}{\partial x} \cdot \cos\alpha + \frac{\partial f(M)}{\partial Y} \cdot \cos\beta$$

formula yordamida hisoblaymiz. M(2;1) nuqta va koordinata boshini tutashtirib ℓ to'g'ri chiziqli hosil qilamiz (9-chizma).



9-chizma.

$|OM| = \sqrt{2^2 + 1^2} = \sqrt{5}$ 9-chizmadan $\cos\alpha = \cos(\pi + \varphi) = -\cos\varphi = -\frac{2}{\sqrt{5}}$

$\cos\beta = \cos\left(\frac{\pi}{2} + \varphi\right) = -\sin\varphi = -\frac{1}{\sqrt{5}}$ ekanligini topamiz.

$$\frac{\partial f(M)}{\partial x} = \left(2xy^2 - y^3\right)\Big|_{\substack{x=2\\y=1}} = 3; \quad \frac{\partial f(M)}{\partial x} = \left(2x^2y - 3xy^2 - 3\right) = -1$$

Topilgan qiymatlarni yuqoridagi formulaga olib borib qo'yamiz:

$$\frac{\partial f(M)}{\partial x} = 3 \cdot \left(-\frac{2}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}} = -\frac{5}{\sqrt{5}} = -\sqrt{5} \ \triangleright$$

9.21-masala. $\frac{\partial u}{\partial x}$ **va** $\frac{\partial u}{\partial y}$ **xususiy hosilalarni hisoblash va f va g funksiyalarning hosilalarini yo'qotish yo'li bilan shunday tenglama tuzingki,** $u(x, y)$ **funksiya uni qanoatlantirsin.**

$$u = x + f(xy)$$

$$\triangleleft \quad \begin{cases} \dfrac{\partial u}{\partial x}=1+y\cdot f'(xy) \\ \dfrac{\partial u}{\partial y}=xf'(xy) \end{cases} \Rightarrow x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x. \ \triangleright$$

10.21-masala. $z=z(x,y)$ **bo'lsa** $\dfrac{\partial z}{\partial x}$ **va** $\dfrac{\partial z}{\partial y}$ **lar topilsin.**

$$F(y-zx, x-zy, z-xy)=0$$

$\triangleleft\ \xi=y-zx,\quad \eta=x-zy,\quad \zeta=z-xy$ **deb belgilab, berilgan tenglamani differensiallash yordamida topamiz:**

$$\begin{cases} F'_\xi\cdot\left(-z-x\dfrac{\partial z}{\partial x}\right)+F'_\eta\cdot\left(1-y\dfrac{\partial z}{\partial x}\right)+F'_\zeta\cdot\left(\dfrac{\partial z}{\partial x}-y\right)=0, \\ F'_\xi\cdot\left(1-x\dfrac{\partial z}{\partial y}\right)+F'_\eta\cdot\left(-z-y\dfrac{\partial z}{\partial y}\right)+F'_\zeta\cdot\left(\dfrac{\partial z}{\partial y}-x\right)=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{\partial z}{\partial x}\cdot\left(-xF_\xi-yF'_\eta+F'_\zeta\right)=z\cdot F'_\xi-F'_\eta+y\cdot F'_\zeta \\ \dfrac{\partial z}{\partial y}\cdot\left(-xF'_\xi-yF'_\eta+F'_\zeta\right)=-F'_\xi+z\cdot F'_\eta+x\cdot F'_\zeta. \end{cases} \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{-z\cdot F'_\xi+F'_\eta-y\cdot F'_\zeta}{x\cdot F'_\xi+y\cdot F'_\eta-F'_\zeta}, \\ \dfrac{\partial z}{\partial y}=\dfrac{F'_\xi-z\cdot F'_\eta-x\cdot F'_\zeta}{x\cdot F'_\xi+y\cdot F'_\eta-F'_\zeta} \end{cases} \triangleright$$

11.21-masala. $u=x^2-2xy+4y^2+6z^2+6yz-6z$ **funksiya ekstremumga tekshirilsin.**

$$\triangleleft \quad \begin{cases} \dfrac{\partial u}{\partial x}=2x-2y, \\ \dfrac{\partial u}{\partial y}=-2x+8y+6z, \\ \dfrac{\partial u}{\partial z}=12z+6y-6 \end{cases} \quad \begin{cases} \dfrac{\partial u}{\partial x}=0 \\ \dfrac{\partial u}{\partial y}=0 \\ \dfrac{\partial u}{\partial z}=0 \end{cases}$$

sistemani yechib, $M_0(-1,-1,\ 1)$ nuqta statsionar nuqta ekanligini topamiz. Endi ikkinchi tartibli xususiy hosilalarni hisoblab, $d^2u\,|_{M_0}$ ning ishorasini aniqlaymiz.

$$a_{11}=\frac{\partial^2 u}{\partial x^2}=2,\quad a_{12}=a_{21}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=-2,\quad a_{13}=a_{31}=\frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}=0,$$

$$a_{22}=\frac{\partial^2 u}{\partial y^2}=8,\quad a_{23}=a_{32}=\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial^2 u}{\partial z\partial y}=6\quad a_{33}=\frac{\partial^2 u}{\partial z^2}=12$$

$$a_{11}=2>0;\quad \begin{vmatrix} a_{11}a_{12}\\ a_{21}a_{22}\end{vmatrix}=\begin{vmatrix}2,-2\\ -2,8\end{vmatrix}=12>0;$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix} 2 & -2 & 0\\ -2 & 8 & 6\\ 0 & 6 & 12\end{vmatrix}=24\cdot\begin{vmatrix} 1 & -1 & 0\\ -1 & 4 & 3\\ 0 & 1 & 2\end{vmatrix}=$$

$$=48>0\Rightarrow d^2u\,\big|_{M_0}>0\Rightarrow u_{\min}=u(-1;-1;\,1)=-3.\ \triangleright$$

12.21-masala. Oshkormas ko'rinishda berilgan $z=z(x,y)$ funktsiyaning ekstremumlari topilsin.

$$5z^2+4zy+y^2-2y+3x^2-6x+4=0.$$

$\triangleleft$ Birinchi navbatda oshkormas funksiyaning xususiy hosilalarini va ular yordamida statsionar nuqtalarni topamiz:

$$\begin{cases}10z\cdot z'_x+4yz'_x+6x-6=0\\ 10z\cdot z'_y+4z+4z'_y\cdot y+2y-2=0\end{cases}\Rightarrow\begin{cases}z_x=\dfrac{3-3x}{5z+2y}\\ z_y=\dfrac{1-y-2z}{5z+2y}\end{cases}\quad\begin{cases}z'_x=0\\ z'_y=0\end{cases}$$

sistema va berilgan tenglamani x, y, z o'zgaruvchilarga nisbatan yechib $M_1(1;\,1;\,0)$ va $M_2(1;\,9;\,-4)$ statsionar nuqtalarini topamiz. Funksiyaning bu nuqtalarida ekstremumga erishishini tekshirish uchun ikkinchi tartibli xususiy hosilalarni hisoblaymiz:

$$z''_{x^2}=(z'_x)'_x=\left(\frac{3-3x}{5z+2y}\right)'_x=\frac{-3\cdot(5z+2y)-5z'_x\cdot(3-3x)}{(5z+2y)^2}$$

$$z''_{xy}=(z'_x)'_y=\left(\frac{3-3x}{5z+2y}\right)'_y=\frac{(3-3x)\cdot(5z'_y+2)}{(5z+2y)^2}$$

$$z''_{y^2}=(z'_y)'_y=\left(\frac{1-y-2z}{5z+2y}\right)'_y=\frac{(-1-2z'_y)\cdot(5z+2y)-(-1-y-2z)\cdot(5z'_y+2)}{(5z+2y)^2}$$

a) $M_1(1;\,1;\,0)$ nuqtada ekstremumga tekshiramiz.

$$a_{11}=z''_{x^2}\,\big|_{M_1}=-\frac{3}{2};\quad a_{12}=z''_{xy}\,\big|_{M_1}=0\quad a_{22}=z''_{y^2}\,\big|_{M_1}=-\frac{1}{2};\ \Rightarrow$$

$\Rightarrow \Delta a_{11}a_{22} - a_{12}^2 = \frac{3}{4}$. Demak $a_{11} < 0$ va $\Delta > 0 \Rightarrow \max \Rightarrow z_{\max} = z(1,\ 1) = 0$

b) $M_2(1;9;-4)$ nuqtada ekstremumga tekshiramiz.

$$a_{11} = z''_{x^2}\Big|_{M_2} = -\frac{3}{2};\quad a_{12} = z''_{xy}\Big|_{M_2} = 0\quad a_{22} = z''_{y^2}\Big|_{M_2} = -\frac{1}{2}; \Rightarrow$$

$\Rightarrow \Delta a_{11}a_{22} - a_{12}^2 = \frac{3}{4}$. Demak $a_{11} > 0$ va $\Delta > 0 \Rightarrow \min \Rightarrow z_{\min} = z(1,\ 9) = -4$

Shunday qilib $z_{\max} = z(1,\ 1) = 0$ va $z_{\min} = z(1,\ 9) = -4$ ▷

13.21-masala. Lagranj usulidan foydalanib $u = x - 2y + 2z$ funksiyaning $x^2 + y^2 + z^2 = 1$ shartni qanoatlantiruvchi ekstremumlari topilsin.

$$\lhd\quad \Phi(x,y,z) = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$$

Lagranj funksiyasini olamiz va bu funksiyaning ekstremumlarini qidiramiz:

$$\begin{cases} \dfrac{\partial\Phi}{\partial x} = 1 + 2\lambda x, \\ \dfrac{\partial\Phi}{\partial y} = -2 + 2\lambda y, \\ \dfrac{\partial\Phi}{\partial z} = 2 + 2\lambda z \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} 1 + 2\lambda x = 0 \\ -2 + 2\lambda y = 0 \\ 2 + 2\lambda z = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -\dfrac{1}{2\lambda}; y = \dfrac{1}{\lambda}; z = -\dfrac{1}{\lambda} \\ \lambda_{1,2} = \pm\dfrac{3}{2} \end{cases}$$

a) $\lambda = \frac{3}{2}$ bo'lsin $\Rightarrow x_1 = -\frac{1}{3};\ y_1 = \frac{2}{3},\ z_1 = -\frac{2}{3}$

$\frac{\partial^2\Phi}{\partial x^2} = 2\lambda$, $\frac{\partial^2\Phi}{\partial y^2} = 2\lambda$ $\frac{\partial^2\Phi}{\partial z^2} = 2\lambda$ va aralash hosilalar nolga teng.

$$\Rightarrow d^2\Phi = 2\lambda\left[(dx)^2 + (dy)^2 + (dz)^2\right] > 0 \Rightarrow \min \Rightarrow$$

$$u_{\min} = u\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) = -3$$

b) $\lambda = -\frac{3}{2}$ bo'lsin $\Rightarrow x_2 = \frac{1}{3};\ y_2 = -\frac{2}{3};\ z_2 = \frac{2}{3}$. Bu holda

$$d^2\Phi<0\Rightarrow \max\Rightarrow u_{\max}=u\left(\frac{1}{3};-\frac{2}{3};\frac{2}{3}\right)=3.\triangleright$$

14.21-masala. u va v larni yangi erkli o‘zgaruvchi sifatida qabul qilib, quyidagi tenglamalardan o‘zgaruvchilarni almashtiring.

$$x^2\cdot\frac{\partial^2 z}{\partial x^2}-y^2\cdot\frac{\partial^2 z}{\partial y^2}=0,\quad u=xy,\quad v=\frac{x}{y}.$$

$$\triangleleft\, z=z(x,y)\to z=z(u,v)\Rightarrow\ \frac{\partial z}{\partial x}=y\cdot\frac{\partial z}{\partial u}+\frac{1}{y}\cdot\frac{\partial z}{\partial v};\quad \frac{\partial z}{\partial y}=x\cdot\frac{\partial z}{\partial u}-\frac{x}{y^2}\cdot\frac{\partial z}{\partial v};\Rightarrow$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+\frac{1}{y}\cdot\frac{\partial z}{\partial v}\right)=y\left(y\cdot\frac{\partial^2 z}{\partial u^2}+\frac{1}{y}\cdot\frac{\partial^2 z}{\partial u\partial v}\right)+$$

$$+\frac{1}{y}\left(\frac{\partial^2 z}{\partial u\partial v}\cdot y+\cdot\frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)=y^2\cdot\frac{\partial^2 z}{\partial u^2}+2\cdot\frac{\partial^2 z}{\partial u\partial v}+\frac{1}{y^2}\cdot\frac{\partial^2 z}{\partial v^2}$$

va

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(x\cdot\frac{\partial z}{\partial u}-\frac{x}{y^2}\cdot\frac{\partial z}{\partial v}\right)=x\left(x\cdot\frac{\partial^2 z}{\partial u^2}-\frac{x}{y^2}\cdot\frac{\partial^2 z}{\partial u\partial v}\right)-$$

$$-\frac{x}{y^2}\left(\frac{\partial^2 z}{\partial u\partial v}\cdot x-\frac{x}{y^2}\cdot\frac{\partial^2 z}{\partial v^2}\right)+\frac{2x}{y^3}\cdot\frac{\partial z}{\partial v}=$$

$$=x^2\cdot\frac{\partial^2 z}{\partial u^2}-2\frac{x^2}{y^2}\cdot\frac{\partial^2 z}{\partial u\partial v}+\frac{x^2}{y^4}\cdot\frac{\partial^2 z}{\partial v^2}+\frac{2x}{y^3}\cdot\frac{\partial z}{\partial v}.$$

Topilgan $\frac{\partial^2 z}{\partial x^2}$ va $\frac{\partial^2 z}{\partial y^2}$ ifodalarning qiymatlarini berilgan tenglamaga olib borib qo‘yamiz.

$$4x^2\cdot\frac{\partial^2 z}{\partial u\partial v}-\frac{2x}{y}\cdot\frac{\partial z}{\partial v}=0\ \Big|\cdot\frac{y}{2x}$$

$$2xy\cdot\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial z}{\partial v}=0\Rightarrow 2u\cdot\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial v}$$

Demak, berilgan tenglama almashtirishdan so‘ng ushbu

$$2u\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial v}$$

ko‘rinishga kelar ekan. $\triangleright$

6-§. 5-MUSTAQIL ISH
Sonli qatorlar

Sonli qatorlar va ularning yaqinlashishi.
Musbat hadli qatorlar va ularning yaqinlashish alomatlari.
Ishorasi o'zgaruvchi qatorlar va ularning yaqinlashish alomatlari.
Cheksiz ko'paytmalar.

-A-
Asosiy tushuncha va teoremalar
1^0. Yaqinlashuvchi qatorlar va ularning xossalari.

Ushbu

$$a_1, a_2, ..., a_n, ...$$

haqiqiy sonlar ketma-ketligi berilgan bo'lsin.

1-ta'rif. Quyidagi

$$a_1 + a_2 + ... + a_n + ... \qquad (1)$$

ifodaga **qator (sonli qator)** deyiladi va u $\sum_{n=1}^{\infty} a_n$ kabi belgilanadi.

Shunday qilib,

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + ... + a_n + ... \qquad (2)$$

ekan. $\{a_n\}$ ketma-ketlikning $a_1, a_2, ..., a_n, ...$ elementlari **qatorning hadlari** deyiladi, a_n esa **qatorning umumiy hadi** deb ataladi. Ushbu

$$S_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, ... \qquad (3)$$

yig'indilar esa (2)-qatorning **qismiy yig'indilari deyiladi.**

2-ta'rif. Agar $\{S_n\}$ ketma-ketlik chekli limitga ega, ya'ni

$$\lim_{n\to\infty} S_n = S;$$

bo'lsa, unda qator **yaqinlashuvchi** deyiladi va bu limitning qiymati S (2)-**qatorning yig'indisi** deb ataladi hamda u

$$S = a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n;$$

kabi yoziladi.

Agar $\{S_n\}$ ketma-ketlik yaqinlashuvchi bo‘lmasa, u holda **uzoqlashuvchi** deyiladi.

3-ta’rif. Ushbu

$$\sum_{n=m+1}^{\infty} a_n = a_{m+1} + a_{m+2} + \dots \qquad (4)$$

qator (2)-qatorning (m-hadidan keyingi) **qoldig‘i** deyiladi.

1-teorema. Agar (2)-qator yaqinlashuvchi bo‘lsa, uning istalgan (4)-qoldig‘i ham yaqinlashuvchi bo‘ladi va aksincha, (4)-qoldiqning yaqinlashuvchi bo‘lishidan berilgan (2)-qatorning yaqinlashuvchi bo‘lishi kelib chiqadi.

1-natija. Agar (2)-qator yaqinlashuvchi bo‘lsa, uning qoldig‘i

$$r_m = a_{m+1} + a_{m+2} + \dots$$

$m \to \infty$ da nolga intiladi.

2-teorema. Agar (2)-qator yaqinlashuvchi bo‘lib, uning yig‘indisi S bo‘lsa, u holda $\sum_{n=1}^{\infty} ca_n$ qator ham yaqinlashuvchi bo‘lib, uning yig‘indisi $c \cdot S$ bo‘ladi, ya’ni

$$\sum_{n=1}^{\infty} ca_n = c \cdot \sum_{n=1}^{\infty} a_n$$

tenglik bajariladi.

3-teorema. Agar $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar yaqinlashuvchi bo‘lsa, unda $\sum_{n=1}^{\infty} (a_n + b_n)$ qator ham yaqinlashuvchi bo‘lib,

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

bo‘ladi.

2 va 3-teoremalardan quyidagi natija kelib chiqadi.

2-natija. Agar $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} (ca_n + d_0 b_n)$ $(c, d - const)$ qator ham yaqinlashuvchi bo‘lib,

$$\sum_{n=1}^{\infty} (c \cdot a_n + d \cdot b_n) = c \cdot \sum_{n=1}^{\infty} a_n + d \cdot \sum_{n=1}^{\infty} b_n$$

bo‘ladi.

4-teorema. **(Qator yaqinlashishining zaruriy sharti).** *Agar (2)-qator yaqinlashuvchi bo'lsa, u holda*

$$\lim_{n\to\infty} a_n = 0 \qquad (5)$$

bo'ladi.

Izoh. 4-teoremaning aksi har doim ham o'rinli bo'lavermaydi. Masalan, $\sum_{n=1}^{\infty}\frac{1}{n}$ uchun $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$, lekin bu qator yaqinlashuvchi emas.

5-teorema. **(Koshi kriteriyasi)** *(2)-qatorning yaqinlashuvchi bo'lishi uchun quyidagi shartning bajarilishi zarur va yetarli:* $\forall\varepsilon > 0$ *son uchun* $\exists n_0(\varepsilon)\in N : \forall n \geq n_0$ *va* $\forall$ *butun* $p \geq 0$ *son uchun*

$$\left|\sum_{k=n}^{n+p} a_k\right| = \left|a_n + a_{n+1} + \ldots + a_{n+p}\right| < \varepsilon \qquad (6)$$

tengsizlik bajariladi.

2°. Musbat hadli qatorlar va ularning yaqinlashishi

Aytaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (7)$$

qator berilgan bo'lsin. Agar $\forall n \in N$ uchun $a_n \geq 0$ bo'lsa, unda (7)-qatorga **musbat hadli qator** yoki qisqacha **musbat qator** deb ataladi.

Bu punktda biz musbat hadli qatorlar uchun yaqinlashish alomatlarini keltiramiz.

1-teorema. **(Veyershtrass kriteriyasi)** *(7)-qator yaqinlashuvchi bo'lishi uchun uning qismiy yig'indilari ketma-ketligi* $\{S_n\}$ *yuqoridan chegaralangan bo'lishi zarur va yetarlidir.*

Misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}} = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \ldots + \frac{1}{n^{\alpha}} + \ldots \qquad (8)$$

umumlashgan garmonik qatorning $\alpha > 1$ da yaqinlashuvchi ekanligi isbotlansin.

$\lhd$ $S_n = \sum_{k=1}^{n}\frac{1}{k^{\alpha}} = 1 + \frac{1}{2^{\alpha}} + \ldots + \frac{1}{n^{\alpha}}$ va $S_{n+1} = S_n + \frac{1}{(n+1)^{\alpha}} \Rightarrow \{S_n\}\uparrow$.

Endi uning yuqoridan chegaralanganligini ko'rsatamiz:

$$S_n < S_{2n+1} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{(2n+1)^\alpha} =$$

$$= 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \frac{1}{5^\alpha}\right) + \ldots + \left[\frac{1}{(2n)^\alpha} + \frac{1}{(2n+1)^\alpha}\right] <$$

$$1 + \left(\frac{1}{2^\alpha} + \frac{1}{2^\alpha}\right) + \left(\frac{1}{4^\alpha} + \frac{1}{4^\alpha}\right) + \ldots + \left[\frac{1}{(2n)^\alpha} + \frac{1}{(2n)^\alpha}\right] =$$

$$= 1 + \frac{2}{2^\alpha} \cdot \left(1 + \frac{1}{2^\alpha} + \ldots + \frac{1}{n^\alpha}\right) = 1 + \frac{1}{2^{\alpha-1}} \cdot S_n \Rightarrow$$

$\Rightarrow S_n < \dfrac{2^{\alpha-1}}{2^{\alpha-1}-1}$ $(n = 1, 2, \ldots) \Rightarrow \{S_n\}$ ketma-ketlik yuqoridan chegaralangan. 1-teoremaga ko‘ra $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ umumlashgan garmonik qator $\alpha > 1$ da yaqinlashadi. ▷

Faraz qilaylik, (7)-qator va ushbu

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \ldots + b_n + \ldots \qquad (9)$$

qatorlar berilgan bo‘lsin. Unda quyidagi taqqoslash teoremalari o‘rinli bo‘ladi.

2-teorema. (Birinchi taqqoslash alomati). Agar n ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun

$$a_n \leq b_n$$

tengsizlik o‘rinli bo‘lsa, unda (9)-qatorning yaqinlashuvchi bo‘lishidan (7) qatorning yaqinlashuvchi bo‘lishi va (7)-qatorning uzoqlashuvchi bo‘lishidan (9)-qatorning uzoqlashuvchi bo‘lishi kelib chiqadi.

3-teorema. Agar

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k \qquad (0 \leq k \leq \infty)$$

bo‘lsa,

a) $k < \infty$ bo‘lganda, (9)-qatorning yaqinlashuvchi bo‘lishidan (7)-qatorning yaqinlashuvchi bo‘lishi;

b) $k > 0$ bo‘lganda, (9)-qatorning uzoqlashuvchi bo‘lishidan (7)-qatorning uzoqlashuvchi bo‘lishi kelib chiqadi.

Natija. Agar $n \to \infty$ da $a_n = 0^*(b_n)$ bo'lsa $\left(y'ni \ \ 0 < k < \infty \ bo'lsa\right)$ unda (7)-qatorning yaqinlashishi (9)-qatorning yaqinlashishiga ekvivalent bo'ladi.

4-teorema. (Ikkinchi taqqoslash alomati). *Agar n ning biror $n_0 (n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

tengsizlik bajarilsa, unda

1) (9)-qator yaqinlashuvchi bo'lsa, (7)-qator yaqinlashuvchi;

2) (7)-qator uzoqlashuvchi bo'lsa, (9)-qator uzoqlashuvchi bo'ladi.

Endi musbat hadli (7)-qator uchun yaqinlashish alomatlarini keltiramiz.

5-teorema. (Dalamber alomati). *Agar (7)-qator uchun*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = d.$$

bo'lib,

1) $d < 1$ bo'lsa, qator yaqinlashuvchi;

2) $d > 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

6-teorema. (Koshi alomati). *Agar (7)-qator uchun*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = q$$

bo'lib,

1) $q < 1$ bo'lsa, qator yaqinlashuvchi;

3) $q > 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

Izoh. 5 va 6-teoremalardagi d va $q = 1$ bo'lsa, qator uzoqlashuvchi ham, yaqinlashuvchi ham bo'lishi mumkin. Masalan, $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator uchun $d = q = 1$ va qator uzoqlashuvchi; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ umumlashgan garmonik qator uchun ham $d = q = 1$, lekin qator yaqinlashuvchi.

7-teorema. (Raabe alomati). *Agar (7)-qator uchun*

$$\lim_{n\to\infty} n \cdot \left(1 - \frac{a_{n+1}}{a_n}\right) = \rho \qquad (11)$$

bo'lib,

1) $\rho > 1$ bo'lsa, qator yaqinlashuvchi;
2) $\rho < 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

8-teorema. (Gauss alomati). *Agar (7)-qator uchun*

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + \frac{\theta_n}{n^{1+\varepsilon}} \tag{12}$$

$|\theta_n| < c$ va $\varepsilon > 0$ bo'lib
1) $\lambda > 1$ bo'lsa, qator yaqinlashuvchi;
2) $\lambda = 1$ va $\mu > 1$ bo'lsa, qator yaqinlashuvchi;
3) $\lambda = 1$ va $\mu \leq 1$ bo'lsa, qator uzoqlashuvchi;
4) $\lambda < 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

9-teorema. (Koshining integral alomati). *Faraz qilaylik, $f(x)$ funksiya $[1;+\infty)$ oroliqda aniqlangan bo'lib, $f(x) > 0$ va monoton kamayuvchi bo'lsin. U holda*

$$\sum_{n=1}^{\infty} f(n)$$

qatorning yaqinlashuvchi bo'lishi uchun

$$\int_{1}^{+\infty} f(x)dx$$

integralning yaqinlashuvchi bo'lishi zarur va yetarli.

3[0] Ixtiyoriy hadli qatorlar va ularning yaqinlashishi

Bizga biror

$$\sum_{n=1}^{\infty} a_n \tag{13}$$

qator berilgan bo'lsin. Agar bu qatorning hadlari $\forall$ ishorani qabul qilishi mumkin bo'lsa, bunday qatorga **ixtiyoriy hadli qator** (yoki **ixtiyoriy qator**) deyiladi.

1-ta'rif. *Agar*

$$\sum_{n=1}^{\infty} |a_n| \tag{14}$$

qator yaqinlashuvchi bo'lsa, u holda (13)-qator ***absolut yaqinlashuvchi*** *qator deyiladi.*

1-teorema. *Agar (14)-qator yaqinlashuvchi bo'lsa, unda (13)-qator ham yaqinlashadi, ya'ni absolut yaqinlashuvchi qator oddiy ma'noda ham yaqinlashuvchi bo'ladi.*

2-ta'rif. *Agar (13)-qator yaqinlashuvchi bo'lib, (14)-qator uzoqlashsa, unda (13)-qator* ***shartli yaqinlashuvchi*** *qator deyiladi.*

Agar sonli qator $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ yoki $\sum_{n=1}^{\infty}(-1)^{n}a_n$ ko'rinishda bo'lib, $a_n>0$ bo'lsa, u holda bunday qatorga **hadlarining ishoralari almashinib keluvchi qator** deyiladi.

2-teorema. (Leybnis alomati). Agar

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \qquad (15)$$

qator berilgan bo'lib,

1) $\{a_n\}\downarrow$, ya'ni $a_n \ge a_{n+1} > 0 \quad (n=1,2,\ldots)$,

2) $\lim\limits_{n\to\infty} a_n = 0$

bo'lsa, u holda (15)-qator yaqinlashuvchi bo'ladi.

Misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

qator Leybnis alomatiga ko'ra yaqinlashuvchi bo'ladi va uning shartli yaqinlashuvchi ekanligini ko'rish qiyin emas.

3-teorema. (Dirixle alomati). *Agar*

$$\sum_{n=1}^{\infty}a_n b_n \qquad (16)$$

qator berilgan bo'lib,

1) $\{a_n\}$ *ketma-ketlik monoton bo'lib nolga intilsa;*

2) $B_n=\sum_{k=1}^{n}b_k \quad (n=1,2,3)K\ldots$, *chegaralangan bo'lsa, u holda (16)-qator yaqinlashuvchi bo'ladi.*

4-teorema. (Abel alomati). *Agar (16)-qator berilgan bo'lib,*

1) $\{a_n\}$ *ketma-ketlik monoton va chegaralangan,*

2) $B_n=\sum_{k=1}^{n}b_k$ *qator yaqinlashuvchi*

bo'lsa, unda (16)-qator yaqinlashuvchi bo'ladi.

Bizga $\forall$ hadli (13)-qator berilgan bo'lsin. Bu qator hadlarini guruhlab quyidagi qatorni tuzamiz:

$$\left(a_1 + a_2 + \ldots + a_{n1}\right) + \left(a_{n1+1} + a_{n1+2} + \ldots + a_{n2}\right) + \ldots, \qquad (17)$$

bu yerda $n_1 < n_2 < \ldots$ **va** $k \to \infty$ da $n_k \to \infty$

5-teorema. *Agar (13)-qator yaqinlashuvchi bo'lib, yig'indisi S soniga teng bo'lsa, unda (17)-qator ham yaqinlashuvchi va uning yig'indisi ham S soniga teng bo'ladi.*

Izoh. 5-teoremaning aksi **har doim ham o'rinli bo'lavermaydi. Masalan,**

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots$$

qator uzoqlashuvchi, lekin bu qatorni guruhlash natijasida hosil bo'lgan

$$(1-1)+(1-1)+(1-1)+\ldots = 0+0+\ldots+0+\ldots$$

qator yaqinlashuvchi.

Endi

$$\sum_{n=1}^{\infty} a_n' = a_1' + a_2' + \ldots + a_n' + \ldots \qquad (18)$$

yordamida (13)-qator hadlarining o'rinlarini almashtirishdan hosil bo'lgan yangi qatorni belgilaymiz.

6-teorema. *Agar (13)-qator absolut yaqinlashuvchi bo'lib, yig'indisi S soniga teng bo'lsa, u holda (18)-qator ham yaqinlashuvchi va uning yig'indisi ham S soniga teng bo'ladi.*

Izoh. 6-teoremadagi (13)-qatorning absolut yaqinlashishi sharti muhim shartdir. Aks holda teoremaning o'rinli bo'lishi shart emas.

$\triangleleft$ Masalan,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \cdot \frac{1}{n} + \ldots$$

qator shartli yaqinlashuvchi va $S = \ln 2$. Darhaqiqat,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1} \cdot \frac{x^n}{n} + r_n(x), \quad x > -1 \qquad (19)$$

yoyilmada $x = 1$ desak,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \cdot \frac{1}{n} + r_n(1) = S_n + r_n(1) \text{ va}$$

$$|r_n(1)| < \frac{1}{n+1} \text{ bo'ladi. } \Rightarrow S = \lim_{n\to\infty} S_n = \ln 2$$

Shunday qilib

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

ekan. $\Rightarrow$ Bu qatorning qismiy yig'indilari

$$S_{2n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k} \right), \quad S_{2n+1} = S_{2n} + \frac{1}{2n+1}$$

chekli S limitga ega:

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1} = S = \ln 2$$

Endi berilgan qatorda hadlarining o'rinlarini almashtirish yordamida quyidagi

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} + \ldots \qquad (20)$$

qatorni hosil qilamiz. (20)-qatorning yig'indisini hisoblaymiz.

$S'_{3n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \right)$ qismiy yig'indini olamiz.

$$\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k} \right) \Rightarrow \lim_{n\to\infty} S'_{3n} = \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) =$$

$$= \frac{1}{2} \lim_{n\to\infty} S_{2n} = \frac{1}{2} S \Rightarrow \lim_{n\to\infty} S'_{3n+1} = \lim_{n\to\infty} \left(S'_{3n} + \frac{1}{2n+1} \right) = \frac{1}{2} S \text{ va } \lim_{n\to\infty} S'_{3n+2} =$$

$$= \lim_{n\to\infty} \left(S'_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2} \right) = \frac{1}{2} S \Rightarrow$$ (20)-qatorning yig'indisi

$S' = \frac{1}{2} S = \frac{1}{2} \ln 2$ ekan. ▷

7-teorema. (Riman teoremasi). *Agar* $\sum_{n=1}^{\infty} a_n$ *qator shartli yaqinlashuvchi bo'lsa, u holda* $\forall A$ *(chekli yoki cheksiz) son olinganda ham berilgan qator hadlarining o'rinlarini shunday almashtirish mumkinki, hosil bo'lgan qatorning yig'indisi xuddi shu A ga teng bo'ladi.*

4^0. Cheksiz ko'paytmalar

Bizga

$$p_1, p_2, \dots, p_n, \dots$$

sonlar ketma-ketligi berilgan bo'lsin. Ulardan tuzilgan

$$p_1 \cdot p_2 \cdot \dots \cdot p_n \cdot \dots = \prod_{n=1}^{\infty} p_n \qquad (21)$$

simvolga **cheksiz ko'paytma** deyiladi. Ushbu

$$P_n = \prod_{k=1}^{n} p_k \quad (n = 1, 2, \dots)$$

ko'paytmalarga **xususiy ko'paytmalar** deb ataladi.

Ta'rif. *Agar P_n xususiy ko'paytmalar $n \to \infty$ da chekli yoki cheksiz P limitga ega bo'lsa*

$$\lim_{n\to\infty} P_n = P,$$

bu limitni (21)-ko'paytmaning ***qiymati*** *deb ataladi va*

$$P = \prod_{n=1}^{\infty} p_n$$

kabi yoziladi. Agar $P \neq 0$ va chekli bo'lsa, u holda ko'paytma ***yaqinlashuvchi,*** *aks holda* ***uzoqlashuvchi*** *deyiladi.*

Bundan buyon cheksiz ko'paytmalarni tekshirayotganimizda $p_n \neq 0$ deb faraz qilamiz.

Cheksiz ko'paytmalarning birinchi m ta hadini tashlab yuborib

$$\pi_m = \prod_{n=m+1}^{\infty} p_n = p_{m+1} \cdot p_{m+2} \cdot \dots \qquad (22)$$

qoldiq ko'paytmani hosil qilamiz.

1-teorema. *Agar (21)-ko'paytma yaqinlashsa, (22)-ko'paytma yaqinlashadi va aksincha, (22)-ko'paytmaning yaqinlashidan (21)-ko'paytmaning yaqinlashishi kelib chiqadi.*

2-teorema. *Agar (21)-cheksiz ko'paytma yaqinlashuvchi bo'lsa, unda*

$$\lim_{m\to\infty} \pi_m = 1$$

bo'ladi.

3-teorema. (Cheksiz ko'paytma yaqinlashishining zaruriy sharti). *Agar (21)-ko'paytma yaqinlashuvchi bo'lsa u holda*

$$\lim_{n\to\infty} p_n = 1$$

bo'ladi.

Yaqinlashuvchi cheksiz ko'paytmalar uchun 3-teoremaga ko'ra $\lim_{n\to\infty} p_n = 1 \Rightarrow$ Biror nomerdan boshlab hamma p_n lar >0 bo'ladi. Demak, umumiylikka ziyon keltirmasdan, barcha p_n lar uchun $p_n > 0$ deb faraz qilishimiz mumkin.

4-teorema. *(21)-cheksiz ko'paytma yaqinlashuvchi bo'lishi uchun*

$$\sum_{n=1}^{\infty} \ln p_n \tag{23}$$

qatorning yaqinlashuvchi bo'lishi zarur va yetarlidir. Agar bu shart bajarilsa va (23)-qatorning yig'indisi S bo'lsa, unda

$$P = e^S$$

bo'ladi.

Agar $p_n = 1 + a_n$ bo'lsa, unda $\prod_{n=1}^{\infty} p_n = \prod_{n=1}^{\infty} (1 + a_n)$ bo'lib, 4-teoremaga ko'ra (21)-ko'paytmaning yaqinlashuvchi bo'lishi uchun ushbu $\sum_{n=1}^{\infty} \ln(1 + a_n)$ qatorning yaqinlashuvchi bo'lishi zarur va yetarli ekanligini hosil qilamiz.

5-teorema. *Agar biror $n_0 \in N$ nomerdan boshlab, barcha $n > n_0$ lar uchun $a_n > 0$ (yoki $a_n < 0$) bo'lsa, (21)-cheksiz ko'paytmaning yaqinlashuvchi bo'lishi uchun*

$$\sum_{n=1}^{\infty} a_n \tag{24}$$

qatorning yaqinlashuvchi bo'lishi zarur va yetarlidir.

Umumiy holda, ya'ni a_n lar ishorani saqlamagan va (24)-qator yaqinlashgan holda, (21)-cheksiz ko'paytmaning yaqinlashuvchi bo'lishi uchun

$$\sum_{n=1}^{\infty} a_n^2 \qquad (25)$$

qatorning yaqinlashuvchi bo'lishi zarur va yetarlidir.

Agar (23)-qator absolut yoki shartli yaqinlashsa, unda (21)-cheksiz ko'paytma **absolut yoki shartli yaqinlashuvchi** deyiladi. $\Rightarrow$ (21)-ko'paytmaning absolut yaqinlashuvchi bo'lishi uchun (24)-qatorning absolut yaqinlashuvchi bo'lishi zarur va yetarli.

Nazorat savollari

1. Sonli qator tushunchasi.
2. Sonli qator yaqinlashishining ta'rifi.
3. Qator yaqinlashishining zaruriy sharti.
4. Qator yaqinlashishi uchun Koshi kriteriyasi.
5. Musbat qatorlar uchun Veyershtrass kriteriyasi.
6. Birinchi taqqoslash alomati.
7. Ikkinchi taqqoslash alomati.
8. Dalamber alomati.
9. Koshi alomati.
10. Raabe alomati.
11. Gauss alomati.
12. Koshining integral alomati.
13. Ixtiyoriy hadli qatorlar va ularning yaqinlashishi.
14. Leybnis alomati.
15. Dirixle alomati.
16. Abel alomati.
17. Absolut yaqinlashuvchi qatorlarning xossalari.
18. Shartli yaqinlashuvchi qatorlar.
19. Riman teoremasi.
20. Cheksiz ko'paytmalar va ularning yaqinlashishi.
21. Cheksiz ko'paytma yaqinlashishining zaruriy sharti.
22. Cheksiz ko'paytma yaqinlashishining zaruriy va yetarli shartlari.

-B-

Mustaqil yechish uchun misol va masalalar

1-masala. Qator yig'indisini toping.

1.1 $\sum_{n=1}^{\infty} \frac{6}{9n^2+12n-5}$.

1.2 $\sum_{n=2}^{\infty} \frac{24}{9n^2-12n-5}$.

1.3 $\sum_{n=1}^{\infty} \frac{6}{9n^2+6n-8}$.

1.4 $\sum_{n=1}^{\infty} \frac{9}{9n^2+21n-8}$.

1.5 $\sum_{n=1}^{\infty} \frac{2}{4n^2+8n+3}$.

1.6 $\sum_{n=1}^{\infty} \frac{14}{49n^2-28n-45}$.

1.7 $\sum_{n=1}^{\infty} \frac{3}{9n^2+3n-2}$.

1.8 $\sum_{n=1}^{\infty} \frac{7}{49n^2-7n-12}$.

1.9 $\sum_{n=2}^{\infty} \frac{1}{n^2+n-2}$.

1.10 $\sum_{n=1}^{\infty} \frac{14}{49n^2-14n-48}$.

1.11 $\sum_{n=1}^{\infty} \frac{6}{36n^2-24n-5}$.

1.12 $\sum_{n=1}^{\infty} \frac{14}{49n^2-84n-13}$.

1.13 $\sum_{n=1}^{\infty} \frac{4}{4n^2+4n-3}$.

1.14 $\sum_{n=1}^{\infty} \frac{7}{49n^2+35n-6}$.

1.15 $\sum_{n=1}^{\infty} \frac{9}{9n^2+3n-20}$.

1.16 $\sum_{n=1}^{\infty} \frac{8}{16n^2-8n-15}$.

1.17 $\sum_{n=1}^{\infty} \frac{7}{49n^2-21n-10}$.

1.18 $\sum_{n=1}^{\infty} \frac{6}{4n^2-9}$.

1.19 $\sum_{n=1}^{\infty} \frac{7}{49n^2-35n-6}$.

1.20 $\sum_{n=1}^{\infty} \frac{12}{36n^2+12n-35}$.

1.21 $\sum_{n=1}^{\infty} \frac{3}{9n^2-3n-2}$.

2-masala. Qator yig'indisini toping.

2.1 $\sum_{n=1}^{\infty} \frac{3n+8}{n(n+1)(n+2)}$.

2.2 $\sum_{n=1}^{\infty} \frac{2-n}{n(n+1)(n+2)}$.

2.3 $\sum_{n=1}^{\infty}\frac{3-n}{(n+3)(n+1)n}$.

2.4 $\sum_{n=1}^{\infty}\frac{n-1}{n(n+1)(n+2)}$.

2.5 $\sum_{n=3}^{\infty}\frac{4}{n(n-1)(n-2)}$.

2.6 $\sum_{n=3}^{\infty}\frac{n-4}{n(n-1)(n-2)}$.

2.7 $\sum_{n=3}^{\infty}\frac{3n+1}{(n-1)n(n+1)}$.

2.8 $\sum_{n=1}^{\infty}\frac{5n+9}{n(n+1)(n+3)}$.

2.9 $\sum_{n=1}^{\infty}\frac{4-n}{n(n+1)(n+2)}$.

2.10 $\sum_{n=3}^{\infty}\frac{8n-10}{(n-1)(n+1)(n-2)}$.

2.11 $\sum_{n=2}^{\infty}\frac{1}{n\cdot(n^2-1)}$.

2.12 $\sum_{n=3}^{\infty}\frac{3n-1}{n\cdot(n^2-1)}$.

2.13 $\sum_{n=1}^{\infty}\frac{1-n}{n(n+1)(n+3)}$.

2.14 $\sum_{n=1}^{\infty}\frac{3n+2}{n(n+1)(n+2)}$.

2.15 $\sum_{n=1}^{\infty}\frac{n+6}{n(n+1)(n+2)}$.

2.16 $\sum_{n=3}^{\infty}\frac{n+5}{(n+2)(n^2-1)}$.

2.17 $\sum_{n=3}^{\infty}\frac{n-2}{(n-1)n(n+1)}$.

2.18 $\sum_{n=3}^{\infty}\frac{n+2}{n(n-1)(n-2)}$.

2.19 $\sum_{n=1}^{\infty}\frac{3n+4}{n(n+1)(n+2)}$.

2.20 $\sum_{n=1}^{\infty}\frac{2}{(n+2)(n+1)\cdot n}$.

2.21 $\sum_{n=2}^{\infty}\frac{5n-2}{(n-1)n(n+2)}$.

3-masala. Qatorning qismiy yig'indisi S_n va yig'indisi S ni toping.

3.1 $\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-3}$.

3.2 $\sum_{n=1}^{\infty}\frac{1}{25n^2+5n-6}$.

3.3 $\sum_{n=1}^{\infty}\frac{1}{36n^2-24n-5}$.

3.4 $\sum_{n=1}^{\infty}\frac{1}{49n^2+7n-12}$.

3.5 $\sum_{n=1}^{\infty}\frac{1}{4n^2+4n-3}$.

3.6 $\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-15}$.

3.7 $\sum_{n=1}^{\infty}\frac{1}{36n^2+12n-35}$.

3.8 $\sum_{n=1}^{\infty}\frac{2n+1}{n^2\cdot(n+1)^2}$.

3.9 $\sum_{n=1}^{\infty}\frac{n}{(2n-1)\cdot(2n+1)^2}$.

3.10 $\sum_{n=1}^{\infty}\frac{n-\sqrt{n^2-1}}{\sqrt{n\cdot(n+1)}}$.

3.11 $\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$.

3.12 $\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$.

3.13 $\sum_{n=2}^{\infty}\ln\left(1-\frac{2}{n(n+1)}\right)$.

3.14 $\sum_{n=2}^{\infty}\ln\frac{n^3-1}{n^3+1}$.

3.15 $\sum_{n=1}^{\infty}\sin\frac{1}{2^n}\cos\frac{3}{2^n}$.

3.16 $\sum_{n=0}^{\infty}\ln\frac{1}{n!\cdot(n+2)}$.

3.17 $\sum_{n=1}^{\infty}arctg\frac{1}{2n^2}$.

3.18 $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^{n-1}}$.

3.19 $\sum_{n=1}^{\infty}\frac{2n-1}{2^n}$.

3.20 $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

3.21 $\sum_{n=1}^{\infty}\sin\frac{\alpha}{2^{n+1}}\sin\frac{3\alpha}{2^{n+1}}$.

4-masala.

Koshi kriteriyasidan foydalanib umumiy hadi a_n ga teng bo'lgan $\sum_{n=1}^{\infty}a_n$ qatorning yaqinlashuvchi ekanligini isbotlang.

4.1 $a_n=\frac{\cos n\alpha}{3^n}$.

4.2 $a_n=\frac{\sin n\alpha}{n\cdot(n+1)}$.

4.3 $a_n=\frac{1}{n^2}$.

4.4 $a_n=\frac{\cos\alpha^n}{n^2}$.

4.5 $a_n=\frac{\cos n\alpha-\cos(n+1)\alpha}{n}$.

4.6 $a_n=b_0+\frac{b_1}{10}+...+\frac{b_n}{10}+...(|b_n|<10)$.

4.7 $a_n=\frac{\sin n\alpha}{2^n}$.

4.8 $a_n=\frac{\sin^3 n\alpha}{(n+1)(n+3)}$.

4.9 $a_n = \frac{2}{n^2 \cdot \sqrt{n}}$.

4.10 $a_n = \frac{n-1}{n^3+1}$.

4.11 $a_n = \frac{1}{n}\sin\frac{\alpha}{n}$.

4.12 $a_n = \frac{1}{n^2+n+1}$.

Koshi kriteriyasidan foydalanih, umumiy hadi a_n ga teng bo'lgan $\sum_{n=1}^{\infty} a_n$ qatorning uzoqlashuvchi ekanligini isbotlang.

4.13 $a_n = \frac{1}{2n+1}$.

4.14 $a_n = \frac{n-1}{n^2+1}$.

4.15 $a_n = \frac{n+1}{n^2+4}$.

4.16 $a_n = \frac{arctgn}{n}$.

4.17 $a_n = \ln\left(1+\frac{1}{n}\right)$.

4.18 $a_n = \frac{1}{\sqrt{n^2+1}}$.

4.19 $a_n = \frac{1}{\sqrt{n}}$.

4.20 $a_n = \frac{1}{3n+2}$.

4.21 $a_n = \frac{1}{\sqrt{n \cdot (n+1)}}$.

5-masala. Qatorning yaqinlashishga tekshiring.

5.1 $\sum_{n=1}^{\infty} \frac{\sin^2 n\sqrt{n}}{n\sqrt{n}}$.

5.2 $\sum_{n=1}^{\infty} n \sin\frac{2+(-1)^n}{n^3}$.

5.3 $\sum_{n=1}^{\infty} \frac{\cos^2 \frac{n\pi}{2}}{n(n+1)(n+2)}$.

5.4 $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt[3]{n^7}}$.

5.5 $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n-\ln n}$.

5.6 $\sum_{n=1}^{\infty} \frac{arctg \frac{1+(-1)^n}{2} n}{n^3+2}$.

5.7 $\sum_{n=1}^{\infty} \frac{n(2+\cos n\pi)}{2n^2-1}$.

5.8 $\sum_{n=2}^{\infty} \frac{\arcsin \frac{n-1}{n}}{\sqrt[3]{n^3-3n}}$.

5.9 $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2+1}$.

5.10 $\sum_{n=2}^{\infty}\frac{\ln\sqrt{n^2+3n}}{\sqrt{n^2-n}}$.

5.11 $\sum_{n=1}^{\infty}\frac{\arccos\frac{(-1)^n\cdot n}{n+1}}{n^2+2}$.

5.12 $\sum_{n=1}^{\infty}\frac{n\cdot\cos^2 n}{n^3+5}$.

5.13 $\sum_{n=1}^{\infty}\frac{n\ln n}{n^2-3}$.

5.14 $\sum_{n=1}^{\infty}\frac{n^2+3}{n^3\cdot\left(2+\sin\frac{n\pi}{2}\right)}$.

5.15 $\sum_{n=2}^{\infty}\frac{1}{\sqrt[4]{n^3}}\sin\frac{2+(-1)^n}{6}\pi$.

5.16 $\sum_{n=1}^{\infty}\frac{\ln n}{n^3+n+1}$.

5.17 $\sum_{n=1}^{\infty}\frac{1+\sin\frac{\pi n}{2}}{n^2}$.

5.18 $\sum_{n=1}^{\infty}\frac{\cos^2\frac{\pi n}{3}}{3^n+2}$.

5.19 $\sum_{n=1}^{\infty}\frac{2+\sin\frac{\pi n}{4}}{n^2}\cdot ctg\frac{1}{\sqrt{n}}$.

5.20 $\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n^5+n}}$.

5.21 $\sum_{n=1}^{\infty}\frac{\left(2+\cos\frac{\pi n}{2}\right)\sqrt{n}}{\sqrt[4]{n^7+5}}$.

6-masala. Yaqinlashishga tekshiring.

6.1 $\sum_{n=1}^{\infty}\frac{2}{5^{n-1}+n-1}$.

6.2 $\sum_{n=1}^{\infty}\frac{1}{n}\cdot tg\frac{1}{\sqrt{n}}$.

6.3 $\sum_{n=1}^{\infty}\ln\frac{n^2+5}{n^2+4}$.

6.4 $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$.

6.5 $\sum_{n=1}^{\infty}\frac{1}{n-1}arctg\frac{1}{\sqrt[3]{n-1}}$.

6.6 $\sum_{n=1}^{\infty}\frac{(n^2+3)^2}{n^5+\ln^4 n}$.

6.7 $\sum_{n=1}^{\infty}\frac{n^3+2}{n^5+\sin 2^n}$.

6.8 $\sum_{n=1}^{\infty}\frac{2^n+\cos n}{3^n+\sin n}$.

6.9 $\sum_{n=1}^{\infty}\frac{1}{n-\cos^2 6n}$.

6.10 $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n+1}}\sin\frac{1}{\sqrt{n}}$.

6.11 $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}arctg\frac{\pi}{4\sqrt{n}}$.

6.12 $\sum_{n=1}^{\infty}\frac{1}{n^2-\ln n}$.

6.13 $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n+5}}\sin\frac{1}{n-1}$.

6.14 $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}+2}arctg\frac{n+3}{n^2+5}$.

6.15 $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}\left(e^{1/\sqrt{n}}-1\right)$.

6.16 $\sum_{n=1}^{\infty}\ln\frac{n^2+1}{n^2+n+2}$.

6.17 $\sum_{n=1}^{\infty}\sqrt[3]{n}arctg\frac{1}{n^3}$.

6.18 $\sum_{n=1}^{\infty}\ln\frac{n^3}{n^3+1}$.

6.19 $\sum_{n=1}^{\infty}n^3tg^5\frac{\pi}{n}$.

6.20 $\sum_{n=1}^{\infty}\sin\frac{\sqrt[3]{n}}{\sqrt{n^5+2}}$..

6.21 $\sum_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right)$.

7-masala. Yaqinlashishga tekshiring.

7.1 $\sum_{n=2}^{\infty}\frac{n+1}{2^n\cdot(n-1)!}$.

7.2 $\sum_{n=1}^{\infty}\frac{(n!)^2}{2^{n^2}}$.

7.3 $\sum_{n=1}^{\infty}\frac{2^{n+1}\cdot(n^3+1)}{(n+1)!}$.

7.4 $\sum_{n=1}^{\infty}\frac{10^n\cdot 2n!}{(2n)!}$.

7.5 $\sum_{n=1}^{\infty}\frac{(2n+2)!}{3n+5}\cdot\frac{1}{2^n}$.

7.6 $\sum_{n=1}^{\infty}\frac{n+5}{n!}\sin\frac{2}{3^n}$.

7.7 $\sum_{n=1}^{\infty}\frac{arctg\frac{5}{n}}{n!}$

7.8 $\sum_{n=1}^{\infty}\frac{n^n}{3^n\cdot n!}$.

7.9 $\sum_{n=1}^{\infty}\frac{6^n\cdot(n^2-1)}{n!}$.

7.10 $\sum_{n=1}^{\infty}\frac{n^2}{(n+2)!}$.

7.11 $\sum_{n=1}^{\infty}\frac{n!}{(n!)^2}$.

7.12 $\sum_{n=1}^{\infty}\frac{7^{2n}}{(2n-1)!}$.

7.13 $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{3^n\cdot(n+1)!}$.

7.14 $\sum_{n=1}^{\infty}\frac{n!}{n^{n-1}}$.

7.15 $\sum_{n=1}^{\infty}\frac{(n!)^2}{(3^n+1)\cdot(2n)!}$.

7.16 $\sum_{n=1}^{\infty}n!\cdot\sin\frac{\pi}{2^n}$

7.17 $\sum_{n=1}^{\infty}\frac{(n+1)!}{n^n}$.

7.18 $\sum_{n=1}^{\infty}\frac{5^n\cdot\sqrt[3]{n^2}}{(n+1)!}$.

7.19 $\sum_{n=1}^{\infty}\frac{3^n}{(n+2)!\cdot4^n}$.

7.20 $\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdot\ldots\cdot(2n+1)}{2\cdot5\cdot8\cdot\ldots\cdot(3n-1)}$.

7.21 $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}\cdot tg\frac{1}{5^n}$.

8-masala. Yaqinlashishga tekshiring.

8.1 $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}\cdot\frac{1}{4^n}$.

8.2 $\sum_{n=1}^{\infty}\left(\frac{2n+2}{3n+1}\right)^n(n+1)^3$.

8.3 $\sum_{n=1}^{\infty}\left(\frac{2n^2+1}{n^2+1}\right)^{n^2}$.

8.4 $\sum_{n=1}^{\infty}\left(\frac{4n-3}{5n+4}\right)^{n^2}$.

8.5 $\sum_{n=1}^{\infty}n^4\cdot\left(\frac{2n}{3n+5}\right)^n$.

8.6 $\sum_{n=1}^{\infty}n\arcsin^n\frac{\pi}{4n}$.

8.7 $\sum_{n=1}^{\infty}\left(\frac{2n+1}{3n-2}\right)^{n^2}$.

8.8 $\sum_{n=1}^{\infty}\left(\frac{n+2}{3n-1}\right)^{n^2}$.

8.9 $\sum_{n=1}^{\infty}\left(\frac{n-1}{n}\right)^n\cdot\frac{n}{5^n}$.

8.10 $\sum_{n=1}^{\infty}\frac{n^5\cdot3^n}{(2n+1)^n}$.

8.11 $\sum_{n=1}^{\infty}\frac{2^{n+1}}{n^n}$.

8.12 $\sum_{n=1}^{\infty}2^{n-1}\cdot e^{-n}$.

8.13 $\sum_{n=1}^{\infty}n^2\sin^n\frac{\pi}{2n}$.

8.14 $\sum_{n=1}^{\infty}n\cdot\left(\frac{3n-1}{4n+2}\right)^{2n}$.

8.15 $\sum_{n=2}^{\infty}\frac{n^3}{(\ln n)^n}$.

8.16 $\sum_{n=1}^{\infty}\frac{n^{n+2}}{(2n^2+1)^{n/2}}$.

8.17 $\sum_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{n^3}$.

8.18 $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n^3}\cdot\frac{1}{2^n}$.

8.19 $\sum_{n=1}^{\infty}n^3\cdot arctg^n\frac{\pi}{3n}$.

8.20 $\sum_{n=1}^{\infty}n^4\cdot arctg^{2n}\frac{\pi}{4n}$.

8.21 $\sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\left(\frac{n}{n+1}\right)^{-n^2}$.

9-masala. Yaqinlashishga tekshiring.

9.1 $\sum_{n=2}^{\infty}\frac{1}{n\ln^2(3n+1)}$.

9.2 $\sum_{n=1}^{\infty}\frac{1}{n\ln^2(2n+1)}$.

9.3 $\sum_{n=1}^{\infty}\frac{1}{(2n+3)\ln^2(2n+1)}$.

9.4 $\sum_{n=3}^{\infty}\frac{1}{(3n-5)\ln^2(4n-7)}$.

9.5 $\sum_{n=1}^{\infty}\frac{1}{(3n+4)\ln^2(5n+2)}$.

9.6 $\sum_{n=1}^{\infty}\frac{1}{(2n+1)\ln^2\left(n\sqrt{5}+2\right)}$.

9.7 $\sum_{n=1}^{\infty}\frac{1}{\left(n\sqrt{2}+1\right)\ln^2\left(n\sqrt{3}+2\right)}$.

9.8 $\sum_{n=5}^{\infty}\frac{1}{(n-2)\ln\ (n-3)}$.

9.9 $\sum_{n=1}^{\infty}\frac{1}{(2n-1)\ln\ (2n)}$.

9.10 $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln\ (2n)}$.

9.11 $\sum_{n=2}^{\infty}\frac{1}{(3n-1)\ln n}$.

9.12 $\sum_{n=2}^{\infty}\frac{1}{(2n-1)\ln\ (n+1)}$.

9.13 $\sum_{n=2}^{\infty}\frac{1}{(2n-3)\ln\ (3n+1)}$.

9.14 $\sum_{n=2}^{\infty}\frac{1}{(n+2)\ln^2 n}$.

9.15 $\sum_{n=2}^{\infty}\frac{1}{(n+3)\ln^2 2n}$.

9.16 $\sum_{n=2}^{\infty}\frac{1}{(2n+3)\ln^2(n+1)}$.

9.17 $\sum_{n=3}^{\infty}\frac{1}{n\ln\ (n-1)}$.

9.18 $\sum_{n=2}^{\infty}\frac{1}{2n\cdot\sqrt{\ln(3n-1)}}$.

9.19 $\sum_{n=5}^{\infty}\frac{1}{(n-2)\cdot\sqrt{\ln(n-3)}}$.

9.20 $\sum_{n=4}^{\infty}\frac{1}{(3n-1)\cdot\sqrt{\ln(n-1)}}$.

9.21 $\sum_{n=2}^{\infty}\frac{1}{(n+5)\ln^2(n+1)}$.

10-masala. Quyidagi tengliklarni isbotlang. Ko'rsatma. Qator yaqinlashishining zaruriy shartidan foydalaning.

10.1 $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

10.2 $\lim_{n\to\infty}\frac{(3n)!}{2^{n^2}}=0$.

10.3 $\lim_{n\to\infty}\frac{n^n}{(2n)!}=0$.

10.4 $\lim_{n\to\infty}\frac{n^n}{(n!)^3}=0$.

10.5 $\lim_{n\to\infty}\frac{(2n)!!}{n^n}=0$.

10.6 $\lim_{n\to\infty}\frac{(n+2)!}{n^n}=0$.

10.7 $\lim_{n\to\infty}\frac{(2n)^n}{(2n-1)!}=0$.

10.8 $\lim_{n\to\infty}\frac{n^n}{(2n-1)!}=0$.

10.9 $\lim_{n\to\infty}\frac{n^n}{(n!)^2}=0$.

10.10 $\lim_{n\to\infty}\frac{(2n+1)!!}{n^n}=0$.

10.11 $\lim_{n\to\infty}\frac{(2n)!!}{5^{n^2}}=0$.

10.12 $\lim_{n\to\infty}\frac{(4n)!}{2^{n^2}}=0$.

10.13 $\lim_{n\to\infty}\frac{n^2}{n!}=0$.

10.14 $\lim_{n\to\infty}\frac{n^n}{[(n+1)!]^2}=0$.

10.15 $\lim_{n\to\infty}\frac{(n+1)!}{n^n}=0$.

10.16 $\lim_{n\to\infty}\frac{(n+3)!}{n^n}=0$

10.17 $\lim_{n\to\infty}\frac{n^n}{(2n+1)!}=0$.

10.18 $\lim_{n\to\infty}\frac{n^n}{(2n+3)!}=0$.

10.19 $\lim_{n\to\infty}\frac{(3n)^n}{(2n-1)!}=0$.

10.20 $\lim_{n\to\infty}\frac{(2n+3)!!}{n^n}=0$.

10.21 $\lim_{n\to\infty}\frac{(2n-1)!!}{n^n}=0$.

11-masala. $\sum_{n=1}^{\infty} a_n$ **qator** α **ning qanday qiymatlarida yaqinlashishini aniqlang.**

11.1 $a_n = \left(1 - n\sin\frac{1}{n}\right)^{\alpha}$.

11.2 $a_n = n^{\alpha}\left[\ln\left(n^2+1\right) - 2\ln n\right]$..

11.3 $a_n = \left[arctg\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right]^{\alpha}$.

11.4 $a_n = \frac{\sqrt[3]{n^2+1} - \sqrt[3]{n^2-1}}{n^{\alpha}}$.

11.5 $a_n = \left(e^{tg\frac{1}{n}} - 1\right)^{\alpha}$.

11.6 $a_n = \left(\frac{1}{n\cdot\sin\frac{1}{n}} - \cos\frac{1}{n}\right)^{\alpha}$.

11.7 $a_n = \left(e^{\frac{1}{n}\cos\frac{1}{n}} - 1 - \frac{1}{n}\right)^{\alpha}$.

11.8 $a_n = \left|\ln\left(arctg\frac{1}{n}\right) - \ln\left(tg\frac{1}{n}\right)\right|^{\alpha}$.

11.9 $a_n = \left(\cos\frac{1}{\sqrt{n}} - \frac{\sqrt{n^2-n}}{n}\right)^{\alpha}$.

11.10 $a_n = \left(1 - \sin\frac{\pi n^2}{2n^2+1}\right)^{\alpha}$.

11.11 $a_n = n\cdot\sin^{\alpha}\left(\frac{1}{n} - arctg\frac{1}{n}\right)$.

11.12 $a_n = \left[1 - \left(\cos\frac{1}{n}\right)^{\frac{1}{n}}\right]^{\alpha}$.

11.13 $a_n = \left(e^{\frac{1}{2n^2}} - \cos\frac{1}{n}\right)^{\alpha}$.

11.14 $a_n = \frac{\left[e - \left(1+\frac{1}{n}\right)^n\right]^{\alpha}}{\left(1-\cos\frac{1}{n}\right)^2}$.

11.15 $a_n = \left|\ln n + \ln\left(\sin\frac{1}{n}\right)\right|^{\alpha}$.

11.16 $a_n = \left(\sqrt[4]{n^2+n+1} - \sqrt{n+\frac{1}{2}}\right)^{\alpha}$.

11.17 $a_n = \left|\ln\frac{n+1}{n-1} - \frac{2}{n-1}\right|^{\alpha}, n \geq 2$.

11.18 $a_n = \left(\frac{1}{\sqrt{n}} - \sqrt{\ln\frac{n+1}{n}}\right)^{\alpha}$.

11.19 $a_n=\left(n\sin\frac{1}{n}-\cos\frac{1}{n\sqrt{3}}\right)^{\alpha}$. **11.20** $a_n=\left(\frac{\pi}{2n+2}-\cos\frac{\pi n}{2n+2}\right)^{\alpha}$.

11.21 $a_n=\left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha}\cdot\ln\frac{2n+1}{2n-1}$.

12-masala. Yaqinlashishga tekshiring.

12.1 $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n\cdot(n+1)}$.

12.2 $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\left(\frac{n}{2n+1}\right)^{n}$.

12.3 $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln(n+1)}$.

12.4 $\sum_{n=3}^{\infty}\frac{(-1)^{n}}{n\cdot(\ln\ln n)\cdot\ln n}$.

12.5 $\sum_{n=2}^{\infty}\frac{(-1)^{n}\cdot 2n^2}{n^4-n^2+1}$.

12.6 $\sum_{n=3}^{\infty}\frac{(-1)^{n}}{(n+1)\cdot\ln n}$.

12.7 $\sum_{n=3}^{\infty}\frac{(-1)^{n}}{n\cdot\ln(n+1)}$.

12.8 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot\sqrt[4]{2n+3}}$.

12.9 $\sum_{n=1}^{\infty}\frac{(-1)^{n}\cdot\sin\frac{\pi}{2\sqrt{n}}}{\sqrt{3n+1}}$.

12.10 $\sum_{n=1}^{\infty}(1-)^{n}\cdot\cos\frac{\pi}{6n}$.

12.11 $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\cdot\ln(2n)}$.

12.12 $\sum_{n=1}^{\infty}(-1)^{n}\cdot tg\frac{1}{n}$.

12.13 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)\cdot 2^{2n}}$.

12.14 $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\cos\frac{\pi}{3\sqrt{n}}\cdot\sqrt[3]{3n+\ln n}}$.

12.15 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)\cdot\left(\frac{3}{2}\right)^{n}}$.

12.16 $\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{2n-1}{3n}$.

12.17 $\sum_{n=1}^{\infty}\frac{(-1)^{n}\cdot(n+3)}{\ln(n+4)}$.

12.18 $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)\cdot 2^{2n+2}}$.

12.19 $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n+1}{\sqrt{n^3}}$.

12.20 $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{\sin\left(n\sqrt{n}\right)}{n\sqrt{n}}$.

12.21 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot tg\frac{\pi}{4\sqrt{n}}}{\sqrt{5n-1}}$.

13-masala. Quyidagi qatorlarning absolut yaqinlashuvchi ekanligini isbotlang.

13.1 $\sum_{n=1}^{\infty}\frac{(n+1)\cos 2n}{\sqrt[3]{n^7+3n+4}}$.

13.2 $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{\sqrt[6]{n}}\right)\cdot arctg\frac{\sin n}{n}$.

13.3 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln^2(n+1)}\cdot\left(1-\cos\frac{1}{\sqrt{n}}\right)$.

13.4 $\sum_{n=1}^{\infty}\frac{\sin\left(2n+\frac{\pi}{4}\right)}{n\cdot\sqrt[3]{n+2}}$.

13.5 $\sum_{n=1}^{\infty}\frac{arctg(-n)^n}{\sqrt[4]{2n^6+3n+1}}$.

13.6 $\sum_{n=1}^{\infty}\frac{\cos\frac{\pi n}{4}}{(n+2)\sqrt{\ln^3(n+3)}}$.

13.7 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\ln^2 n}{2^n}$.

13.8 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[5]{n}}\cdot\arcsin\frac{\pi}{4n}$.

13.9 $\sum_{n=1}^{\infty}\cos^3 n\cdot arctg\frac{n+1}{n^3+2}$.

13.10 $\sum_{n=1}^{\infty}n^3\sin n\cdot e^{-\sqrt{n}}$.

13.11 $\sum_{n=1}^{\infty}\frac{(-n)^n}{(2n)!}$.

13.12 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(2n)!!}{(n+1)^n}$.

13.13 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n^2(n+1)}{n\sqrt{n+1}}$.

13.14 $\sum_{n=1}^{\infty}\left(\frac{1}{n\cdot\sin\frac{1}{n}}-\cos\frac{1}{n}\right)\cdot\cos\pi n$.

13.15 $\sum_{n=1}^{\infty}(-1)^{\frac{n\cdot(n+1)}{2}}\cdot\frac{2^n+n^2}{3^n+n^3}$.

13.16 $\sum_{n=1}^{\infty}(-1)^n\left(arctg\frac{1}{\sqrt{n}}-\arcsin\frac{1}{\sqrt{n}}\right)$.

13.17 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\sin 3n}{n\cdot\ln(n+1)\cdot\ln^2(n+2)}$.

13.18 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}arctg\frac{1}{2n}$.

13.19 $\sum_{n=1}^{\infty}\left(\frac{\sin n}{\sqrt[3]{n^2}}-\sin\left(\frac{\sin n}{\sqrt[3]{n^2}}\right)\right)$.

13.20 $\sum_{n=1}^{\infty}n^2\cdot\cos n\cdot e^{-\sqrt[3]{n}}$.

13.21 $\sum_{n=1}^{\infty}\sqrt{\frac{n^2+3}{n^3+4n}}\cdot\ln\left[1+\frac{(-1)^n}{n}\right]$.

14-masala. Quyidagi qatorlarni yaqinlashishga tekshiring.

14.1 $\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\cdot\frac{1}{\sqrt{n}}$.

14.2 $\sum_{n=1}^{\infty}\frac{\sin n}{\sqrt[5]{n}}$.

14.3 $\sum_{n=1}^{\infty}\frac{\cos 2n}{\sqrt{n}}$.

14.4 $\sum_{n=1}^{\infty}\frac{\cos 3n}{\sqrt[3]{n}}$.

14.5 $\sum_{n=1}^{\infty}\frac{\sin 2n}{\sqrt[3]{2n}}$.

14.6 $\sum_{n=1}^{\infty}\frac{\cos\left(n+\frac{\pi}{4}\right)}{\ln^2(n+1)}$.

14.7 $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{\cos^2 2n}{\sqrt{n}}$.

14.8 $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{\sin^2\frac{n}{2}}{\sqrt[5]{n+1}}$.

14.9 $\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\cdot\frac{1}{\sqrt{n+1}}\cdot\left(1+\frac{2}{n}\right)^n$.

14.10 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n+1}}$.

14.11 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\ln n}{\sqrt{n}}$.

14.12 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\ln\ln(n+2)}{\ln(n+1)}$.

14.13 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n}{(n+2)\cdot\sqrt[4]{n+1}}$.

14.14 $\sum_{n=1}^{\infty}\cos\left(\frac{\pi}{4}+\pi n\right)\cdot\sin\frac{1}{n}$.

14.15 $\sum_{n=1}^{\infty}(-1)^n\cdot\left(1-\cos\frac{\pi}{\sqrt{n}}\right)$.

14.16 $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{\sqrt[n]{n^2+1}}$.

14.17 $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n+2}{\sqrt{n^2+4}}arctg\frac{\pi}{\sqrt{n}}$.

14.18 $\sum_{n=1}^{\infty}\frac{\sin^3 n}{\sqrt{n}}$.

14.19 $\sum_{n=1}^{\infty}\frac{\sin n}{\sqrt{n}+\sin n}$.

14.20 $\sum_{n=1}^{\infty}\sin\left(\pi\sqrt{n^2+1}\right)$..

14.21 $\sum_{n=1}^{\infty}\frac{\sin 2n}{\ln\ln(n+2)}\cdot\cos\frac{1}{n}$.

15-masala. Quyidagi qatorlar α ning qanday qiymatlarida
a) absolut yaqinlashuvchi,
b) shartli yaqinlashuvchi bo'lishini aniqlang.

15.1 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\sin^{2n}\alpha}{n}$.

15.2 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\cdot\cos^{2n}\alpha}{\sqrt{n}}$.

15.3 $\sum_{n=1}^{\infty}\frac{\sin 2n\cdot\ln^2 n}{n^\alpha}$.

15.4 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\left[\sqrt{n+1}+(-1)^n\right]^\alpha}$.

15.5 $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\left[\frac{(2n-1)!!}{(2n)!!}\right]^\alpha$.

15.6 $\sum_{n=2}^{\infty}\frac{(-1)^n}{\left[\sqrt[3]{n}+(-1)^n\right]^\alpha}$

15.7 $\sum_{n=1}^{\infty}\frac{\alpha\cdot(\alpha-1)(\alpha-2)\cdot\ldots\cdot[\alpha-(n-1)]}{n!}$.

15.8 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[n\ln n+(-1)^n\right]^\alpha}$.

15.9 $\sum_{n=1}^{\infty}\frac{\sin\frac{n}{2}}{n\ln^\alpha(n+1)}$.

15.10 $\sum_{n=1}^{\infty}(-1)^n\frac{2+(-1)^n}{n\cdot\ln^\alpha(n+1)}$.

15.11 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\alpha}$.

15.12 $\sum_{n=1}^{\infty}\frac{\sin n}{n^\alpha}$.

15.13 $\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\alpha}.$

15.14 $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^\alpha+\frac{(-1)^n}{n^\alpha}}.$

15.15 $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\alpha+\frac{1}{n}}}.$

15.16 $\sum_{n=1}^{\infty}\frac{\alpha\cdot(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-2n)}{(2n+2)!!}.$

15.17 $\sum_{n=1}^{\infty}\ln\left[1+\frac{(-1)^{n-1}}{(n+1)^\alpha}\right].$

15.18 $\sum_{n=1}^{\infty}(-1)^n\left[\frac{\sqrt{n^2+n-1}-\sqrt{n^2-n+1}}{n}\right]^\alpha.$

15.19 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[2n+(-1)^n\right]^\alpha}.$

15.20 $\sum_{n=2}^{\infty}\frac{(-1)^n}{\left[n^2\ln n+(-1)^n\right]^\alpha}.$

15.21 $\sum_{n=1}^{\infty}\frac{\cos n}{n^\alpha}.$

16-masala.
Tengliklar isbotlansin.

16.1 $\prod_{n=0}^{\infty}\left[1+\left(\frac{1}{2}\right)^{2^n}\right]=2.$

16.2 $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}.$

16.3 $\prod_{n=1}^{\infty}\cos\frac{\pi}{2^{n+1}}=\frac{2}{\pi}.$

16.4 $\prod_{n=2}^{\infty}\left[1-\frac{2}{n\cdot(n+1)}\right]=\frac{1}{3}.$

16.5 $\prod_{n=1}^{\infty}\cos\frac{x}{2^n}=\frac{\sin x}{x}.$

16.6 $\prod_{n=0}^{\infty}\left(1+x^{2^n}\right)=\frac{1}{1-x}.\ (|x|<1).$

16.7 $\frac{\pi}{2}=\frac{2}{\sqrt{2}}\cdot\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdot\ldots$

16.8 $\prod_{n=1}^{\infty}\left(\frac{3n}{3n-1}\cdot\frac{3n}{3n+1}\right)=\frac{2\pi}{3\sqrt{3}}.$

16.9 $\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$ (Vallis formulasi)

16.10 $\prod_{n=1}^{\infty}\left[1-\frac{1}{(2n+1)^2}\right]=\frac{\pi}{4}.$

16.11 $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)=\frac{2}{\pi}.$

16.12 $\prod_{n=1}^{\infty}ch\frac{x}{2^n}=\frac{shx}{x}.$

Quyidagi cheksiz ko'paytmalarning yaqinlashuvchiligini isbotlang va ularning qiymatlarini toping.

16.13 $\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}$. **16.14** $\prod_{n=1}^{\infty}\frac{(2n+1)(2n+7)}{(2n+3)(2n+5)}$.

16.15 $\prod_{n=1}^{\infty}\left[1+\frac{1}{n(n+2)}\right]$. **16.16** $\prod_{n=1}^{\infty}2^{\frac{(-1)^n}{n}}$.

Quyidagi cheksiz ko'paytmalarni absolut va shartli yaqinlashishga tekshiring.

16.17 $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]$. **16.18** $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{\sqrt{n}}\right]$.

16.19 $\prod_{n=2}^{\infty}\frac{\sqrt{n}}{\sqrt{n}+(-1)^n}$. **16.20** $\prod_{n=2}^{\infty}\left[1+\frac{(-1)^n}{\ln n}\right]$.

16.21 $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n^p}\right]$.

-D-

Namunaviy variant yechimi

1.21-masala. Ushbu

$$\sum_{n=1}^{\infty}\frac{3}{9n^2-3n-2}$$

qator yig'indisini toping.

$$\triangleleft\quad a_n=\frac{3}{9n^2-3n-2}=\frac{3}{9\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right)}=\frac{3}{(3n+1)\cdot(3n-2)}=\frac{1}{3n-2}-\frac{1}{3n+1}\Rightarrow$$

$$\Rightarrow S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)=1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\ldots+\frac{1}{3n-2}-\frac{1}{3n+1}=1-\frac{1}{3n+1}.$$

Demak, $S:=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\frac{1}{3n+1}\right)=1.\ \triangleright$

2.21-masala. Ushbu

$$\sum_{n=2}^{\infty}\frac{5n-2}{(n-1)n(n+2)};$$

qator yig'indisini toping.

◁ Birinchi navbatda bu qatorning umumiy hadini noma'lum koeffitsientlar usulidan foydalanib, sodda kasrlarga yoyamiz:

$$a_n=\frac{5n-2}{(n-1)n(n+2)}=\frac{1}{n-1}+\frac{1}{n}-\frac{2}{n+2}=\left(\frac{1}{n-1}-\frac{1}{n}\right)+2\cdot\left(\frac{1}{n}-\frac{1}{n+2}\right)=b_n+c_n.$$

$$S_n=\sum_{k=2}^{n}a_k=\sum_{k=2}^{n}(b_k+c_k)=\sum_{k=2}^{n}b_k+\sum_{k=2}^{n}c_k=\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)+2\sum_{k=2}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)=$$

$$\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n-1}-\frac{1}{n}\right)+2\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{n}-\frac{1}{n+2}\right)=$$

$$\left(1-\frac{1}{n}\right)+2\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{8}{3}-\frac{1}{n}-\frac{2}{n+1}-\frac{2}{n+2}.\Rightarrow S=\lim_{n\to\infty}S_n=\frac{8}{3}=2\frac{2}{3}.\ \triangleright$$

3.21-masala. Quyidagi

$$\sum_{n=1}^{\infty}\sin\frac{\alpha}{2^{n+1}}\cdot\sin\frac{3\alpha}{2^{n+1}}.$$

qatorning qismiy yig'indisi S_n va yig'indisi S ni toping.

◁ Bu masalani yechishda

$$\sin x\cdot\sin y=\frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right];$$

formuladan foydalanamiz.

$$S_n=\sum_{k=1}^{n}\sin\frac{\alpha}{2^{k+1}}\cdot\sin\frac{3\alpha}{2^{k+1}}=\frac{1}{2}\sum_{k=1}^{n}\left(\cos\frac{\alpha}{2^k}-\cos\frac{\alpha}{2^{k-1}}\right)=$$

$$=\frac{1}{2}\left(\cos\frac{\alpha}{2}-\cos\alpha+\cos\frac{\alpha}{2^2}-\cos\frac{\alpha}{2}+\cos\frac{\alpha}{2^3}-\cos\frac{\alpha}{2^2}+\ldots+\cos\frac{\alpha}{2^n}-\cos\frac{\alpha}{2^{n-1}}\right)=$$

$$\frac{1}{2}\left(\cos\frac{\alpha}{2^n}-\cos\alpha\right)\Rightarrow S=\lim_{n\to\infty}S_n=\frac{1}{2}(1-\cos\alpha)=\sin^2\frac{\alpha}{2}.\ \triangleright$$

4.21-masala. Koshi kriteriyasidan foydalanib, umumiy hadi

$$a_n=\frac{1}{\sqrt{n(n+1)}}$$

bo'lgan $\sum_{n=1}^{\infty}a_n$ qatorning uzoqlashuvchi ekanligini isbotlang.

◁ Ma'lumki, $\exists\varepsilon>0$ son topilsaki , $\forall n_0\in N$ olinganda ham $\exists n\geq n_0$ va butun $p\geq 0$ sonlar mavjud bo'lib,

$$\left|\sum_{k=n}^{n+p} a_k\right| \geq \varepsilon$$

tengsizlik bajarilsa, unda $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo'ladi.

Agar $\varepsilon = 1$ va $p = n$ deb olsak, unda $\forall n_0 \in N$ olinganda ham $\exists n > n_0$ topiladiki va

$$\left|\sum_{k=n}^{n+p} a_k\right| = \sum_{k=n}^{n+p} \frac{1}{\sqrt{k\cdot(k+1)}} = \frac{1}{\sqrt{n\cdot(n+1)}} + \frac{1}{\sqrt{(n+1)\cdot(n+2)}} + \ldots + \frac{1}{\sqrt{(n+p)(n+p+1)}} >$$

$> \dfrac{p}{\sqrt{(n+p)(n+p+1)}} > \dfrac{p}{n+p+1} \geq \dfrac{p}{2n+p} \overset{p=n}{=} \dfrac{n}{2n+n} = \dfrac{1}{3} = \varepsilon$ bo'ladi $\Rightarrow \sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi. ▷

5.21-masala. $\sum_{n=1}^{\infty} \dfrac{\left(2+\cos\frac{n\pi}{2}\right)\sqrt{n}}{\sqrt[4]{n^7}+5}$ **qatorni yaqinlashishga tekshiring.**

◁ $a_n = \dfrac{\left(2+\cos\frac{n\pi}{2}\right)\sqrt{n}}{\sqrt[4]{n^7}+5} < \dfrac{(2+1)\cdot\sqrt{n}}{\sqrt[4]{n^7}} = \dfrac{3}{n^{5/4}} = b_n$ bo'lib, $\sum_{n=1}^{\infty} b_n = 3\sum_{n=1}^{\infty} \dfrac{1}{n^{5/4}}$

qator yaqinlashuvchi, chunki $\dfrac{5}{4} > 1$. Unda taqqoslash alomatiga ko'ra berilgan qator ham yaqinlashuvchi. ▷

◁ **6.21-masala.** $\sum_{n=1}^{\infty}\left(1-\cos\dfrac{\pi}{n}\right)$ **qatorni yaqinlashishga tekshiring.**

$a_n = 1-\cos\dfrac{\pi}{n} = 2\cdot\sin^2\dfrac{\pi}{2n}$. Agar $b_n = \dfrac{1}{n^2}$ desak, x_0 da $a_n = 0^*(b_n)$ bo'ladi. Darhaqiqat, M $(M \subset R)$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ -yaqinlashuvchi $\Rightarrow$ **taqqoslash alomatiga** ko'ra $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(1-\cos\dfrac{\pi}{n}\right)$ qator ham yaqinlashuvchi. ▷

7.21-masala. $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}tg\frac{1}{5^n}$ **qatorni yaqinlashishga tekshiring.**

◁ Dalamber alomatidan foydalanib, tekshiramiz.

$$a_n=\frac{n!}{(2n)!}tg\frac{1}{5^n};$$

$$a_{n+1}=\frac{(n+1)!}{(2n+2)!}tg\frac{1}{5^{n+1}}=\frac{n!\cdot(n+1)}{(2n)!\cdot(2n+1)\cdot[2n+2]}tg\frac{1}{5^{n+1}}\Rightarrow;$$

$$a=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2\cdot(2n+1)}\cdot\frac{tg\frac{1}{5^{n+1}}}{tg\frac{1}{5^n}}=\frac{1}{10}\lim_{n\to\infty}\frac{1}{2n+1}=0<1\Rightarrow$$

Berilgan qator yaqinlashuvchi. ▷

8.21-masala. $|f(x)-f_n(x)|<\varepsilon$ **qatorni yaqinlashishga tekshiring.**

◁Koshi alomatidan foydalanib, tekshiramiz:

$$q=\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{1}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{-n}=\frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\frac{e}{3}<1\Rightarrow$$

Berilgan qator yaqinlashuvchi. ▷

9.21-masala. $\sum_{n=2}^{\infty}\frac{1}{(n+5)\ln^2(n+1)}$ **qatorni yaqinlashishga tekshiring.**

◁Bu qatorni yaqinlashishga taqqoslash va Koshining integral alomatlaridan foydalanib, tekshiramiz:

$$a_n=\frac{1}{(n+5)\ln^2(n+1)}<\frac{1}{(n+1)\ln^2(n+1)}<\frac{1}{n\cdot\ln^2 n}=b_n, n\geq 2.$$

$\int_2^{+\infty}\frac{dx}{x\ln^2 x}=f(x)$ bo‘lgani uchun Koshining integral alomatiga ko‘ra $\sum_{n=2}^{\infty}b_n$ qator yaqinlashuvchi va taqqoslash alomatiga ko‘ra berilgan qator ham yaqinlashuvchi. ▷

10.21-masala. Quyidagi

$$\lim_{n\to\infty}\frac{(2n-1)!!}{n^n}=0$$

tenglikni qator yaqinlashishining zaruriy shartidan foydalanib, isbotlang.

◁ Umumiy hadi $a_n = \frac{(2n-1)!!}{n^n}$ bo'lgan $\sum_{n=1}^{\infty} a_n$ qatorni yaqinlashishga tekshiramiz. Bunda Dalamber alomatidan foydalanamiz:

$$d = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!}\right] = \lim_{n\to\infty}\left[\frac{2n+1}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = \frac{2}{e} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$$

yaqinlashuvchi $\Rightarrow$ Qator yaqinlashishining zaruriy sharti, ya'ni $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n-1)!!}{n^n} = 0$ tenglik bajariladi. ▷

11.21-masala. $\sum_{n=1}^{\infty} a_n$ **qator α ning qanday qiymatlarida yaqinlashishini aniqlang.**

$$a_n = \left(\sqrt{n+1} - \sqrt{n}\right)^{\alpha} \cdot \ln\frac{2n+1}{2n-1}.$$

◁ $$\sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0^*\left(\frac{1}{\sqrt{n}}\right)$$ va

$$\ln\frac{2n+1}{2n-1} = \ln\left(1+\frac{2}{2n-1}\right) = 0^*\left(\frac{1}{n}\right) \Rightarrow a_n = 0^*\left(\frac{1}{n^{1+\frac{\alpha}{2}}}\right).$$

Agar $b_n = \frac{1}{n^{1+\frac{\alpha}{2}}}$ deb belgilasak, $\sum_{n=1}^{\infty} b_n$ qator $1+\frac{\alpha}{2} > 1$, ya'ni $\alpha > 0$ bo'lganda yaqinlashadi. $\Rightarrow \sum_{n=1}^{\infty} a_n$ qator ham $n \in N$ da yaqinlashadi. ▷

12.21-masala. $\sum_{n=1}^{\infty} \frac{(-1)^n tg\frac{\pi}{4\sqrt{n}}}{\sqrt{5n-1}}$ **qatorni yaqinlashishga tekshiring.**

◁ Bu qator ishorasi almashinuvchi qator bo'lib, uning yaqinlashishini Leybnis alomatidan foydalanib, ko'rsatish mumkin.

$$a_n=\frac{tg\dfrac{\pi}{4\sqrt{n}}}{\sqrt{5n-1}}$$ deb belgilasak

1) $\forall n\in N$ uchun $a_n\geq a_{n+1}>0$, ya'ni $\sum_{n=1}^{\infty}u_n(x)$ va

2) $$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{tg\dfrac{\pi}{4\sqrt{n}}}{\dfrac{\pi}{4\sqrt{n}}}\cdot\frac{1}{4\sqrt{n}\cdot\sqrt{5n-1}}=0;$$

bo'ladi $\Rightarrow$ Leybnis alomatiga ko'ra berilgan qator yaqinlashuvchi. $\triangleright$

13.21-masala. Quyidagi

$$\sum_{n=1}^{\infty}\sqrt{\frac{n^2+3}{n^4+4n}}\cdot\ln\left[1+\frac{(-1)^n}{n}\right]$$

qatorning absolut yaqinlashuvchi ekanligini isbotlang.

$\triangleleft$ $a_n=\sqrt{\dfrac{n^2+3}{n^4+4n}}\cdot\ln\left[1+\dfrac{(-1)^n}{n}\right]$ deb belgilaymiz. Unda quyidagi munosabatlar o'rinli bo'ladi.

$$|a_n|\leq\sqrt{\frac{n^2+3}{n^4+4n}}\cdot\ln\left(1+\frac{1}{n}\right)<\sqrt{\frac{n^2+3n^2}{n^4}}\cdot\frac{1}{n}=\frac{2}{n^2}=b_n$$

$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{2}{n^2}$ qator yaqinlashuvchi $\Rightarrow$ taqqoslash alomatiga ko'ra $x_0\in X$ yaqinlashuvchi, ya'ni berilgan $\sum_{n=1}^{\infty}a_n$ qator absolut yaqinlashuvchi. $\triangleright$

14.21-masala. Quyidagi

$$\sum_{n=1}^{\infty}\frac{\sin 2n}{\ln\ln(n+2)}\cdot\cos\frac{1}{n}$$

qatorni yaqinlashishga tekshiring.

$\triangleleft$Bu qatorning yaqinlashishini Abel alomati yordamida aniqlaymiz: $a_n=\cos\frac{1}{n}$ va $b_n=\dfrac{\sin 2n}{\ln\ln(n+2)}$ deb belgilab, Abel teoremasining shartlari bajarilishini tekshiramiz:

1) x_0 ketma-ketlik monoton (monoton o'suvchi) va chegaralangan $\left(0<\cos\frac{1}{n}<1\right)$;

2) $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{\sin 2n}{\ln\ln(n+2)}$ qator Dirixle alomatiga ko'ra yaqinlashuvchi bo'ladi. Darhaqiqat, agar

$$a_n'=\frac{1}{\ln\ln(n+2)} \text{ va } b_n=\sin 2n$$

deb belgilasak,

a) $\{a_n'\}\downarrow$ va $\forall x_0\in M$

b) $B_n=\sum_{k=1}^{n}b_k'=\sum_{k=1}^{n}\sin 2k=\frac{\sin(n+1)\cdot\sin n}{\sin 1}$ chegaralangan bo'ladi $\left(|B_n|\le\frac{1}{\sin 1}\right)\Rightarrow$ Dirixle alomatiga ko'ra $\sum_{n=1}^{\infty}u_n(x)$ qator yaqinlashuvchi.

Shunday qilib, berilgan qator uchun Abel teoremasining shartlari bajarilar ekan $\Rightarrow\sum_{n=1}^{\infty}a_nb_n=\sum_{n=1}^{\infty}\frac{\sin 2n}{\ln\ln(n+2)}\cos\frac{1}{n}$ qator yaqinlashuvchi. ▷

Izoh. Bu misolni yechishda elementar matematika kursidan ma'lum bo'lgan ushbu

$$S(x)=\sum_{k=1}^{n}\sin kx=\frac{\sin\frac{(n+1)x}{2}\cdot\sin\frac{nx}{2}}{\sin\frac{x}{2}},\ x\ne 2m\pi,\ m\in Z$$

formuladan foydalanildi.

15.21-masala. Ushbu

$$\sum_{n=1}^{\infty}\frac{\cos n}{n^{\alpha}}$$

qator α ning qanday qiymatlarida

a) absolut yaqinlashuvchi,

b) shartli yaqinlashuvchi bo'lishini aniqlang.

◁ Birinchi navbatda berilgan qator α ning qanday qiymatlarida yaqinlashuvchi bo'lishini aniqlaymiz. Bunda Dirixle alomatidan foydalanamiz. Agar

$a_n=\frac{1}{n^{\alpha}}$ va $b_n=\cos n$ deb belgilasak

1) $\alpha>0$ bo'lganda $\{a_n\}\downarrow$ va $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0$,

2) $B_n=\sum_{k=1}^{n} b_k=\dfrac{\cos\dfrac{n+1}{2}\cdot\cos\dfrac{n}{2}}{\sin\dfrac{1}{2}}$ va $|B_n|\le\dfrac{1}{\sin\dfrac{1}{2}}$ bo'ladi. $\Rightarrow$ Dirixle alomatiga ko'ra $\sum_{n=1}^{\infty} a_n b_n=\sum_{n=1}^{\infty}\dfrac{\cos n}{n^{\alpha}}$ qator $\alpha>0$ bo'lganda yaqinlashadi. $\alpha\le 0$ bo'lganda esa bu qator uzoqlashadi, chunki $\alpha\le 0$ bo'lganda qator yaqinlashishining zaruriy sharti bajarilmaydi.

Endi qatorni absolut yaqinlashishga tekshiramiz. $\left|\dfrac{\cos n}{n^{\alpha}}\right|\le\dfrac{1}{n^{\alpha}}$ va $\sum_{n=1}^{\infty}\dfrac{1}{n^{\alpha}}$ umumlashgan garmonik qatorning $\alpha>1$ da yaqinlashuvchi bo'lishidan $\alpha>1$ da $\sum_{n=1}^{\infty}\dfrac{|\cos n|}{n^{\alpha}}$ qatorning yaqinlashishini hosil qilamiz.

Endi $0<\alpha\le 1$ bo'lganda berilgan qatorning absolut yaqinlashuvchi emasligini, ya'ni $\sum_{n=1}^{\infty}\dfrac{|\cos n|}{n^{\alpha}}$ qatorning uzoqlashishini ko'rsatamiz.

$$\frac{|\cos n|}{n^{\alpha}}\ge\frac{\cos^2 n}{n^{\alpha}}=\frac{1+\cos 2n}{2n^{\alpha}}=\frac{1}{2n^{\alpha}}+\frac{\cos 2n}{2n^{\alpha}}$$

tengsizlik hamda $\sum_{n=1}^{\infty}\dfrac{\cos 2n}{2n^{\alpha}}$ qatorning Dirixle alomatiga ko'ra yaqinlashuvchi bo'lishi va x_0 qatorning uzoqlashuvchi ekanligidan $\{S_n(x)\}$: qatorning ham uzoqlashuvchi ekanligini, taqqoslash alomatiga ko'ra $S_n(x)=$ qatorning uzoqlashuvchiligini hosil qilamiz.

Shunday qilib, $\sum_{n=1}^{\infty}\dfrac{\cos n}{n^{\alpha}}$ qator

a) $\alpha>1$ da absolut yaqinlashuvchi,

b) $0<\alpha\le 1$ da shartli yaqinlashuvchi bo'lar ekan. ▷

16.21-masala. Quyidagi

$$\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n^{p}}\right]$$

cheksiz ko'paytmani absolut va shartli yaqinlashishga tekshiring.

$P_n = 1 + \frac{(-1)^{n+1}}{n^p} = 1 + a_n \Rightarrow a_n = \frac{(-1)^{n+1}}{n^p}$. 4^0 punktga ko'ra berilgan cheksiz ko'paytma absolut yaqinlashuvchi bo'lishi uchun $\sum_{n=1}^{\infty} a_n$ qatorning absolut yaqinlashuvchi bo'lishi zarur va yetarli.

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p} - \begin{cases} yaqinlashuvchi, & p>1, \\ uzoqlashuvchi, & p \le 1. \end{cases}$$

Cheksiz ko'paytmani shartli yaqinlashishga tekshirishda 4^0 – punktdagi 5-teoremadan foydalanamiz.

Unga ko'ra cheksiz ko'paytma yaqinlashuvchi bo'lishi uchun $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo'lgan holda $\sum_{n=1}^{\infty} a_n^2$ qatorning yaqinlashuvchi bo'lishi zarur va yetarli edi.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ qator $p>0$ bo'lganda Leybnis alomatiga ko'ra yaqinlashadi.

$a_n \ge 0$ qator esa $p > \frac{1}{2}$ da yaqinlashadi, $\sum_{n=1}^{\infty} u_n(x)$ da esa uzoqlashadi.

Shunday qilib, berilgan $\prod_{n=1}^{\infty}\left[1 + \frac{(-1)^{n+1}}{n^p}\right]$ cheksiz ko'paytma

a) $p > 1$ da absolut va

b) $\frac{1}{2} < p \le 1$ da shartli yaqinlashadi. ▷

7-§. 6-MUSTAQIL ISH
Funksional ketma-ketliklar va qatorlar

Funksional ketma-ketlik tushunchasi.
Funksional ketma-ketliklarning yaqinlashishi va tekis yaqinlashishi.
Funksional qator tushunchasi.
Funksional qatorlarning yaqinlashishi va tekis yaqinlashishi.
Funksional qator yig'indisi va funksional ketma-ketlik limitining xossalari.
Darajali qatorlar.
Teylor qatori. Elementar funksiyalarni Teylor qatoriga yoyish.
Darajali qatorlarning tatbiqlari.

-A-
Asosiy tushuncha va teoremalar
1⁰. Funksional ketma-ketliklar, ularning yaqinlashishi va tekis yaqinlashishi.

$X \subset R$ to'plam berilgan bo'lib, unda

$$f_1(x), f_2(x), ..., f_n(x), ...$$

funksiyalar aniqlangan bo'lsin. Ana shu funksiyalardan tuzilgan ketma-ketlikka X to'plamda berilgan **funksional ketma-ketlik** deyiladi va u $\{f_n(x)\}$ kabi belgilanadi:

$$\{f_n(x)\} : f_1(x), f_2(x), ..., f_n(x), ... \qquad (1)$$

$f_n(x)$ ga funksional ketma-ketlikning **umumiy hadi** deyiladi.

Ixtiyoriy $x_0 \in X$ nuqta olib, ushbu

$$\{f_n(x_0)\} : f_1(x_0), f_2(x_0), ..., f_n(x_0), ... \qquad (2)$$

sonli ketma-ketlikni qaraymiz. Agar bu sonli ketma-ketlik **yaqinlashuvchi (uzoqlashuvchi)** bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada **yaqinlashuvchi (uzoqlashuvchi)** deyiladi, x_0 nuqta esa funksional ketma-ketlikning **yaqinlashish (uzoqlashish) nuqtasi** deb ataladi.

$\{f_n(x)\}$ funksional ketma-ketlikning barcha yaqinlashish nuqtalaridan iborat M $(M \subset R)$ to'plam $\{f_n(x)\}$ funksional ketma-ketlikning **yaqinlashish sohasi deyiladi.** $\Rightarrow \forall x \in M$ uchun ushbu

$\lim\limits_{n\to\infty} f_n(x) - \exists$ bo'ladi. Agar $\forall x \in M$ uchun unga mos keluvchi $\lim\limits_{n\to\infty} f_n(x)$ ni mos qo'ysak, ya'ni

$$f : x \to \lim_{n\to\infty} f_n(x)$$

bo'lsa, unda M to'plamda aniqlangan $f(x)$ funksiya hosil bo'ladi. Bu $f(x)$ funksiya $\{f_n(x)\}$ ketma-ketlikning **limit funksiyasi** deyiladi. Demak,

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in M) \tag{3}$$

Ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham $\exists n_0 = n_0(\varepsilon) \in N : \forall n > n_0$ va $\forall x \in M$ uchun

$$|f(x) - f_n(x)| < \varepsilon \tag{4}$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik M to'plamda $f(x)$ limit funksiyaga **tekis yaqinlashadi** deyiladi va $f_n(x) \rightrightarrows f(x)$ $(x \in M)$ kabi belgilandi. Aks holda, ya'ni $\exists \varepsilon_0 > 0$ $\forall n \in N$ olinganda ham $\exists m > n$ va $\exists x_0 \in M$ lar mavjud bo'lsaki

$$|f(x_0) - f_m(x_0)| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik M to'plamda $f(x)$ limit funksiyaga **tekis yaqinlashmaydi** yoki **notekis yaqinlashadi** deyiladi.

1-teorema. $\{f_n(x)\}$ funksional ketma-ketlikning M toplamda $f(x)$ *ga tekis yaqinlashishi uchun*

$$\lim_{n\to\infty} \sup_{x\in M} |f(x) - f_n(x)| = 0 \tag{5}$$

tenglikning bajarilishi zarur va yetarli.

2-teorema. (Koshi kriteriyasi). $\{f_n(x)\}$ funksional ketma-ketlikning M to'plamda $f(x)$ ga tekis yaqinlashishi uchun quyidagi shartning bajarilishi zarur va yetarlidir: $\forall \varepsilon > 0$ uchun $\exists n_0 = n_0(\varepsilon) \in N : \forall n > n_0$ va $\forall$ butun $p \geq 0$ sonlari hamda barcha $x \in M$ lar uchun

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \tag{6}$$

tengsizlik bajariladi.

3-teorema. (Veyershtrass alomati). Agar $\exists\{a_n\}$ sonlar ketma-ketligi mavjud bo'lib,

1) $\forall n \in N$ uchun $a_n \geq 0$ va $\lim\limits_{n\to\infty} a_n = 0$;

2) $\forall x \in M$ va barcha $n \in N$ lar uchun

$$\left|f_{n+p}(x) - f_n(x)\right| \leq a_n$$

bo'lsa, unda M to'plamda $f_n \rightrightarrows f(x)$ bo'ladi.

2⁰. Funksional qatorlarning yaqinlashishi va tekis yaqinlashishi.

Biror $X \subset R$ to'plamda $\{u_n(x)\}$ funksional ketma-ketlik berilgan bo'lsin. Quyidagi

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

ifodaga **funksional qator** deyiladi va u $\sum\limits_{n=1}^{\infty} u_n(x)$ kabi belgilanadi.

$$\sum_{n=1}^{\infty} u_n(x_0) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \qquad (7)$$

$u_1(x), u_2(x), \ldots, u_n(x), \ldots$ larga funksional qatorning **hadlari,** $u_n(x)$ ga esa funksional qatorning **umumiy hadi** deyiladi.

Ixtiyoriy $x_0 \in X$ nuqta olib, ushbu

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x_0) + u_2(x_0) + \ldots + u_n(x_0) + \ldots \qquad (8)$$

sonli qatorni qaraymiz. Agar bu sonli qator **yaqinlashuvchi (uzoqlashuvchi)** bo'lsa, $\sum\limits_{n=1}^{\infty} u_n(x)$ funksional qator x_0 nuqtada **yaqinlashuvchi (uzoqlashuvchi)** deyiladi, x_0 nuqta esa funksional qatorning **yaqinlashish (uzoqlashish) nuqtasi** deb ataladi.

$\sum\limits_{n=1}^{\infty} u_n(x)$ funksional qatorning barcha yaqinlashish nuqtalaridan iborat M $(M \subset R)$ to'plam bu funksional qatorning **yaqinlashish sohasi** deyiladi. $\Rightarrow$ $\forall x_0 \in M$ nuqta olib, $\sum\limits_{n=1}^{\infty} u_n(x_0)$ sonli qatorni ko'rsak, u yaqinlashuvchi bo'ladi. Uning yig'indisini $S(x_0)$ deb belgilaymiz. Xuddi shunga o'xshash $\forall x \in M$ olib, unga $\sum_{n=1}^{\infty} u_n(x)$ qatorning yig'indisini mos qo'ysak, u holda M to'plamda aniqlangan $S(x)$ funksiya hosil bo'ladi. Bu $S(x)$ funksiya (7)-funksional qatorning **yig'indisi** deyiladi:

$$S(x) = \sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$

Ushbu

$$S_n(x) = \sum_{k=1}^{n} u_k(x), \quad n = 1, 2, ...$$

yig'indilarga (7)-funksional qatorning **qismiy yig'indilar** deyiladi.

Shunday qilib, (7)-qatorga mos keluvchi

$$\{S_n(x)\} : S_1(x), S_2(x), ..., S_n(x), ... \tag{9}$$

funksional ketma-ketlikni hosil qildik va aksincha, (9)-qismiy yig'indilari ketma-ketligi berilgan holda har doim hadlari (7)-funksional qatorning hadlariga teng bo'lgan quyidagi

$$S_1(x) + [S_2(x) - S_1(x)] + ... + [S_n(x) - S_{n-1}(x)] + ...$$

funksional qatorni hosil qilish mumkin. $\Rightarrow$ Agar (9)-ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'lsa, u holda (7)-qator ham x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'ladi va

$$S(x) = \lim_{n \to \infty} S_n(x)$$

tenglik bajariladi.

Demak, funksional qator yoki funksional ketma-ketlikdan birining xossalarini batafsil o'rganish yetarlidir.

Ta'rif. Agar (7)-funksional qatorning qismiy yig'indilaridan tuzilgan $\{S_n(x)\}$ funksional ketma-ketlik M to'plamda qatorning yig'indisi $S(x)$ ga tekis yaqinlashsa, unda (7)-funksional qator M to'plamda **tekis yaqinlashadi** deyiladi.

$r_n(x) = S(x) - S_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ deb belgilaymiz.

1-teorema. (7)-funksional qatorning M to'plamda tekis yaqinlashuvchi bo'lishi uchun

$$\lim_{n\to\infty} \sup_{x\in M} |r_n(x)| = 0 \tag{10}$$

tenglikning bajarilishi zarur va yetarli.

2-teorema. (Koshi kriteriyasi). *(7)-funksional qatorning M to'plamda tekis yaqinlashuvchi bo'lishi uchun quyidagi shartning bajarilishi zarur va yetarli:* $\forall \varepsilon > 0$ *uchun* $\exists n_0 = n_0(\varepsilon) \in N : \forall n \geq n_0$ *va* $\forall$ *butun* $p \geq 0$ *hamda barcha* $x \in M$ *lar uchun*

$$\left|\sum_{k=n}^{n+p} u_k(x)\right| < \varepsilon \tag{11}$$

bo'ladi.

Natija. (Funksional qator yaqinlashishining zaruriy sharti). Agar (7)-funksional qator M to'plamda tekis yaqinlashsa, u holda shu to'plamda $u_n(x) \rightrightarrows 0$ bo'ladi.

3-teorema. (Veyershtrass alomati). *Bizga* $\sum_{n=1}^{\infty} u_n(x)$ *funksional va*

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0 \tag{12}$$

sonli qator berilgan bolsin. Agar $\forall x \in M$ *uchun*

$$|u_n(x)| \leq a_n, \quad n = 1, 2, \ldots$$

tengsizlik bajarilsa va (12)-sonli qator yaqinlashsa, unda $\sum_{n=1}^{\infty} u_n(x)$ *funksional qator M to'plamda absolut va tekis yaqinlashadi.*

Aytaylik, ushbu

$$\sum_{n=1}^{\infty} a_n(x) \cdot b_n(x) \tag{13}$$

funksional qator berilgan bo'lsin.

4-teorema. (Dirixle alomati). *Agar*

1) har bir $x \in M$ *uchun* $\{a_n(x)\}$ *monoton va M to'plamda* $a_n(x)$ *0 ga tekis yaqinlashsa;*

2) $B_n(x) = \sum_{k=1}^{n} b_k(x)$ *qismiy yig'indilar M to'plamda birgalikda chegaralangan ya'ni* $\exists K \quad \forall x \in M \quad |B_n(x)| \leq K$ *bo'lsa, u holda (13)-qator M to'plamda tekis yaqinlashadi.*

5-teorema. (Abel alomati). *Agar*

1) har bir $x \in M$ *uchun* $\{a_n(x)\}$ *monoton va* $\{a_n(x)\}$ *ketma-*

ketlik M to'plamda chegaralangan;

2) $\sum_{n=1}^{\infty} b_k(x)$ *funksional qator M to'plamda tekis yaqinlashuvchi bo'lsa, unda (13)-qator M to'plamda tekis yaqinlashadi.*

3°. Tekis yaqinlashuvchi funksional ketma-ketlik va qatorlarning xossalari

Funksional qatorlarda (ketma-ketliklarda) shuni ta'kidlash lozimki, ularning har bir hadi uzluksiz bo'lgan taqdirda ham qatorning yig'indisi (ketma-ketlikning limit funksiyasi) uzluksiz bo'lishi shart emas.

Misol. $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$ funksional qator berilgan bo'lsin. Bu funksional qatorda $u_n(x) = \frac{x^2}{(1+x^2)^n} \in C(-\infty, +\infty)$. Berilgan qatorning yig'indisi topamiz:

$$S_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k} = x^2 \cdot \left[1 + \frac{1}{1+x^2} + \ldots + \frac{1}{(1+x^2)^n}\right] \Rightarrow S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 0, x = 0, \\ x^2 + 1, x \neq 0 \end{cases}.$$

Bu tenglikdan ko'rinadiki $\lim_{x\to 0} S(x) = \lim_{x\to 0}(x^2+1) = 1$ va $S(0) = 0 \Rightarrow S(x)$ funksiya $x = 0$ nuqtada uzluksiz emas. Berilgan qator uchun ushbu

$$\lim_{x\to 0} \sum_{n=0}^{\infty} u_n(x) \neq \sum_{n=0}^{\infty} \lim_{x\to 0} u_n(x)$$

munosabat o'rinli. ▷

Tabiiy savol tug'iladi: qanday shartlar bajarilganda funksional qatorlarda hadlab limitga o'tish, ularni hadlab differensiallash va integrallash mumkin?

Bu savollarga quyidagi teoremalar javob beradi.

Bizga M to'plamda yaqinlashuvchi (7)-funksional qator berilgan bo'lib, bu qatorning yig'indisi $S(x)$ bo'lsin.

1-teorema. *Agar* $\forall n \in N$ *uchun* $u_n(x) \in C(M)$ *bo'lib, (7)-qator M to'plamda tekis yaqinlashsa,* $S(x) \in C(M)$ *bo'ladi, ya'ni* $\forall x_0 \in M$ *uchun*

$$\lim_{x\to x_0} S(x) = \lim_{x\to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x\to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0) = S(x_0)$$

tenglik bajariladi.

Agar M to'plamda yaqinlashuvchi (1)-funksional ketma-ketlik berilgan bo'lib, $f(x)$ funksiya uning limit funksiyasi bo'lsa, unda quyidagi teorema o'rinli bo'ladi.

2-teorema. *Agar* $f_n(x)\in C(M)$, $n=1,2,..$ *bo'lib, M to'plamda* $f_n(x)\rightrightarrows f(x)$ *bo'lsa,* $f(x)\in C(M)$ *bo'ladi.*

3-teorema. *Agar (7)-funksional qator M to'plamda tekis yaqinlashuvchi va* x_0 *nuqta M to'plamning limit nuqtasi bo'lib,*

$$\lim_{x\to x_0} u_n(x)=c_n\ (n=1,2,...)$$

bo'lsa, u holda

$$\sum_{m=1}^{\infty} c_m = c_1+c_2+...+c_n+...$$

qator ham yaqinlashuvchi, uning yig'indisi C esa $S(x)$ *ning* $x\to x_0$ *dagi limitiga teng bo'ladi:*

$$\lim_{x\to x_0} S(x)=\lim_{x\to x_0}\sum_{n=1}^{\infty} u_n(x)=\sum_{n=1}^{\infty}\lim_{x\to x_0} u_n(x)=\sum_{n=1}^{\infty} c_n = C$$

Faraz qilaylik, $[a,b]$ kesmada yaqinlashuvchi (7)-funksional qator berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin.

4-teorema. Agar (7)-qator $[a,b]$ kesmada tekis yaqinlashuvchi bo'lib, $u_n(x)\in C[a,b]$ $(n=1,2,..)$ bo'lsa, u holda quyidagi

$$\int_a^b u_1(x)dx+\int_a^b u_2(x)dx+...+\int_a^b u_n(x)dx$$

qator ham yaqinlashuvchi va uning yig'indisi $\int_a^b S(x)dx$ ga teng bo'ladi:

$$\int_a^b S(x)dx=\int_a^b\left[\sum_{n=1}^{\infty} u_n(x)\right]dx=\sum_{n=1}^{\infty}\int_a^b u_n(x)dx.$$

Izoh. 4-teoremadagi (7)-qatorning tekis yaqinlashuvchanligi sharti yetarli shart bo'lib, u zaruriy shart emas, ya'ni ba'zan tekis yaqinlashmaydigan qatorlarni ham hadlab integrallash mumkin.

Misol. $\sum_{k=1}^{\infty}\left(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}}\right)$ $(0\le x\le 1)$ funksional qator berilgan bo'lsin.

$$S_n(x)=\sum_{k=1}^{n}\left(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}}\right)=-x+x^{\frac{1}{2n+1}}\Rightarrow S(x)=\lim_{n\to\infty}S_n(x)=\begin{cases}0, x=0,\\ 1-x, 0<x\le 1\end{cases}\Rightarrow$$

$S_n(x)$ $[0,1]$ da $S(x)$ ga tekis yaqinlashmaydi, lekin

$$\int_0^1 S(x)dx=\int_0^1(1-x)dx=\frac{1}{2} \text{ va } \sum_{n=1}^{\infty}\int_0^1 u_n(x)dx=$$

$$=\sum_{n=1}^{\infty}\left[\int_0^1\left(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}}\right)dx\right]=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=\frac{1}{2}.$$

Demak, $\int_0^1 S(x)dx=\sum_{n=1}^{\infty}\int_0^1 u_n(x)dx=\frac{1}{2}$, lekin $\sum_{n=1}^{\infty}u_n(x)$ qator $[0,1]$ kesmada tekis yaqinlashmaydi. ▷

5-teorema. *Agar (7)-funksional qatorning har bir $u_n(x)$ hadi $[a,b]$ kesmada uzluksiz $u_n'(x)$ hosilaga ega bo'lib,*

$$\sum_{n=1}^{\infty}u_n'(x)=u_1'(x)+u_2'(x)+...+u_n'(x)+... \qquad (14)$$

funksional qator $[a,b]$ da tekis yaqinlashuvchi bo'lsa, u holda berilgan (7)-qatorning yig'indisi $S(x)$ shu $[a,b]$ da $S'(x)$ hosilaga ega va

$$S'(x)=\left[\sum_{n=1}^{\infty}u_n(x)\right]'=\sum_{n=1}^{\infty}u_n'(x)$$

tenglik o'rinli bo'ladi.

Izoh. Bu teoremada ham (14)-funksional qatorning tekis yaqinlashuvchanlik sharti yetarli shart bo'lib, zaruriy shart emas.

4⁰. Darajali qatorlar

1-ta'rif. *Quyidagi*

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n \qquad (15)$$

ko'rinishdagi funksional qatorga ***darajali qator*** *deyiladi.* Bu yerda $a_1,a_2,...,a_n,...,x_0$ lar o'zgarmas haqiqiy sonlar.

Agar (15) da $\xi=x-x_0$ deb belgilash kiritsak,

$$\sum_{n=0}^{\infty} a_n \xi^n \qquad (16)$$

darajali qatorga kelamiz. Demak (16)-ko'rinishdagi darajali qatorlarni o'rganish kifoyadir.

1-teorema. ***(Abelning birinchi teoremasi).*** *Agar*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad (17)$$

darajali qator $x = x_0 \neq 0$ ***nuqtada yaqinlashsa, u holda qator*** x ***ning*** $|x| < |x_0|$ *tengsizlikni qanoatlantiruvchi barcha qiymatlarida* ***absolut yaqinlashuvchi*** *bo'ladi.*

Natija. Agar (17)-qator $x = x_0$ nuqtada uzoqlashuvchi bo'lsa, u holda bu qator $\{|x| > |x_0|\}$ da ham uzoqlashuvchi bo'ladi.

2-ta'rif. *Agar* $\sum_{n=0}^{\infty} a_n x^n$ *darajali qator* $\{|x| < R\}$ *da yaqinlashib,* $\{|x| > R\}$ *da uzoqlashsa, u holda shu* $R \geq 0$ *soniga darajali qatorning* ***yaqinlashish radiusi,*** $(-R,\ R)$ *oraliqqa esa* ***yaqinlashish intervali*** *deyiladi.*

2-teorema. *Ixtiyoriy darajali qatorning yaqinlashish radiusi R mavjud bo'lib, bu qator* $\{|x| < R\}$ *da absolut va* $\forall r < R$ *uchun* $\{|x| \leq r\}$ *da tekis yaqinlashadi.*

Izoh. Darajali qator yaqinlashish oralig'ining chegaraviy $x = \pm R$ nuqtalarida yaqinlashishi ham, uzoqlashishi ham mumkin. Darajali qatorni bu nuqtalarda alohida tekshirish lozim.

Darajali qatorning yaqinlashish radiusini quyidagi teoremalardan foydalanib, topish mumkin.

3-teorema. ***(Dalamber).*** *Agar* $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ *mavjud bo'lsa, u holda*

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (18)$$

bo'ladi.

4-teorema. *(Koshi). Agar* $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ *mavjud bo'lsa, u holda*

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}} \tag{19}$$

bo'ladi.

5-teorema. (Koshi-Adamar) *Agar R soni (17)-darajali qatorning yaqinlashish radiusi bo'lsa, u holda*

$$R=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} \tag{20}$$

formula **(Koshi-Adamar formulasi)** o'rinli bo'ladi.

Darajali qatorlar quyidagi xossalarga ega.

6-teorema. *Darajali qatorning yig'indisi $S(x)$ yaqinlashish oralig'iga tegishli bo'lgan $\forall$ nuqtada uzluksiz bo'ladi.*

7-teorema. ***(Abelning ikkinchi teoremasi).*** *Agar (17)-qator $x=R$ $(x=-R)$ nuqtada yaqinlashsa, unda bu qator $[0;R]$ $([-R;0])$ kesmada tekis yaqinlashuvchi bo'ladi.*

Natija. *Agar (17)-qator $x=R$ $(x=-R)$ nuqtada yaqinlashsa, u holda $S(x)$ yig'indi $[0;R]$ $([-R;0])$ kesmada uzluksiz bo'ladi.*

Endi $\sum\limits_{n=0}^{\infty}a_n(x-x_0)^n$ ko'rinishidagi darajali qatorni ko'ramiz. Bu qatorning yaqinlashish radiusi $\sum\limits_{n=0}^{\infty}a_nx^n$ qatorning yaqinlashish radiusini hisoblash formulalari yordamida topiladi, faqat bu yerda yaqinlashish oralig'i $\{|x-x_0|<R\}=(x_0-R,x_0+R)$ interval bo'ladi.

8-teorema. *Agar $R>0$ soni quyidagi*

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n \tag{21}$$

darajali qatorning yaqinlashish radiusi bo'lsa, u holda

1) $f(x)$ funksiya $(x_0-R,\ x_0+R)$ intervalda ixtiyoriy tartibli hosilalarga ega bo'ladi va u hosilalar (21)-darajali qatorni hadlab differensiallash yordamida topiladi;

2) bu qatorni $\forall[a,b]\subset(x_0-R,\ x_0+R)$ oraliqda hadlab integrallash mumkin.

3) (21)-darajali qatorni hadlab differensiallash yoki integrallash dan hosil bo'lgan yangi qatorlarning yaqinlashish radiuslari ham (21) qatornning yaqinlashish radiusi R ga teng bo'ladi.

Izoh. Agar $f(x)$ funksiya (21)-tenglik yordamida ifodalanib, $R>0$ bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada (aniqrog'i, x_0 nuqtaning atrofida) analitik funksiya deyiladi. 8-teoremadan analitik funksiyaning cheksiz differensiallanuvchi ekanligi kelib chiqadi. Lekin, ixtiyoriy cheksiz differensiallanuvchi funksiya analitik bo'lishi shart emas. Bunga misol tariqasida $f(x)=\exp\left(-\frac{1}{x^2}\right)$ funksiyani olish mumkin.

9-teorema. *Agar $f(x)$ funksiya x_0 nuqtada analitik bo'lsa, ya'ni*

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

tenglik x_0 nuqtaning biror atrofida o'rinli bo'lsa, u holda

$$a_n=\frac{f^{(n)}x_0}{n!},n=0,1.2,..$$

bo'ladi, ya'ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}x_0}{n!}(x-x_0)^n$$

tenglik ham x_0 nuqtaning o'sha atrofida o'rinli bo'ladi.

5^0. Teylor qatori. Elementar funksiyalarni Teylor qatoriga yoyish

Ta'rif. *Faraz qilaylik, $f(x)$ funksiya x_0 nuqtaning biror atrofida aniqlangan va shu nuqtada ixtiyoriy tartibdagi hosilalarga ega bo'lsin. U holda quyidagi*

$$\sum_{n=0}^{\infty}\frac{f^{(n)}x_0}{n!}(x-x_0)^n \qquad (22)$$

qatorga $f(x)$ funksiyaning x_0 nuqtadagi Teylor qatori deyiladi.

Izoh. (22)-qatorning yig'indisi har doim ham $f(x)$ bilan ustma-ust tushavermaydi.

Masalan, $f(x)=\exp\left(-\frac{1}{x^2}\right)$ funksiya uchun barcha hosilalar $f^{(n)}(0)=0$ va (22) qatorning yig'indisi $0\neq f(x)$

Lekin ba'zi bir shartlar bajarilsa ular orasida tenglik o'rnatish mumkin.

Teorema. (Teylor). *Faraz qilaylik* (x_0-h, x_0+h) *intervalda* $f(x)$ *funksiyaning o'zi va barcha tartibdagi hosilalari birgalikda chegaralangan bo'lsin, ya'ni* $\exists M>0: \forall x\in(x_0-h, x_0+h)$ *uchun*

$$\left|f^{(n)}(x)\right|\le M, \quad n=0,1,2,3,...$$

tengsizlik bajarilsin. U holda $(x_0-h,\ x_0+h)$ oraliqda $f(x)$ funksiya Teylor qatoriga yoyiladi, ya'ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}x_0}{n!}(x-x_0)^n, |x-x_0|<h, \tag{23}$$

tenglik o'rinli bo'ladi.

Agar Teylor qatorida $x_0=0$ bo'lsa, u holda hosil bo'lgan qatorga **Makloren qatori** deyiladi.

Endi asosiy elementar funksiyalarning Makloren qatoriga yoyilmalarini keltiramiz.

1. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...\quad(-\infty<x<+\infty).$

2. $shx=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+...+\frac{x^{2n+1}}{(2n+1)!}+...\quad(-\infty<x<+\infty).$

3. $chx=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+...+\frac{x^{2n}}{(2n)!}+...\quad(-\infty<x<+\infty).$

4. $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}+...+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+...\quad(-\infty<x<+\infty).$

5. $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}+...+\frac{(-1)^n x^{2n}}{(2n)!}+...\quad(-\infty<x<+\infty).$

6. $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}=x-\frac{x^2}{2!}+\frac{x^3}{3!}-...+(-1)^{n+1}\frac{x^n}{n}+...\quad(-1<x\le 1).$

7. $(1+x)^\alpha=1+\sum_{n=1}^{\infty}\frac{\alpha\cdot(\alpha-1)\cdot...\cdot(\alpha-n+1)}{n!}x^n=1+\alpha x+\frac{\alpha\cdot(\alpha-1)}{2!}x^2+$
$+\frac{\alpha\cdot(\alpha-1)(\alpha-2)}{3!}x^3+...+\frac{\alpha(\alpha-1)\cdot...\cdot(\alpha-n+1)}{n!}x^n+...(-1<x<1).$

6^0. Darajali qatorlarning tatbiqlari

a) Darajali qatorlar yordamida differensial tenglamalarni yechish. Aytaylik,

$$y'' + p(x)y' + q(x)y = f(x) \tag{24}$$

differensial tenglamaning ushbu

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{25}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topish talab qilinsin.

Agar $p(x)$, $q(x)$, $f(x)$ funksiyalarni x_0 nuqtaning biror atrofida shu funksiyalarga yaqinlashuvchi

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

ko'rinishida ifodalash mumkin bo'lsa, unda yuqoridagi Koshi masalasi yagona yechimga ega bo'lib, uni

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{26}$$

ifodalash mumkin. (26)-qatordagi noma'lum a_n koeffitsientlarni topish uchun (24)-tenglamadagi y, y', y'', p, q, f lar o'rniga ularning yoyilmalari olib borib qo'yiladi va noma'lum koeffitsientlar usulidan foydaniladi.

Misol.

$$y'' - xy = 0 \tag{27}$$

tenglamaning ushbu

$$y(0) = 1, \quad y'(0) = 0 \tag{28}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

◁ (27)-tenglamaning yechimini

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{29}$$

ko'rinishda qidiramiz. Unda

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n,$$

$$xy = x \cdot \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

bo'lib, (27)-tenglama quyidagi ko'rinishga keladi:

$$2a_2+\sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}x^n=\sum_{n=1}^{\infty}a_{n-1}x^n.$$

Bu tenglikdagi x ning mos darajalari oldidagi mos koeffitsientlarni tenglash yordamida

$$a_2=0,\quad (n+1)(n+2)a_{n+2}=a_{n-1},\quad n\in N \tag{30}$$

rekkurent formulani hosil qilamiz. $a_2=0$ bo'lganligi sababli bu rekkurent formuladan $a_5=0$, $a_8=0$ va umuman

$$a_{3n-1}=0,\quad n\in N$$

ekanligini topamiz. Shu formuladan yana

$$a_{3n}=\frac{a_0}{(2\cdot3)\cdot(5\cdot6)\cdot\ldots\cdot[(3n-1)3n]},$$

$$a_{3n+1}=\frac{a_0}{(3\cdot4)\cdot(6\cdot7)\cdot\ldots\cdot[3n\cdot(3n+1)]},$$

tengliklar o'rinli bo'lishi kelib chiqadi. (28)-shartlar va (29)-tenglikdan $\Rightarrow a_0=1, a_1=0.$

Demak, (27)-tenglamaning (28)-shartlarni qanoatlantiruvchi yechimi quyidagi ko'rinishga ega ekan:

$$y=1+\frac{x^3}{2\cdot3}+\frac{x^6}{(2\cdot3)\cdot(5\cdot6)}+\ldots+\frac{x^{3n}}{(2\cdot3)\cdot(5\cdot6)\cdot\ldots\cdot[(3n-1)\cdot3n]}+\ldots\rhd$$

b) Darajali qatorlar yordamida integrallarni hisoblash.

Integrallarni hisoblashda ham integral ostidagi funksiyani darajali qatorga yoyish ko'p hollarda yaxshi natija beradi.

Misol. Ushbu

$$I=\int_0^1\frac{\ln(1+x)}{x}dx$$

integral hisoblansin.

$\lhd$ Avvalgi punktdagi $\ln(1+x)$ning Makloren qatoriga yoyilmasidan foydalanamiz:

$$I=\int_0^1\frac{\ln(1+x)}{x}dx=\int_0^1\frac{1}{x}\left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\right]dx=\int_0^1\left[\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{x^{n-1}}{n}\right]dx=\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^1\frac{x^{n-1}}{n}dx=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}-\frac{\pi^4}{24}=\frac{\pi^2}{12}.\rhd$$

Izoh. Sonli qatorlarning yig'indilarini hisoblashda ko'p hollarda quyidagi tengliklar katta yordam beradi.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2, \tag{31}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{32}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \tag{33}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \tag{34}$$

Yuqoridagi misolni yechishda (33) va (32)-tengliklardan foydalanildi.

Nazorat savollari

1. Funksional ketma-ketlik tushunchasi.
2. Funksional ketma-ketlikning limit funksiyasi tushunchasi.
3. Funksional ketma-ketlikning tekis yaqinlashishi ta'rifi.
4. Funksional ketma-ketlik tekis yaqinlashishining zaruriy va yetarli sharti.
5. Funksional qator tushunchasi.
6. Funksional qatorning yaqinlashishi tushunchasi.
7. Funksional qator tekis yaqinlashishining ta'rifi.
8. Funksional qator tekis yaqinlashishining zaruriy va yetarli sharti.
9. Funksional qator tekis yaqinlashishining zaruriy sharti.
10. Funksional qatorning tekis yaqinlashishi haqidagi Veyershtrass alomati.
11. Dirixle alomati.
12. Abel alomati.
13. Funksional ketma-ketlik limit funksiyasining uzluksizligi.
14. Funksional qator yig'indisining uzluksizligi.
15. Funksional qatorlarni hadlab integrallash.
16. Funksional qatorlarni hadlab differensiallash.
17. Darajali qator tushunchasi.
18. Abelning birinchi teoremasi.
19. Darajali qatorning yaqinlashish radiusi va yaqinlashish oralig'i.
20. Darajali qator yaqinlashish radiusini topish uchun Dalamber formulasi.
21. Darajali qator yaqinlashish formulasini topish uchun Koshi formulasi.
22. Koshi-Adamar formulasi.
23. Teylor qatori va Teylor teoremasi.
24. Elementar funksiyalarni Teylor qatoriga yoyish.
25. Darajali qatorlar yordamida differensial tenglamalarni yechish.
26. Darajali qatorlar yordamida integrallarni hisoblash.

-B-
Mustaqil yechish uchun misol va masalalar

1-masala. $\{f_n(x)\}$ **funksional ketma-ketlikning** M **to'plamdagi limit funksiyasini toping.**

1.1 $f_n(x)=x^n-3x^{n+2}+2x^{n+3}, M=[0;1]$.

1.2 $f_n(x)=\dfrac{nx^2}{x+3n+2}, M=[0;+\infty)$.

1.3 $f_n(x)=\sqrt{x^2+\dfrac{1}{\sqrt{n}}},\ M=R$.

1.4 $f_n(x)=(x-1)\,arctgx^n, M=(0;+\infty]$.

1.5 $f_n(x)=\sqrt[n]{1+x^n},\ M=[0;2]$.

1.6 $f_n(x)=\dfrac{nx}{1+n^2x^2}, M=R$.

1.7 $f_n(x)=\sin^n x, M=[0;\pi]$.

1.8 $f_n(x)=\dfrac{\ln^2 n+x^2}{2\ln^2 n+x^2+x\ln n}, M=(0;+\infty)$.

1.9 $f_n(x)=\sqrt[n]{x\sin x}, M=\left[0;\dfrac{\pi}{2}\right]$.

1.10 $f_n(x)=\sqrt[n]{\cos x}, M=\left[-\dfrac{\pi}{2};\dfrac{\pi}{2}\right]$.

1.11 $f_n(x)=n^3x^2\cdot e^{-nx}, M=[0;+\infty)$.

1.12 $f_n(x)=n\left(\sqrt{x^2+\dfrac{1}{n}}-x\right), M=(0;+\infty)$.

1.13 $f_n(x)=n\left(x^{\frac{1}{n}}-1\right), M=[1;3]$.

1.14 $f_n(x)=narctgnx^2, M=(0;+\infty)$.

1.15 $f_n(x)=n\left(x^{\frac{1}{n}}-x^{\frac{1}{2n}}\right), M=(0;+\infty).$

1.16 $f_n(x)=\sqrt[n]{1+x^n+\left(\frac{x^2}{2}\right)^n}, M=[0;+\infty).$

1.17 $f_n(x)=\frac{\sin n\sqrt{x}}{\ln(n+1)}, M=[0,+\infty).$

1.18 $f_n(x)=\frac{n}{x^3+n^3}\, arctg\sqrt{nx}, M=[0;+\infty).$

1.19 $f_n(x)=\ln\left(1+\frac{\cos nx}{\sqrt{n+x}}\right), M=[0;+\infty).$

1.20 $f_n(x)=n^{3/4}\cdot x\cdot e^{-\sqrt{nx}}, M=[0;+\infty).$

1.21 $f_n(x)=\ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right), M=[0;+\infty).$

2-masala. Berilgan funksional ketma-ketlikni ko'rsatilgan oraliqda tekis yaqinlashishga tekshiring.

2.1 $f_n(x)=\frac{x}{n}\ln\frac{x}{n};0<x<1.$

2.2 $f_n(x)=x^n;0\le x\le\frac{1}{4}.$

2.3 $f_n(x)=e^{-(x-n)^2};-1<x<1.$

2.4 $f_n(x)=x^n-x^{n+1};0\le x\le 1.$

2.5 $f_n(x)=e^{n(x-1)};0<x<1.$

2.6 $f_n(x)=x^n;0\le x\le 1.$

2.7 $f_n(x)=x\,arctg\,nx, 0<x<+\infty.$

2.8 $f_n(x)=x^n-x^{2n};0\le x\le 1.$

2.9 $f_n(x)=arctg\,nx;0<x<+\infty.$

2.10 $f_n(x)=\frac{1}{x+n};0<x<+\infty.$

2.11 $f_n(x)=\sin\frac{x}{n};-\infty<x<+\infty.$

2.12 $f_n(x)=\frac{nx}{1+n+x};0\le x\le 1.$

2.13 $f_n(x)=\frac{\sin nx}{n};-\infty<x<+\infty.$

2.14 $f_n(x)=\frac{x^n}{1+x^n};0\le x\le 1-\varepsilon,\varepsilon>0.$

2.15 $f_n(x)=n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right);0<x<+\infty.$ 2.16 $f_n(x)=\frac{x^n}{1+x^n};1-\varepsilon\le x\le 1+\varepsilon,\varepsilon>0.$

2.17 $f_n(x)=\sqrt{x^2+\frac{1}{n^2}};-\infty<x<+\infty.$ 2.18 $f_n(x)=\frac{x^n}{1+x^n};2\le x<+\infty.$

2.19 $f_n(x)=\frac{2nx}{1+n^2x^2};1<x<+\infty.$ 2.20 $f_n(x)=\frac{2nx}{1+n^2x^2};0\le x\le 1.$

2.21 $f_n(x)=\frac{n+x}{n+x+\sqrt{nx}},a)0\le x<+\infty,b)0\le x\le 1.$

3-masala. Veyershtrass alomatidan foydalanib, berilgan funksional qatorlarni ko'rsatilgan oraliqlarda tekis yaqinlashishini ko'rsating.

3.1 $\sum_{n=1}^{\infty} arctg\frac{2x}{x^2+n^3};|x|<+\infty.$ 3.2 $\sum_{n=1}^{\infty} x^2e^{-nx};0\le x<+\infty.$

3.3 $\sum_{n=1}^{\infty}\ln\left(1+\frac{x^2}{n\ln^2 n}\right);|x|<3.$ 3.4 $\sum_{n=1}^{\infty}\frac{\sin nx}{n\sqrt{n}};|x|<+\infty.$

3.5 $\sum_{n=1}^{\infty}\frac{\cos nx}{n^2};|x|<+\infty.$ 3.6 $\sum_{n=1}^{\infty}\frac{\sin nx}{\sqrt[3]{n^4+x^4}};|x|<+\infty.$

3.7 $\sum_{n=1}^{\infty}\frac{x^n}{n!};|x|<2.$ 3.8 $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n!}}\left(x^n+x^{-n}\right);\frac{1}{2}\le x\le 2.$

3.9 $\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2};|x|<+\infty.$ 3.10 $\sum_{n=1}^{\infty}\frac{x}{1+n^4x^2};0\le x<+\infty.$

3.11 $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+2^n};-2<x<+\infty.$ 3.12 $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2};-\infty<x<+\infty.$

3.13 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x+\sqrt{n^3}};0\le x<+\infty.$ 3.14 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{\sqrt{n^3}};0\le x\le 1.$

3.15 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n};-1<x<1.$ 3.16 $\sum_{n=1}^{\infty}\frac{1}{(x+2n-1)\cdot(x+2n+1)};0\le x<+\infty.$

3.17 $\sum_{n=1}^{\infty}\sin\frac{1}{\sqrt{n}}\,arctg\frac{2x}{x^2+n^2};-\infty<x<+\infty.$ **3.18** $\sum_{n=1}^{\infty}\ln\left(1+\frac{nx^2}{2+n^3x^2}\right);-\infty<x<+\infty.$

3.19 $\sum_{n=1}^{\infty}\frac{(x-1)^n}{(3n+1)\cdot 3^n};-1\le x<3.$ **3.20** $\sum_{n=1}^{\infty}\frac{n\ln(1+nx)}{x^n},2<x<+\infty.$

3.21 $\sum_{n=1}^{\infty}\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right];0\le x\le 2.$

4-masala. Berilgan funksional qatorning ko'rsatilgan oraliqda tekis yoki notekis yaqinlashuvchiligini aniqlang.

4.1 $\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)...(1+nx)};\ \ 0\le x\le 1.$

4.2 $\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)...(1+nx)};1\le x<+\infty.$

4.3 $\sum_{n=1}^{\infty}\frac{1}{(x+n)(x+n+1)};0<x<+\infty.$

4.4 $\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1](nx+1)};0<x<+\infty.$

4.5 $\sum_{n=1}^{\infty}\left(\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right);-1\le x\le 1.$ **4.6** $\sum_{n=0}^{\infty}(1-x)x^n;0\le x\le 1.$

4.7 $\sum_{n=0}^{\infty}\frac{x^n}{n!};0<x<+\infty.$ **4.8** $\sum_{n=1}^{\infty}\frac{x^n}{n^3};-1\le x\le 1.$

4.9 $\sum_{n=1}^{\infty}\frac{\sin nx}{n};\frac{\pi}{2}\le x\le\frac{3\pi}{2}.$ **4.10** $\sum_{n=1}^{\infty}\frac{\sin nx}{n};0\le x\le 2\pi.$

4.11 $\sum_{n=1}^{\infty}2^n\cdot\sin\frac{1}{3^n x};0<x<+\infty.$ **4.12** $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+n};0<x<+\infty.$

4.13 $\sum_{n=2}^{\infty}\frac{(-1)^n}{n+\sin x};0\le x\le 2\pi.$ **4.14** $\sum_{n=1}^{\infty}\frac{\cos\frac{2n\pi}{3}}{\sqrt{n^2+x^2}};-\infty<x<+\infty.$

4.15 $\sum\limits_{n=1}^{\infty} e^{-n\,\mathrm{tg}\,x}, 0 < x < \frac{\pi}{2}.$

4.16 $\sum\limits_{n=1}^{\infty} \ln^2\left(1+\frac{x}{1+n^2x^2}\right); 0 \le x < +\infty.$

4.17 $\sum\limits_{n=1}^{\infty} (-1)^n \frac{x^n}{n\cdot\left(x^n+1\right)}; 1 \le x < +\infty.$

4.18 $\sum\limits_{n=1}^{\infty} \frac{n\sqrt{x}}{1+n^3x^3}; 0 \le x < +\infty.$

4.19 $\sum\limits_{n=1}^{\infty} \frac{n\sqrt{x}}{1+n^3x^3}; 1 \le x < +\infty.$

4.20 $\sum\limits_{n=1}^{\infty} \frac{x^2}{(1+nx)^4}; \quad 0 \le x < +\infty.$

4.21 $\sum\limits_{n=1}^{\infty} \frac{n}{(1+2x^2)(1+4x^2)\ldots(1+2nx^2)}; \quad 1 \le x < +\infty.$

5-masala. Berilgan funksional qatorning yaqinlashish sohasini toping.

5.1 $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^{-1/5}};$

5.2 $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\cdot\left(\frac{1-x}{1+x}\right)^n;$

5.3 $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}\cdot\frac{1}{\left(3x^2+4x+2\right)^n}.$

5.4 $\sum\limits_{n=1}^{\infty} \frac{n+1}{3^n}\cdot\left(x^2-4x+6\right)^n.$

5.5 $\sum\limits_{n=1}^{\infty} \frac{x^n}{1-x^n}.$

5.6 $\sum\limits_{n=1}^{\infty} \frac{n+3}{n+1}\cdot\frac{1}{\left(27x^2+12x+2\right)^n}.$

5.7 $\sum\limits_{n=1}^{\infty} \frac{x^n}{1+x^{2n}}.$

5.8 $\sum\limits_{n=1}^{\infty} \frac{n\cdot 2^n}{n+1}\cdot\frac{1}{\left(3x^2+8x+6\right)^n}.$

5.9 $\sum\limits_{n=1}^{\infty} \frac{1}{n+3}\left(\frac{1+x}{1-x}\right)^n.$

5.10 $\sum\limits_{n=1}^{\infty} \frac{\left(x^2-6x+12\right)^n}{4^n\cdot\left(n^2+1\right)}.$

5.11 $\sum\limits_{n=1}^{\infty} \frac{1}{\left(\sqrt[3]{n^2}+\sqrt{n}+1\right)^{2x+1}}.$

5.12 $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(x+n)^3}.$

5.13 $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{x+n}}.$

5.14 $\sum\limits_{n=1}^{\infty} \frac{\left(x^2-5x+11\right)^n}{5^n\cdot\left(n^2+5\right)}.$

5.15 $\sum\limits_{n=1}^{\infty} \frac{(n+x)^n}{n^n}.$

5.16 $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+x)}.$

5.17 $\sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^2}$.

5.18 $\sum_{n=1}^{\infty}\frac{1+x^n}{1-x^n}$.

5.19 $\sum_{n=1}^{\infty}\frac{n+1}{x\cdot n^x}$.

5.20 $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^{x^2-1}}$.

5.21 $\sum_{n=1}^{\infty}\frac{n^2}{2^n\cdot(n^2+1)}\cdot(25x^2+1)^n$.

6-masala. Berilgan funksional qatorning yaqinlashish sohasini toping.

6.1 $\sum_{n=1}^{\infty}\frac{9^n}{n}x^{2n}\cdot\sin(x+\pi n)$.

6.2 $\sum_{n=1}^{\infty}\frac{4^n}{n}x^{4n}\cdot\sin(2x-\pi n)$.

6.3 $\sum_{n=1}^{\infty}\frac{3^n}{n}x^{4n}\cdot\cos(x+\pi n)$.

6.4 $\sum_{n=1}^{\infty}\left(\frac{5}{3}\right)^n\cdot\frac{1}{\sqrt{n}}x^{2n}\cdot\cos(x-\pi n)$.

6.5 $\sum_{n=1}^{\infty}\frac{2^{3n}}{\sqrt[3]{n}}x^{4n}\cdot\sin(3x+\pi n)$.

6.6 $\sum_{n=1}^{\infty}\frac{6^n}{n}x^{2n}\cdot\sin(5x-\pi n)$.

6.7 $\sum_{n=1}^{\infty}\frac{5^n}{\sqrt[4]{3n}}x^{2n}\cdot\cos(x+\pi n)$.

6.8 $\sum_{n=1}^{\infty}\frac{9^n}{2n}x^{2n}\cdot\sin(3x-\pi n)$.

6.9 $\sum_{n=1}^{\infty}2^n\cdot x^{3n}\cdot\sin\frac{x}{n}$.

6.10 $\sum_{n=1}^{\infty}3^{2n}\cdot x^n\cdot\sin\frac{x}{2n}$.

6.11 $\sum_{n=1}^{\infty}2^{3n}\cdot x^n\cdot\sin\frac{2x}{n}$.

6.12 $\sum_{n=1}^{\infty}3^n\cdot x^{2n}\cdot\sin\frac{3x}{\sqrt{n}}$.

6.13 $\sum_{n=1}^{\infty}3^n\cdot x^n\cdot tg\frac{3x}{n}$.

6.14 $\sum_{n=1}^{\infty}8^n\cdot x^{3n}\cdot tg\frac{x}{4\sqrt{n}}$.

6.15 $\sum_{n=1}^{\infty}x^{3n}\cdot tg\frac{2x}{3n}$.

6.16 $\sum_{n=1}^{\infty}2^n\cdot x^{3n}\cdot\arcsin\frac{x}{3n}$.

6.17 $\sum_{n=1}^{\infty}16^n\cdot x^{3n}\cdot\arcsin\frac{x}{\sqrt[3]{n}}$.

6.18 $\sum_{n=1}^{\infty}32^n\cdot x^{5n}\cdot\arcsin\frac{x}{\sqrt{n}}$.

6.19 $\sum_{n=1}^{\infty}2^n\cdot x^n\cdot arctg\frac{2x}{n+1}$.

6.20 $\sum_{n=1}^{\infty}2^n\cdot x^{3n}\cdot arctg\frac{x}{2\cdot(n+3)}$.

6.21 $\sum_{n=1}^{\infty}27^n\cdot x^{3n}\cdot arctg\frac{3x}{2n+1}$.

7-masala. Berilgan funksional qatorning yaqinlashish sohasini toping.

7.1 $\sum_{n=1}^{\infty} 2n^2\sqrt{x-2}\cdot e^{-\frac{n^2}{(x-1)^3}}$ 7.2 $\sum_{n=1}^{\infty} \frac{\ln^n\left(x+\frac{1}{n}\right)}{\sqrt{x-e}}$.

7.3 $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^n\cdot 5^{-\frac{n}{(x+1)^2}}$. 7.4 $\sum_{n=1}^{\infty} n^2\sqrt{x-1}\cdot e^{-\frac{n}{x}}$

7.5 $\sum_{n=1}^{\infty} e^{-\left(1-x\sqrt{n}\right)^2}$. 7.6 $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n\cdot 3^{-\frac{n}{x-1}}$.

7.7 $\sum_{n=1}^{\infty} 5^{-n^3\sin\frac{x^2+1}{n}}$. 7.8 $\sum_{n=1}^{\infty}\frac{1}{\ln^n(x-1)}$.

7.9 $\sum_{n=1}^{\infty} 5^{nx}\cdot arctg\frac{x}{7^{nx}\cdot(x-1)}$. 7.10 $\sum_{n=1}^{\infty}\frac{1}{\ln^n(x+2)}$.

7.11 $\sum_{n=1}^{\infty}\left(1+\frac{5}{n}\right)^n\cdot 3^{-\frac{n}{x^2}}$. 7.12 $\sum_{n=1}^{\infty}\frac{1}{\ln^n(x+e)}$.

7.13 $\sum_{n=1}^{\infty} e^{n^2\sin\frac{x^2+1}{n}}$. 7.14 $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot e^{-\frac{n}{\cos x}}$.

7.15 $\sum_{n=1}^{\infty}\frac{\left[\ln\left(1+\frac{1}{n}\right)+\ln\ln x\right]^n}{\sqrt{x-e^{1/e}}}$. 7.16 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\ln|x|}}$.

7.17 $\sum_{n=1}^{\infty}\frac{1}{\ln^n\left(x+\frac{1}{e}\right)}$. 7.18 $\sum_{n=1}^{\infty}\sin^n\frac{n\ln n}{x-n}$.

7.19 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{e^{n\sin x}}$. 7.20 $\sum_{n=1}^{\infty}(-1)^n\cdot 5^{-n^2 arctg\frac{1}{n\cdot|x|}}$.

7.21 $\sum_{n=1}^{\infty}(-1)^n\cdot 3^{-n^2\ln\left(1+\frac{x}{n}\right)}$.

8-masala. Berilgan funksional qatorning yaqinlashish sohasini toping.

8.1 $\sum_{n=1}^{\infty}\frac{(n-2)^3\cdot(x+3)^{2n}}{2n+3}$.

8.2 $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(x-3)^n}{(n+1)\cdot 5^n}$.

8.3 $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n\cdot 9^n}$.

8.4 $\sum_{n=1}^{\infty}\frac{2n+3}{(n+1)^5\cdot x^{2n}}$.

8.5 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{2n}$.

8.6 $\sum_{n=1}^{\infty}\frac{(x-5)^{2n+1}}{3n+8}$.

8.7 $\sum_{n=1}^{\infty}\frac{n^3+1}{3^n\cdot(x-2)^n}$.

8.8 $\sum_{n=1}^{\infty}\frac{n!}{x^n}$.

8.9 $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-1}}{4^n\cdot(2n-1)}$.

8.10 $\sum_{n=1}^{\infty}\frac{(x-7)^{2n-1}}{(2n^2-5n)\cdot 4^n}$.

8.11 $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(3n+1)\cdot 2^n}$.

8.12 $\sum_{n=1}^{\infty}\frac{3n\cdot(x-2)^{3n}}{(5n-8)^3}$.

8.13 $\sum_{n=1}^{\infty}(x+5)^n\cdot tg\frac{1}{3^n}$.

8.14 $\sum_{n=1}^{\infty}\sin\frac{\sqrt{n}}{n^2+1}(x-2)^n$.

8.15 $\sum_{n=1}^{\infty}\frac{1}{n\cdot 9^n(x-1)^{2n}}$.

8.16 $\sum_{n=1}^{\infty}3^{n^2}\cdot x^{n^2}$.

8.17 $\sum_{n=1}^{\infty}\frac{(x+2)^{n^2}}{n^n}$.

8.18 $\sum_{n=1}^{\infty}\frac{n^5}{(n+1)!}(x+5)^{2n+1}$.

8.19 $\sum_{n=1}^{\infty}\frac{(3n-2)\cdot(x-3)^n}{(n+1)^2\cdot 2^{n+1}}$.

8.20 $\sum_{n=1}^{\infty}\frac{(x-5)^n}{(n+4)\cdot\ln(n+4)}$.

8.21 $\sum_{n=1}^{\infty}\frac{1}{(n+2)\cdot\ln(n+2)\cdot(x-3)^{2n}}$.

9-masala Ta'rifdan foydalanib, berilgan funksional qatorning $[0;1]$ kesmada tekis yaqinlashishini isbotlang $n_0(\varepsilon)-?$. n ning qanday qiymatlarida qatorning qoldig'i $\forall x \in [0,1]$ uchun 0,1 dan katta bo'lmaydi?

9.1 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{7n-11}$.

9.2 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{5n-6}$.

9.3 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{4n-6}$.

9.4 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt[3]{n^3-5}}$.

9.5 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{4n-5}$.

9.6 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{5n-9}$.

9.7 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{3n-4}$.

9.8 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt[3]{n^3-2}}$.

9.9 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{6n-11}$.

9.10 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt[3]{n^3-7}}$.

9.11 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{7n-10}$.

9.12 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{6n-8}$.

9.13 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt[3]{n^3-4}}$.

9.14 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{2n-3}$.

9.15 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{8n-12}$.

9.16 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{6n-7}$.

9.17 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{5n-8}$.

9.18 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{6n-10}$.

9.19 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{4n-7}$.

9.20 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{5n-7}$.

9.21 $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{7n-13}$.

10-masala. Berilgan qator uchun uni majorirlovchi qatorni toping va ko'rsatilgan oraliqda tekis yaqinlashishini isbotlang.

10.1 $\sum_{n=1}^{\infty}\frac{\sqrt{x+1}\cdot\cos nx}{\sqrt[3]{n^5+1}};[0,2]$

10.2 $\sum_{n=1}^{\infty}\frac{x^n}{n\cdot 2^n},\left[-\frac{3}{2},\frac{3}{2}\right]$.

10.3 $\sum_{n=1}^{\infty}\frac{x^n}{n^n},\ [-2,2]$.

10.4 $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n,\ \left[-\frac{3}{2},\frac{3}{2}\right]$.

10.5 $\sum_{n=1}^{\infty}x^{n!},\ \left[-\frac{1}{2},\frac{1}{2}\right]$.

10.6 $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n\cdot 5^n},\ [-1,6]$.

10.7 $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(x-3)^n}{(2n+1)\cdot\sqrt{n+1}}[2,4]$.

10.8 $\sum_{n=0}^{\infty}\frac{(\pi-x)\cdot\cos^2 nx}{\sqrt[4]{n^7+1}},\ [0,\pi]$.

10.9 $\sum_{n=1}^{\infty}\frac{(x-1)^{2n}}{n\cdot 9^n},\ [-1,3]$.

10.10 $\sum_{n=1}^{\infty}\frac{n!(x+3)^n}{n^n},\ [-5,-1]$.

10.11 $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(x-2)^{2n}}{(n+1)^2\cdot\ln(n+1)},\ [1,3]$.

10.12 $\sum_{n=1}^{\infty}\frac{x^n}{n!},\ [-3,3]$.

10.13 $\sum_{n=1}^{\infty}\frac{2^{n-1}\cdot x^{2n-2}}{(4n-3)^2},\ \left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$.

10.14 $\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\cdot 3^n\cdot\ln n},\ [-2,2]$.

10.15 $\sum_{n=1}^{\infty}\frac{(x+5)^{2n-2}}{n^2\cdot 4^n},\ [-7,-3]$.

10.16 $\sum_{n=1}^{\infty}\frac{(x+2)^{n^2}}{n^n},\ [-3,-1]$.

10.17 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot x^n}{n},\ \left[-\frac{1}{2},\frac{1}{2}\right]$.

10.18 $\sum_{n=0}^{\infty}\frac{(n+1)^4\cdot x^{2n}}{2n+1},\ \left[-\frac{1}{2},\frac{1}{2}\right]$.

10.19 $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{(x-2)^{2n}}{n},\ \left[\frac{3}{2},\frac{5}{2}\right]$.

10.20 $\sum_{n=1}^{\infty}\frac{(x+5)^n}{n^2},\ [-6,-4]$.

10.21 $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\cdot 2^n},\ [1,3]$.

11-masala. Berilgan funksiyani Makloren qatoriga yoying.

11.1 $f(x)=(1+x)\cdot e^{-x}$.

11.2 $f(x)=\sin^2 x\cdot\cos^2 x$.

11.3 $f(x)=\frac{x^2-3x+1}{x^2-5x+6}$.

11.4 $f(x)=arctg\frac{1-x}{1+x}$.

11.5 $f(x)=arctg\frac{1+x}{1-x}$.

11.6 $f(x)=\arcsin\frac{x}{\sqrt{1+x^2}}$.

11.7 $f(x)=\arccos(1-2x^2)$.

11.8 $f(x)=\cos^2 x$.

11.9 $f(x)=\sin^3 x$.

11.10 $f(x)=\frac{x^{10}}{1-x}$.

11.11 $f(x)=\frac{1}{(1-x)^2}$.

11.12 $f(x)=\frac{x}{\sqrt{1-2x}}$.

11.13 $f(x)=\ln\sqrt{\frac{1+x}{1-x}}$.

11.14 $f(x)=\frac{x}{1+x-2x^2}$.

11.15 $f(x)=\frac{1}{(x^2+2)^2}$.

11.16 $f(x)=\frac{x}{9+x^2}$.

11.17 $f(x)=\frac{3x-5}{x^2-4x+3}$.

11.18 $f(x)=e^{2x}+2e^{-x}$.

11.19 $f(x)=\frac{x}{(1-x)(1-x^2)}$.

11.20 $f(x)=\frac{12-5x}{6-5x-x^2}$.

11.21 $f(x)=\ln(x^2+3x+2)$.

12-masala. Berilgan funksiyani ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va bu qatorning yaqinlashish sohasini toping.

12.1 $f(x)=\sqrt{x}; x_0=4$.

12.2 $f(x)=e^x; x_0=-2$.

12.3 $f(x)=\cos x; x_0=\frac{\pi}{4}$.

12.4 $f(x)=\frac{1}{x}; x_0=-2$.

12.5 $f(x)=\frac{1}{\sqrt{x^2-4x+8}}; x_0=2$.

12.6 $f(x)=\frac{1}{x^2-5x+6}; x_0=1$.

12.7 $f(x)=\cos^4 x; x_0=-\frac{\pi}{2}$.

12.8 $f(x)=\sin^4 x; x_0=\frac{\pi}{2}$.

12.9 $f(x)=\frac{1}{x-1}; x_0=3$.

12.10 $f(x)=e^{2x}; x_0=-1$.

12.11 $f(x)=\sin x; x_0=\frac{\pi}{4}.$

12.12 $f(x)=\sqrt{x}; x_0=3.$

12.13 $f(x)=\sin^2 x; x_0=-\frac{\pi}{2}.$

12.14 $f(x)=\cos^2 x; x_0=\frac{\pi}{2}.$

12.15 $f(x)=\frac{1}{x^2-5x+6}; x_0=-1.$

12.16 $f(x)=\sqrt[3]{x}; x_0=8.$

12.17 $f(x)=\sin^3 x; x_0=\frac{\pi}{4}.$

12.18 $f(x)=\cos^3 x; x_0=\frac{\pi}{4}.$

12.19 $f(x)=\frac{1}{1-x}; x_0=-1.$

12.20 $f(x)=\frac{1}{1-2x-x^2}; x_0=2.$

12.21 $f(x)=\cos^4 x; x_0=\frac{\pi}{4}.$

13-masala. Quyidagi qatorning yig'indisini toping.

13.1 $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\ln^n x}{2^n\cdot n!}.$

13.2 $\sum_{n=0}^{\infty}(2n+1)x^n.$

13.3 $\sum_{n=0}^{\infty}\frac{(3n+1)x^{3n}}{n!}.$

13.4 $\sum_{n=1}^{\infty}\frac{x^n}{n\cdot(n+1)}.$

13.5 $\sum_{n=1}^{\infty}n^4x^n.$

13.6 $\sum_{n=1}^{\infty}\frac{\sin nx}{n!}.$

13.7 $\sum_{n=1}^{\infty}\frac{x^n}{n}.$

13.8 $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$

13.9 $\sum_{n=0}^{\infty}(n+1)x^n.$

13.10 $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot(2n-1)x^{2n-2}.$

13.11 $\sum_{n=1}^{\infty}n\cdot(n+1)x^{n-1}.$

13.12 $\sum_{n=1}^{\infty}\frac{x^{4n-3}}{4n-3}.$

13.13 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)\cdot 3^{n-1}}.$

13.14 $\sum_{n=1}^{\infty}\frac{2n-1}{2^n}.$

13.15 $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}.$

13.16 $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1}.$

13.17 $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$

13.18 $x-4x^2+9x^3-16x^4+\ldots$

13.19 $1\cdot 2x+2\cdot 3x^2+3\cdot 4x^3+\ldots.$

13.20 $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots$

13.21 $\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\ldots$

14-masala. Berilgan qatorning yig'indisini toping.

14.1 $\sum_{n=1}^{\infty}(-1)^{n-1}\left(1+\frac{1}{n}\right)x^{n-1}.$

14.2 $\sum_{n=2}^{\infty}\frac{x^{2n}}{(2n-3)(2n-2)}.$

14.3 $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+2}\right)x^{n+2}.$

14.4 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot x^{2n-1}}{4^n\cdot(2n-2)}.$

14.5 $\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2n+1}x^{2n+1}.$

14.6 $\sum_{n=1}^{\infty}(-1)^{n-1}\left(1-\frac{1}{n}\right)\frac{1}{x^n}.$

14.7 $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}x^n}{n\cdot(n-1)}.$

14.8 $\sum_{n=0}^{\infty}\frac{1+(-1)^{n-1}}{2n+1}x^{2n+1}.$

14.9 $\sum_{n=1}^{\infty}\frac{x^n}{n\cdot(n+1)}.$

14.10 $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{16^n\cdot(2n+1)}.$

14.11 $\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+1)(2n+2)}.$

14.12 $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}+\frac{1}{n+1}\right)x^n.$

14.13 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n(n+1)}.$

14.14 $\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}.$

14.15 $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n\cdot(2n-1)}.$

14.16 $\sum_{n=1}^{\infty}\left[(-1)^n+\frac{1}{n}\right]x^{2n}$

14.17 $\sum_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]x^{n-1}$

14.18 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot(n+1)x^{n+1}}.$

14.19 $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)(n+2)}.$

14.20 $\sum_{n=2}^{\infty}\frac{\sin^n x}{n\cdot(n-1)}.$

14.21 $\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n\cdot(2n+1)}.$

15-masala. Quyidagi qatorning yig‘indisini toping.

15.1 $\sum_{n=0}^{\infty}(4n^2+9n+5)x^{n+1}$.

15.2 $\sum_{n=0}^{\infty}(3n^2+7n+4)x^{n}$.

15.3 $\sum_{n=0}^{\infty}(n^2+n+1)x^{n+3}$.

15.4 $\sum_{n=0}^{\infty}(2n^2+4n+3)x^{n+2}$.

15.5 $\sum_{n=0}^{\infty}(n^2+5n+3)x^{n}$.

15.6 $\sum_{n=0}^{\infty}(2n^2+5n+3)x^{n+1}$.

15.7 $\sum_{n=0}^{\infty}(3n^2+8n+5)x^{n+2}$.

15.8 $\sum_{n=0}^{\infty}(2n^2+8n+5)x^{n}$.

15.9 $\sum_{n=0}^{\infty}(2n^2+7n+5)x^{n+1}$.

15.10 $\sum_{n=0}^{\infty}(3n^2+7n+5)x^{n}$.

15.11 $\sum_{n=0}^{\infty}n\cdot(2n-1)x^{n+2}$.

15.12 $\sum_{n=0}^{\infty}(n^2-n+1)x^{n}$.

15.13 $\sum_{n=0}^{\infty}(2n^2-n-1)x^{n}$.

15.14 $\sum_{n=0}^{\infty}(3n^2+5n+4)x^{n+1}$.

15.15 $\sum_{n=0}^{\infty}(n^2+7n+4)x^{n}$.

15.16 $\sum_{n=0}^{\infty}(2n^2-n-2)x^{n+1}$.

15.17 $\sum_{n=0}^{\infty}(2n^2+2n+1)x^{n}$.

15.18 $\sum_{n=0}^{\infty}(n^2+2n-1)x^{n+1}$.

15.19 $\sum_{n=0}^{\infty}(n^2+2n+2)x^{n+2}$.

15.20 $\sum_{n=0}^{\infty}(n^2+4n+3)x^{n+1}$.

15.21 $\sum_{n=0}^{\infty}(n^2+5n+4)x^{n+2}$.

16-masala.

Integral ostidagi funksiyani qatorga yoyish yordamida berilgan integralni hisoblang.

16.1 $\int_0^1 \ln\frac{1}{1-x}dx$.

16.2 $\int_0^1 \ln x\cdot\ln(1-x)dx$.

16.3 $\int_0^{+\infty} \frac{x}{e^{2\pi x}-1}dx.$

16.4 $\int_0^{+\infty} \frac{xdx}{e^x+1}.$

16.5 $\int_0^1 \frac{\ln\left(x+\sqrt{1+x^2}\right)}{x}dx.$

Darajali qatorlar yordamida quyidagi differensial tenglamaning berilgan boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

16.6 $y'-y=0,\ y(0)=1.$

16.7 $\left(1+x^2\right)y'-1=0,\ y(0)=0.$

16.8 $y'+\lambda^2 y=0,\ y(0)=1,\ y'(0)=\lambda.$

16.9 $y'-xy=0,\ y(0)=1,\ y'(0)=0.$

16.10 $\left(1-x^2\right)y''-xy'=0,\ y(0)=0,\ y'(0)=1.$

16.11 $\left(1-x^2\right)y''-5xy'-4y=0,\ y(0)=1,\ y'(0)=0.$

16.12 $y''-xy=0,\ y(0)=0,\ y'(0)=1.$

Integral ostidagi funksiyani darajali qatorga yoyish usuli yordamida berilgan integralni 0,001 aniqlikda hisoblang.

16.13 $\int_0^1 \frac{\sin x}{x}dx.$

16.14 $\int_0^1 \sqrt[3]{x}\cos x dx.$

16.15 $\int_0^1 \sin x^2 dx.$

16.16 $\int_0^1 \frac{dx}{\sqrt{1+x^4}}.$

16.17 $\int_2^4 e^{1/x}dx.$

16.18 $\int_0^{1/2} \frac{arctgx}{x}dx.$

16.19 $\int_0^{1/3} \frac{dx}{\sqrt[3]{1-x^2}}.$

16.20 $\int_5^{10} \frac{\ln\left(1+x^2\right)}{x^2}dx.$

16.21 $\int_0^1 e^{-x^2}dx.$

-D-
Namunaviy variant yechimi

1.21-masala. Ushbu $f_n(x)=\ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right)$ **funksional ketma-ketlikning** $M=[0;+\infty)$ **to'plamdagi limit funksiyasini' toping.**

$$\triangleleft f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right)=\ln 3+\lim_{n\to\infty}\ln\left[1+\frac{n^2e^x}{3\cdot\left(n^4+e^{2x}\right)}\right]=$$

$$=\ln 3+\lim_{n\to\infty}\frac{\ln\left(1+\frac{n^2e^x}{3\left(n^4+e^{2x}\right)}\right)}{\frac{n^2e^x}{3\left(n^4+e^{2x}\right)}}\cdot\lim_{n\to\infty}\frac{n^2e^x}{3\left(n^4+e^{2x}\right)}=\ln 3+1\cdot 0=\ln 3\,\triangleright.$$

2.21-masala. $f_n(x)=\frac{n+x}{n+x+\sqrt{nx}}$ **funksional ketma-ketlikni a)** $0\le x<+\infty$ **b)** $0\le x\le 1$ **oraliqlarda tekis yaqinlashishga tekshiring.**

$\triangleleft$ Ikkala oraliqda ham $f_n(x)$ ketma-ketlik yaqinlashuvchi bo'lib,

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{n+x}{n+x+\sqrt{nx}}=\lim_{n\to\infty}\frac{1+\frac{x}{n}}{1+\frac{x}{n}+\sqrt{\frac{x}{n}}}=1;$$

bo'ladi. Endi tekis yaqinlashishga tekshirish uchun 1^0-punktdagi 1-teoremadan foydalanamiz.

$$r_n(x)=\left|f(x)-f_n(x)\right|=\left|1-\frac{n+x}{n+x+\sqrt{nx}}\right|=\frac{\sqrt{nx}}{n+x+\sqrt{nx}}$$

deb belgilasak, 1-teoremaga ko'ra $\{f_n(x)\}$ ketma-ketlik M to'plamda tekis yaqinlashishi uchun ushbu

$$\lim_{n\to\infty}\sup_{x\in M}r_n(x)=0;$$

munosabatning bajarilishi zarur va yetarli.

a) $0\le x<+\infty$ bo'lsin.

$$r_n'(x)=\frac{\sqrt{n}\cdot(n-x)}{2\sqrt{x}\cdot\left(n+x+\sqrt{nx}\right)}\Rightarrow\sup_{x\in[0,+\infty]}r_n(x)=r_n(n)=$$

$$=\frac{\sqrt{n\cdot n}}{n+n+\sqrt{n\cdot n}}=\frac{n}{3n}=\frac{1}{3}\Rightarrow \lim_{n\to\infty}\sup_{x\in[0,+\infty)} r_n(x)=\frac{1}{3}\neq 0\Rightarrow f_n(x)\not\rightrightarrows 1.$$

b) $0\le x\le 1$ bo'lsin. Bu oraliqda $r_n'(x)\ge 0$ bo'lgani uchun

$$\{r_n(x)\}\uparrow \Rightarrow \sup_{0\le x\le 1} r_n(x)=r_n(1)=\frac{\sqrt{n}}{n+1+\sqrt{n}} \Rightarrow \lim_{n\to\infty}\sup_{0\le x\le 1} r_n(x)=\lim_{n\to\infty}\frac{\sqrt{n}}{n+1+\sqrt{n}}=0 \Rightarrow$$

$\Rightarrow$ $f_n(x)$ funksional ketma-ketlik 1 ga tekis yaqinlashmaydi.

Demak, berilgan funksional ketma-ketlik $0\le x<+\infty$ to'plamda notekis, $0\le x\le 1$ to'plamda esa tekis yaqinlashar ekan. ▷

3.21-masala. Veyershtrass alomatidan foydalanib, $\sum_{n=1}^{\infty}\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right]$ funksional qatorning $0\le x\le 2$ oraliqda tekis yaqinlashishini ko'rsating.

◁ $u_n(x)=\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right]$ Berilgan $0\le x\le 2$ oraliqda quyidagi tengsizliklar o'rinli.

$$|u_n(x)|=\left|\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right]\right|=\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right]\le\frac{x}{n\cdot\ln^2(n=1)}\le\frac{2}{n\cdot\ln^2(n+1)}.$$

Agar $a_n=\frac{2}{n\cdot\ln^2(n+1)}$ deb belgilasak, Koshining integral alomatiga ko'ra $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{n\cdot\ln^2(n+1)}$ sonli qator yaqinlashuvchi bo'ladi. Unda Veyershtras alomatiga ko'ra berilgan funksional qator $0\le x\le 2$ oraliqda tekis yaqinlashuvchi. ▷

4.21-masala. Berilgan ushbu

$$\sum_{n=1}^{\infty}\frac{n}{(1+2x^2)\cdot(1+4x^2)\cdot\ldots\cdot(1+2nx^2)};$$

funksional qatorning $1\le x<+\infty$ oraliqda tekis yoki notekis yaqinlashuvchiligini aniqlang.

◁ Bu qatorning tekis yaqinlashishini tekshirish uchun 2^0-punktdagi 1-teoremadan, ya'ni (10)-tenglikdan foydalanamiz.

$$u_n(x)=\frac{n}{\left(1+2x^2\right)\cdot\left(1+4x^2\right)\cdot\ldots\cdot\left(1+2nx^2\right)}=$$

$$=\frac{1}{2x^2}\left[\frac{1}{\left(1+2x^2\right)\cdot\left(1+4x^2\right)\cdot\ldots\cdot\left(1+2(n-1)x^2\right)}-\frac{1}{\left(1+2x^2\right)\left(1+4x^2\right)\cdot\ldots\cdot\left(1+2nx^2\right)}\right],\quad n=2,3,\ldots$$

va

$$u_1(x)=\frac{1}{2x^2}\cdot\left(1-\frac{1}{1+2x^2}\right)\Rightarrow S_n(x)=\sum_{k=1}^{n}u_k(x)=\frac{1}{2x^2}\left[1-\frac{1}{\left(1+2x^2\right)\cdot\left(1+4x^2\right)\cdot\ldots\cdot\left(1=2nx^2\right)}\right]\Rightarrow$$

$\Rightarrow \forall x\in[1,+\infty)$ uchun $S(x)=\lim\limits_{n\to\infty}S_n(x)=\dfrac{1}{2x^2}$ va $r_n(x)=S(x)-S_n(x)=$

$$=\frac{1}{\left(1+2x^2\right)\cdot\left(1+4x^2\right)\cdot\ldots\cdot\left(1+2nx^2\right)}\Rightarrow \sup_{x\in[1,+\infty)}\left|r_n(x)\right|=\frac{1}{(1+2)(1+4)\cdot\ldots\cdot(1+2n)}\Rightarrow$$

$\Rightarrow \lim\limits_{n\to\infty}\sup\limits_{x\in[1,+\infty)}\left|r_n(x)\right|=0\Rightarrow$ Berilgan qator $[1,+\infty)$ oraliqda tekis yaqinlashadi. ▷

5.21-masala. $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n\cdot\left(n^2+1\right)}\cdot\left(25x^2+1\right)^n$ **funksional qatorning yaqinlashish sohasini toping.**

◁ Yaqinlashish sohasini Koshi alomatidan foydalanib, topamiz:

$$\lim_{n\to\infty}\sqrt[n]{\left|u_n(x)\right|}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n\cdot\left(n^2+1\right)}\cdot\left(25x^2+1\right)^n}=\frac{25x^2+1}{2}<1;$$

bo‘lsa, yaqinlashadi. $\Rightarrow x^2<\dfrac{1}{25}\Rightarrow |x|<\dfrac{1}{5}\Rightarrow x\in\left(-\dfrac{1}{5};\dfrac{1}{5}\right)$ da qator yaqinlashadi. Chegaraviy $x=\pm\dfrac{1}{5}$ nuqtada esa $u_n\left(\pm\dfrac{1}{5}\right)=\dfrac{n^2}{n^2+1}$ bo‘lib,

$$\lim_{n\to\infty}u_n\left(\pm\frac{1}{n}\right)=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1\neq 0$$

qator yaqinlashishining zaruriy sharti bajarilmaydi $\Rightarrow$ yaqinlashish sohasi $\left(-\dfrac{1}{5};\dfrac{1}{5}\right)$ interval ekan. ▷

6.21-masala. Ushbu

$$\sum_{n=1}^{\infty} 27^n \cdot x^{3n} \cdot arctg\frac{3x}{2n+3}$$

funksional qatorning yaqinlashish sohasini toping.

◁ Bu qatorning yaqinlashish sohasini Dalamber alomatidan foydalanib, topamiz:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\frac{\left|27^{n+1}\cdot x^{3n+3}\cdot arctg\frac{3x}{2n+5}\right|}{\left|27^n\cdot x^{3n}\cdot arctg\frac{3x}{2n+3}\right|} = 27|x|^3\cdot\lim_{n\to\infty}\left(\frac{3|x|}{2n+5}\cdot\frac{2n+3}{3\cdot|x|}\right) = 27\cdot|x|^3 < 1;$$

bo'lsa yaqinlashadi. $\Rightarrow |x| < \frac{1}{3}$ yoki $x \in \left(-\frac{1}{3};\frac{1}{3}\right)$ da yaqinlashadi.

Chegaraviy nuqtalarda tekshiramiz.

1) $x=\frac{1}{3}$ bo'lsin $\Rightarrow u_n\left(\frac{1}{3}\right) = arctg\frac{1}{2n+3} = 0^*\left(\frac{1}{2n+3}\right)$ va $\sum_{n=1}^{\infty}\frac{1}{2n+3}$ uzoqlashuvchi $\Rightarrow$ berilgan qator $x=\frac{1}{3}$ nuqtada uzoqlashadi.

2) $x=-\frac{1}{3}$ bo'lsin $\Rightarrow u_n\left(-\frac{1}{3}\right) = (-1)^{n+1} arctg\frac{1}{2n+3}$ va $\sum_{n=1}^{\infty}(-1)^{n+1} arctg\frac{1}{2n+3}$ qator Lebnis alomatiga ko'ra yaqinlashuvchi $\Rightarrow$ berilgan qator $x=-\frac{1}{3}$ nuqtada yaqinlashuvchi.

Demak, berilgan funksional qatorning yaqinlashish sohasi $\left[-\frac{1}{3};\frac{1}{3}\right)$ yarim interval. ▷

7.21-masala. Berilgan

$$\sum_{n=1}^{\infty}(-1)^n 3^{-n^2\ln\left(1+\frac{x}{n}\right)}.$$

qatorning yaqinlashish sohasini toping.

◁ Koshi alomatiga ko'ra

$$\lim_{n\to\infty}\sqrt[n]{|u_n(x)|} = \lim_{n\to\infty} 3^{-n\ln\left(1+\frac{x}{n}\right)} = 3^{-x} < 1;$$

bo‘lsa, ya’ni $x>0$ bo‘lganda berilgan qator yaqinlashadi. Chegaraviy $x=0$ nuqtada $u_n(0)=(-1)^n$ bo‘lib, $\sum_{n=1}^{\infty}(-1)^n$ qator uzoqlashadi.

Demak, berilgan qatorning yaqinlashish sohasi $(0,+\infty)$ oraliqda iborat ekan. ▷

8.21-masala. $\sum_{n=1}^{\infty}\frac{1}{(n+2)\cdot\ln(n+2)\cdot(x-3)^{2n}}$. **funksional qatorning yaqinlashish sohasini toping.**

◁ Qo‘yilgan masalani Dalamber alomatidan foydalanib, yechamiz. Agar

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim_{n\to\infty}\frac{(n+2)\ln(n+2)\cdot(x-3)^{2n}}{(n+3)\ln(n+3)\cdot(x-3)^{2n+2}}=\frac{1}{(x-3)^2}<1;$$

bo‘lsa, unda berilgan funksional qator yaqinlashadi $\Rightarrow(x-3)^2>1\Rightarrow|x-3|>1\Rightarrow x\in(-\infty;2)\cup(4;+\infty)$ to‘plamda berilgan qator yaqinlashadi. Chegaraviy $x=2$ va $x=4$ nuqtalarda

$$u_n(x)=\frac{1}{(n+2)\ln(n+2)}.$$

bo‘lib, $\sum_{n=1}^{\infty}\frac{1}{(n+2)\ln(n+2)}$ sonli qator Koshining integral alomatiga ko‘ra uzoqlashadi $\Rightarrow$ Berilgan funksional qatorning yaqinlashish sohasi $(-\infty;2)\cup(4;+\infty)$ to‘plamdan iborat. ▷

9.21-masala. Ta’rifdan foydalanib,

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{7n-13}.$$

funksional qatorning $[0;\ 1]$ **kesmada tekis yaqinlashishini isbotlang** $(n_0(\varepsilon)-?)$. **n ning qanday qiymatlrida qatorning qoldig‘i** $\forall x\in[0,\ 1]$ **uchun 0,1 dan katta bo‘lmaydi?**

◁ Bu masalani yechish uchun $\forall\varepsilon>0$ olinganda ham $\exists n_0=n_0(\varepsilon)\in N$ topishimiz kerakki, $\forall n>n_0$ va barcha $x\in[0,1]$ uchun $|r_n(x)|=\left|\sum_{k=n}^{\infty}u_k(x)<\varepsilon\right|$ tengsizlik bajarilishi lozim. $\forall\varepsilon>0$ son olamiz va quyidagi baholashlarni amalga oshiramiz:

$$|r_n(x)|=\left|\sum_{k=n}^{\infty}u_k(x)\right|=\left|\sum_{k=n}^{\infty}(-1)^k\frac{x^k}{7k-13}\right|=\left|(-1)^n\cdot\frac{x^n}{7n-13}+(-1)^{n+1}\cdot\frac{x^{n+1}}{7\cdot(n+1)-13}+(-1)^{n+2}\frac{x^{n+2}}{7(n+2)-13}+\ldots\right|=$$

$$=\left|(-1)^n\cdot x^n\left[\frac{1}{7n-13}-\left(\frac{1}{7(n+1)-13}-\frac{1}{7(n+2)-13}\right)-\left(\frac{1}{7(n+3)-13}-\frac{1}{7(n+4)-13}\right)-...\right]\right|<$$

$$<\frac{|x|^n}{7n-13}\le\frac{1}{7n-13}<\varepsilon\Rightarrow 7n-13>\frac{1}{\varepsilon}\Rightarrow 7n>\frac{1}{\varepsilon}+13\Rightarrow n>\frac{1}{7}\left(\frac{1}{\varepsilon}+13\right)\Rightarrow\forall\varepsilon>0;$$

olinganda ham $n_0(\varepsilon)=\left[\frac{1}{7}\left(\frac{1}{\varepsilon}+13\right)\right]$ deb olsak, $\forall n>n_0$ va $\forall x\in[0,1]$ lar uchun $\left|\sum_{k=n}^{+\infty}u_k(x)\right|<\varepsilon$ tengsizlik bajariladi. Bu esa $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{7n-13}$ funksional qator $[0,1]$ kesmada tekis yaqinlashishini anglatadi.

Masalaning ikkinchi qismini yechish uchun $\varepsilon=0{,}1$ deyish kifoya $\Rightarrow$ $n_0(\varepsilon)=\left[\frac{1}{7}\left(\frac{1}{0{,}1}+13\right)\right]=\left[\frac{23}{7}\right]=3\Rightarrow$ barcha $n>3$ lar uchun $|r_n(x)|\le 0{,}1$ bo‘ladi. ▷

10.21-masala. $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\cdot 2^n}$ **funksional qator uchun uni** $[1;3]$ **kesmada majorirlovchi qatorni toping va ko‘rsatilgan oraliqda tekis yaqinlashishini isbotlang.**

◁ $\forall x\in[1,3]$ uchun $|u_n(x)|=\frac{|x-2|^n}{(2n-1)\cdot 2^n}\le\frac{1}{(2n-1)\cdot 2^n}$ bo‘lib, $a_n=\frac{1}{(2n-1)\cdot 2^n}$ desak, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cdot 2^n}$ sonli qator berilgan qator uchun uni majorirlovchi qator bo‘ladi. $\sum_{n=1}^{\infty}a_n$ qator yaqinlashuvchi bo‘lgani uchun Veyershtrass alomatiga ko‘ra berilgan $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\cdot 2^n}$ qatorning $[1;\ 3]$ kesmada tekis yaqinlashuvchi ekanligini hosil qilamiz. ▷

11.21-masala. $f(x)=\ln(x^2+3x+2)$. **funksiyani Makloren qatoriga yoying.**

◁ Bu masalani yechish uchun

$$f(x)=\ln(x^2+3x+2).=\ln[(x+1)(x+2)]=$$

$$=\ln(1+x)+\ln(2+x)=\ln(1+x)+\ln 2+\ln\left(1+\frac{x}{2}\right);$$

deb olib, 5^0-punktda keltirilgan

tenglikdan foydalanamiz:

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, x\in(-1,\ 1];$$

$$\ln\left(x^2+3x+2\right).=\ln 2+\ln(1+x)+\ln\left(1+\frac{x}{2}\right)=$$

$$=\ln 2+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\left(\frac{x}{2}\right)^n=$$

$$=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(1+\frac{1}{2^n}\right)x^n. \triangleright$$

12.21-masala. $f(x)=\cos^4 x$ **funksiyani** $x_0=\frac{\pi}{4}$ **nuqta atrofida Teylor qatoriga yoying va bu qatorning yaqinlashish sohasini toping.**

$\triangleleft$ Avvalo $\cos^2 x=\frac{1}{2}(1+\cos 2x)$ ekanini e'tiborga olib,

$$f(x)=\cos^4 x=\left(\cos^2 x\right)^2=\frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)=$$

$$=\frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

bo'lishini topamiz. So'ngra $x=t+\frac{\pi}{4}$ almashtirishni bajaramiz:

$$f(x)=f\left(t+\frac{\pi}{4}\right)=\frac{3}{8}+\frac{1}{2}\cos\left(2t+\frac{\pi}{2}\right)+\frac{1}{8}\cos(4t+\pi)=\frac{3}{8}-\frac{1}{2}\sin 2t-\frac{1}{8}\cos 4t.$$

Endi $\sin x$ hamda $\cos x$ larning 5^0 punktda keltirilgan yoyilmalaridan foydalanib, ushbu

$$\sin 2t=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n+1}}{(2n+1)!}t^{2n+1},$$

$$\cos 4t=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 4^{2n}}{(2n)!}t^{2n},$$

tengliklarga ega bo'lamiz. Natijada

$$f(x)=\frac{3}{8}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}-\frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n}}{(2n)!}\cdot\left(x-\frac{\pi}{4}\right)^{2n}=$$

$$=\frac{1}{4}-\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}}{(2n+1)!}\cdot\left(x-\frac{\pi}{4}\right)^{2n+1}-\sum_{n=1}^{\infty}\frac{(-1)^n 2^{4n-3}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}$$

qatorni hosil qilamiz. Bu qatorning yaqinlashish sohasi $(-\infty,+\infty)$ ekanligini ko'rish qiyin emas. ▷

13.21-masala. Quyidagi

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\dots$$

qatorning yig'indisini toping.

◁ Berilgan $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot(n+1)}$ qator uchun $\left|\frac{(-1)^{n+1}}{n(n+1)}\right|=\frac{1}{n(n+1)}=0^{*}\left(\frac{1}{n^2}\right)\Rightarrow$ taqqoslash amlomatiga ko'ra u absolut yaqinlshuvchi $\Rightarrow$ Chekli yig'indiga ega.

$$S=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)};$$

deb belgilaymiz.

Ushbu

$$S(x)=\sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)} \qquad (1)$$

yordamchi qatorni kiritamiz. Bu qator $|x|\le 1$ da absolut va tekis yaqinlashadi. Abelning 2-teoremasiga ko'ra (5^0 punktdagi 7-teorema va uning natijasiga qarang)

$$S=\lim_{x\to -1+0}S(x);$$

bo'ladi. $S(x)$ funksiyani topish uchun (1)-tenglikni 2 marta differensiallaymiz.

$$S'(x)=\sum_{n=1}^{\infty}\frac{x^n}{n},$$

$$S''(x)=\sum_{n=1}^{\infty}x^{n-1}=1+x+x^2+x^3+\dots=\frac{1}{1-x},|x|<1\Rightarrow$$

$$\Rightarrow\quad S'(x)=\ln\frac{1}{1-x}+c_1 \text{ va } S''(0)=0\Rightarrow$$

$$\Rightarrow c_1=0\Rightarrow S(x)=(1-x)\cdot\ln(1-x)+x+c_2 \text{ va } S(0)=0\Rightarrow c_2=0;$$

Demak, $S(x)=(1-x)\cdot\ln(1-x)+x$ ekan $\Rightarrow S=\lim_{x\to -1+0}S(x)=2\ln 2-1.$ ▷

14.21-masala. $\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n\cdot(2n+1)}$. **qatorning yig'indisini toping.**

◁ Bu qatorning yaqinlashish sohasi $[-1;\ 1]$ kesmadan iborat bo‘lib, bu kesmaning ichki nuqtalarida qatorni hadlab, differensiallash mumkin:

$$S(x)=\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n\cdot(2n+1)}.\Rightarrow S'(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}\Rightarrow S''(x)=\sum_{n=1}^{\infty}x^{2n-1}=x+x^3+x^5+...=\frac{x}{1-x^2},$$

$$x\in(-1;1)\quad S'(x)=\int\frac{x}{1-x^2}dx+c_1=-\frac{1}{2}\ln\left(1-x^2\right)+c_1 \text{ va } S'(0)=0\Rightarrow c_1=0\Rightarrow$$

$S(x)=\int S'(x)dx+c_2=-\frac{1}{2}\int\ln\left(1-x^2\right)dx+c_2=$((bo‘laklab integrallash usulidan foydalanamiz)) $=-\frac{x}{2}\ln\left(1-x^2\right)+x+\frac{1}{2}\ln\frac{1-x}{1+x}+c_2$ va $S(0)=0\Rightarrow\ c_2=0$.

Demak, $\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n\cdot(2n+1)}=x+\frac{1}{2}\ln\frac{1-x}{1+x}-\frac{x}{2}\ln\left(1-x^2\right)$. ekan. Tenglik $(-1;\ 1)$ intervalda o‘rinli. ▷

15.21-masala. $\sum_{n=0}^{\infty}\left(n^2+5n+4\right)x^{n+2}$. **qatorning yig‘indisini toping.**

◁ Berilgan qator $(-1;\ 1)$ intervalda absolut va tekis yaqinlashadi va shu intervaldagi $\forall$ oraliqda bu qatorni hadlab integrallash mumkin:

$$S(x)=\sum_{n=0}^{\infty}\left(n^2+5n+4\right)x^{n+2}=\sum_{n=0}^{\infty}(n+1)(n+4)x^{n+2}\Rightarrow x\cdot S(x)=\sum_{n=0}^{\infty}(n+1)(n+4)x^{n+3}\Rightarrow$$

$$\Rightarrow\int x\cdot S(x)dx=\sum_{n=0}^{\infty}(n+1)x^{n+4}+c_1=x^4\sum_{n=0}^{\infty}(n+1)x^n+c_1=x^4\cdot S_1(x)+c_1$$

$x=0$ da $S(0)=S_1(0)=0\Rightarrow c_1=0$

Demak,

$$\int x\cdot S(x)dx=x^4\cdot S_1(x) \text{ va } S_1(x)=\sum_{n=0}^{\infty}(n+1)x^n\Rightarrow\int S_1(x)dx=$$

$$=\sum_{n=0}^{\infty}x^{n+1}=x+x^2+x^3+...=\frac{x}{1-x}+c_2\quad x=0 \text{ da } S_1(0)=0\Rightarrow c_2=0.$$

$\int S_1(x)dx=\frac{x}{1-x}\Rightarrow S_1(x)=\left(\frac{x}{1-x}\right)'=\frac{1}{(1-x)^2}$. Bu tenglik va

$\int x\cdot S(x)dx=x^4\cdot S_1(x)$ dan

$$\Rightarrow x\cdot S(x)=\left(x^4\cdot S_1(x)\right)'=\left[\frac{x^4}{(1-x)^4}\right]'=\frac{4x^3\cdot(1-x)^2+2x^4\cdot(1-x)}{(1-x)^4}=$$

$$=\frac{2x^3\cdot(2-x)}{(1-x)^3}\Rightarrow S(x)=\frac{2x^2\cdot(2-x)}{(1-x)^3}.$$

Shunday qilib,

$$\sum_{n=0}^{\infty}\left(n^2+5n+4\right)x^{n+2}=\frac{2x^2\cdot(2-x)}{(1-x)^3},\quad x\in(-1;\ 1).\triangleright$$

16.21-masala. Integral ostidagi funksiyani darajali qatorga yoyish usuli yordamida $\int_0^1 e^{-x^2}dx$ integralni 0,001 aniqlikda hisoblang.

$\triangleleft$ Agar 5⁰-punktda e^x uchun keltirilgan yoyilmadan foydalansak,

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{n!};$$

ekanligini, bu yerdan esa

$$\int_0^1 e^{-x^2}dx=\int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}\right]dx=\sum_{n=0}^{\infty}\int_0^1\frac{(-1)^n\cdot x^{2n}}{n!}dx=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n+1}}{n!(2n+1)}\Bigg|_0^1=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\cdot(2n+1)}.$$

bo‘lishini topamiz. Bu hosil bo‘lgan qator Leybnis qatori bo‘lib, uning m-hadidan keyingi qoldig‘i

$$r_m=\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!\cdot(2n+1)};$$

uchun

$$|r_m|\le\frac{1}{(m+1)!\cdot(2m+3)};$$

bo‘lishi bizga ma’lum. $|r_m|\le 0{,}001$ bajarilishi uchun oxirgi tengsizlikdan $m\ge 4$ bo‘lishi kifoyaligini aniqlaymiz. Demak,

$$\int_0^1 e^{-x^2}dx\approx 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{42}+\frac{1}{216}\approx 0{,}747.\triangleright$$

8-§. 7-MUSTAQIL ISH
Xosmas va parametrga bog'liq integrallar

1-tur xosmas integrallar va ularning yaqinlashishi.

2-tur xosmas integrallar va ularning yaqinlashishi.

Xosmas integralning Koshi ma'nosidagi bosh qiymati.

Parametrga bog'liq bo'lgan xos integrallar va ularning funksional xossalari.

Parametrga bog'liq bo'lgan xosmas integrallar va ularning tekis yaqinlashishi.

Parametrga bog'liq bo'lgan xosmas integrallar va ularning funksional xossalari.

Eyler integrallari.

-A-
Asosiy tushuncha va teoremalar

Biz 1-kursda $\int_a^b f(x)dx$ aniq integralni o'rganish jarayonida unga 2 ta shart qo'ydik:

1) a va b lar chekli sonlar,

2) $[a,b]$ da berilgan $f(x)$ funksiya shu kesmada chegaralangan.

Endi biz aniq integralni quyidagi umumiyroq hollarda o'rganamiz.

1-hol. Oraliq cheksiz, lekin funksiya chegaralangan,

2-hol. Oraliq chekli, lekin funksiya chegaralanmagan.

1-holda hosil bo'lgan integralga **I-tur xosmas integral**, 2-holda hosil bo'lgan integralga esa **II-tur xosmas integral** deyiladi.

Birinchi va ikkinchi tur xosmas integrallar va ularning xossalarini alohida-alohida va batafsilroq o'rganamiz.

1⁰. Chegaralari cheksiz xosmas integrallar (I-tur xosmas integrallar)

Integrallash oralig'i cheksiz bo'lgan holni ko'raylik. Bunda 3 ta vaziyat yuz berishi mumkin:

1) $a \le x < +\infty$;

2) $-\infty < x \le b$;

3) $-\infty < x < +\infty$.

Aniqlik uchun 1-vaziyatni to‘liq ko‘rib chiqaylik.

Faraz qilaylik, $f(x)$ funksiya $[a,+\infty)$ nurda aniqlangan bo‘lib, $\forall A \geq a$ soni uchun $\int_a^A f(x)dx$ mavjud bo‘lsin.

$$F(A)=\int_a^A f(x)dx; \qquad (1)$$

deb belgilaymiz.

1-ta’rif. *Agar ushbu*

$$\lim_{A\to+\infty} F(A)=\lim_{A\to+\infty}\int_a^A f(x)dx;$$

limit mavjud va chekli bo‘lsa, uni $f(x)$ funksiyaning $[a,+\infty)$ ***ora-liqdagi I-tur xosmas integrali*** *deyiladi va u*

$$\int_a^{+\infty} f(x)dx; \qquad (2)$$

kabi belgilanadi hamda (2)-xosmas integral ***yaqinlashuvchi,*** *aks hol-da esa* ***uzoqlashuvchi*** *deb ataladi.*

Shunday qilib,

$$\int_a^{+\infty} f(x)dx := \lim_{A\to+\infty}\int_a^A f(x)dx.$$

Qolgan 2 ta vaziyatda ham I-tur xosmas integral shunga o‘xshash ta’riflanadi:

$$\int_{-\infty}^{b} f(x)dx := \lim_{A\to-\infty}\int_A^b f(x)dx,$$

$$\int_{-\infty}^{+\infty} f(x)dx := \lim_{\substack{A\to-\infty \\ B\to+\infty}}\int_A^B f(x)dx,$$

Agar $\int_{-\infty}^{a} f(x)dx$ va $\int_a^{+\infty} f(x)dx$ xosmas integrallar yaqinlashsa, u holda $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integral ham yaqinlashadi va

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx;$$

bo‘ladi.

Misol. $\int_a^{+\infty}\frac{dx}{x^\lambda}$ *($a>0$ va λ – $\forall$ haqiqiy son) xosmas integralni yaqinlashishga tekshiring.*

$\triangleleft$ $\forall\ A>a$ *olamiz*

$$F(A)=\int_a^A\frac{dx}{x^\lambda}=\begin{cases}\left.\frac{x^{1-\lambda}}{1-\lambda}\right|_a^A=\frac{A^{1-\lambda}-a^{1-\lambda}}{1-\lambda}, & \lambda\neq 1,\\ \ln x|_a^A=\ln A-\ln a, \lambda=1.\end{cases}$$

$\lambda>1$ *bo'lsa* $\lim\limits_{A\to+\infty}F(A)=\frac{a^{1-\lambda}}{\lambda-1}$ *bo'lib, integral yaqinlashadi.* $\lambda\leq 1$ *bo'lsa* $\lim\limits_{A\to+\infty}F(A)=\infty$ *bo'lib, integral uzoqlashadi.*

Shunday qilib,

$$\int_a^{+\infty}\frac{dx}{x^\lambda}=\begin{cases}\text{yaqinlashadi, agar } \lambda>1 \text{ bo'lsa,}\\ \text{uzoqlashadi, agar } \lambda\leq 1 \text{ bo'lsa.}\end{cases} \triangleright$$

1-teorema. (Koshi kriteriyasi). *(2)-xosmas integralning yaqinlashuvchi bo'lishi uchun quyidagi shartning bajarilishi zarur va yetarlidir:* $\forall\varepsilon>0$ *uchun* $\exists B>a:\forall A_1>B$ *va* $A_2>B$ *lar uchun*

$$\left|\int_{A_1}^{A_2}f(x)dx\right|<\varepsilon;$$

bo'ladi.

Ko'p hollarda Koshi shartini tekshirish qiyin bo'ladi. Shuning uchun tekshirish oson bo'lgan alomatlarni keltiramiz.

Bundan buyon biz har doim $\forall A\geq a$ uchun $\int_a^A f(x)dx$ mavjud deb faraz qilamiz.

2-teorema. ***(Umumiy taqqoslash alomati).*** *Faraz qilaylik,* $[a,+\infty)$ *nurda*

$$|f(x)|\leq g(x)$$

bo'lib, $\int_a^{+\infty}g(x)dx$ *xosmas integral yaqinlashsin. Unda* $\int_a^{+\infty}f(x)dx$ *xosmas integral ham yaqinlashadi.*

3-teorema. (Xususiy taqqoslash alomati). *Faraz qilaylik* $0<a\leq x<+\infty$ *nurda* $|f(x)|\leq\frac{c}{x^\lambda}$, *c,* $\lambda-const$ *va* $\lambda>1$ *bo'lsin. U* *holda* $\int\limits_a^{+\infty} f(x)dx$ *xosmas integral yaqinlashadi. Agar* $\exists c>0$:

$$|f(x)| \geq \frac{c}{x^{\lambda}},$$

bo'lib, $\lambda \leq 1$ *bo'lsa,* $\int\limits_a^{+\infty} f(x)dx$ *xosmas integral uzoqlashadi.*

2-ta'rif. *Agar* $\int\limits_a^{+\infty} |f(x)|dx$ *yaqinlashuvchi bo'lsa, u holda* $\int\limits_a^{+\infty} f(x)dx$ *xosmas integral* ***absolut yaqinlashuvchi*** *deyiladi.*

2-teoremaga ko'ra absolut yaqinlashuvchi integral oddiy ma'noda ham yaqinlashuvchi bo'ladi.

3-ta'rif. *Agar* $\int\limits_a^{+\infty} f(x)dx$ *yaqinlashib* $\int\limits_a^{+\infty} |f(x)|dx$ *uzoqlashsa,* $\int\limits_a^{+\infty} f(x)dx$ *xosmas integral* ***shartli yaqinlashuvchi*** *deyiladi.*

4-teorema. *$f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda aniqlangan bo'lib, $f(x) \geq 0$ va $g(x) \geq 0$ bo'lsin.*

Agar $x \to +\infty$ da

$$f(x) = O^{*}(g(x));$$

bo'lsa, $\int\limits_a^{+\infty} f(x)dx$ va $\int\limits_a^{+\infty} g(x)dx$ xosmas integrallar yoki bir vaqtda yaqinlashadi yoki uzoqlashadi.

5-teorema. *$f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda aniqlangan bo'lib, ular quyidagi shartlarni bajarsin:*

(Abel alomati) *a)* $\int\limits_a^{+\infty} f(x)dx$ *yaqinlashuvchi,*

b) $g(x)$ *funksiya* $[a,+\infty)$ *da monoton va chegaralangan;*

(Dirixle alomati) *a)* $\exists K \ \ \forall A>a \ \ \left|\int\limits_a^{A} f(x)dx\right| \leq K,$

b) $g(x)$ *funksiya* $[a,+\infty)$ *da monoton*

$$va \quad \lim_{x\to+\infty} g(x)=0 .$$

U holda $\int_a^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo‘ladi.

Misollar. 1) $\int_1^{+\infty} \sin x^2 dx$ **xosmas integral yaqinlashishga tekshirilsin.**

◁ $\int_1^{+\infty} \sin x^2 dx = \int_1^{+\infty} x\sin x^2 \cdot \frac{1}{x} dx$ deb olib, $f(x)=x\sin x^2$, $g(x)=\frac{1}{x}$ deb belgilaymiz va Dirixle alomatining shartlarini tekshiramiz:

1) $f(x)=x\sin x^2 \in C[1,+\infty)$ va $F(x)=-\frac{1}{2}\cos x^2 -$ chegaralangan;

2) $g(x)=\frac{1}{x}\downarrow$ *va* $\lim_{x\to+\infty} g(x)=0$;

$$\Rightarrow \quad \int_1^{+\infty} f(x)g(x)dx = \int_1^{+\infty} \sin x^2 dx - \text{yaqinlashuvchi.} \,\triangleright$$

2) $\int_1^{+\infty} \frac{\sin x}{x} dx$ **xosmas integralning shartli yaqinlashuvchi ekanligi ko‘rsatilsin.**

◁ Agar $f(x)=\sin x$ va $g(x)=\frac{1}{x}$ desak, Dirixle alomatiga ko‘ra yaqinlashuvchi ekanligini hosil qilamiz.

Endi

$$\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx = \int_1^{+\infty} \frac{|\sin x|}{x} dx ;$$

xosmas integralning uzoqlashuvchi ekanligini ko‘rsatamiz.

$$|\sin x| \geq \sin^2 x = \frac{1-\cos 2x}{2} .$$

Unda $\forall A>1$ uchun

$$\int_1^{A} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{2}\int_1^{A} \frac{dx}{x} - \frac{1}{2}\int_1^{A} \frac{\cos 2x}{x} dx ;$$

bo‘ladi. Ma’lumki,

$$\lim_{A\to+\infty} \int_1^{A} \frac{dx}{x} = \int_1^{+\infty} \frac{dx}{x} - \text{uzoqlashuvchi va } \lim_{A\to+\infty} \int_1^{A} \frac{\cos 2x}{x} = \int_1^{+\infty} \frac{\cos 2x}{x} dx -$$

Dirixle alomatiga ko'ra yaqinlashuvchi. Shularga asosan oxirga tengsizlikda $A \to +\infty$ da limitga o'tib, $\int_1^{+\infty}\left|\frac{\sin x}{x}\right|dx$ xosmas integralning uzoqlashuvchiligini topamiz. $\Rightarrow$ $\int_1^{+\infty}\frac{\sin x}{x}dx$ integral shartli yaqinlashuvchi. ▷

Eslatma: Birinchi tur xosmas integrallarda ham ma'lum shartlar bajarilganda aniq integrallarni hisoblashda qo'llaniladigan o'zgaruvchilarni almashtirish, Nyuton-Leybnis, bo'laklab integrallash va shu kabi boshqa formulalar o'rinli bo'ladi. Ularning shartlarida va ifodalanishida printsipial farq bo'lmaganligi sababli biz ularga to'xtalmaymiz.

2⁰. Chegaralanmagan funksiyaning xosmas integrali (II-tur xosmas integral)

Faraz qilaylik, $f(x)$ funksiya $[a,b)$ yarim segmentda berilgan bo'lsin. Agar $\forall\alpha>0$ soni uchun $f(x)$ funksiya $[a,b-\alpha)\subset[a,b)$ da chegaralangan bo'lib, $[a,b)$ da chegaralanmagan bo'lsa, u holda b nuqta $f(x)$ funksiya uchun ***maxsus nuqta deyiladi.***

Aytaylik b nuqta $[a,b)$ oraliqda berilgan $f(x)$ funksiya uchun maxsus nuqta bo'lib, $f(x)$ funksiya $[a,\ b-\alpha]$ kesmada integrallanuvchi bo'lsin.

$$F(\alpha)=\int_a^{b-\alpha}f(x)dx;$$

deb belgilaymiz. Bu funksiya $(0,b-a]$ yarim segmentda aniqlangan.

Ta'rif. *Agar ushbu*

$$\lim_{\alpha\to+0}F(\alpha)=\lim_{\alpha\to+0}\int_a^{b-\alpha}f(x)dx;$$

limit mavjud va chekli bo'lsa, uning qiymatiga $f(x)$ funksiyaning $[a,b)$ dagi ***II tur xosmas integrali*** *deyiladi va*

$$\int_a^b f(x)dx; \qquad (3)$$

kabi belgilanadi hamda (3)-xosmas integral ***yaqinlashuvchi*** *, aks holda esa* ***uzoqlashuvchi*** *deb ataladi.*

Shunday qilib,

$$\int_a^b f(x)dx := \lim_{\alpha\to+0}\int_a^{b-\alpha} f(x)dx;$$

Xuddi yuqoridagidek, a nuqta $f(x)$ funksiyaning maxsus nuqtasi bo'lganda $(a,b]$ oraliq bo'yicha xosmas integral, a va b nuqtalar funksiyaning maxsus nuqtalari bo'landa (a,b) oraliq bo'yicha xosmas integrallar quyidagi tengliklar yordamida aniqlanadi:

$$\int_a^b f(x)dx := \lim_{\alpha\to+0}\int_{a+\alpha}^{b} f(x)dx;$$

$$\int_a^b f(x)dx := \lim_{\substack{\alpha\to+0\\ \beta\to+0}}\int_{a+\alpha}^{b-\beta} f(x)dx.$$

Misol. $\int_a^b \frac{1}{(b-x)^\lambda}dx$ $(\lambda>0)$ **xosmas integral** $\lambda<1$ **bo'lganda yaqinlashadi va** $\lambda\geq 1$ **bo'lganda uzoqlashadi.**

Ikkinchi tur xosmas integrallar uchun ham birinchi tur xosmas integrallarda o'rinli bo'lgan ularni hisoblash usullari va yaqinlashish alomatlari o'rinli. Ularning hammasiga to'xtalmay, asosiylarini keltiramiz.

1-teorema. (Koshi kriteriyasi). *(3)-xosmas integralning yaqinlashuvchi bo'lishi uchun quyidagi shartning bajarilishi zarur va yetarlidir:* $\forall\varepsilon>0$ *uchun* $\exists\delta>0$: $0<\alpha''<\alpha'<\delta$ *tengsizlikni qanoatlantiruvchi* $\forall$ α' *va* α'' *lar uchun*

$$\left|\int_{u-\alpha'}^{u-\alpha''} f(x)dx\right|<\varepsilon;$$

tengsizlik bajariladi.

2-teorema. $f(x)$ *va* $g(x)$ *funksiyalar* $[a,b)$ *da berilgan bo'lib, b shu funksiyalarning maxsus nuqtasi bo'lsin. Agar* $\forall x\in[a,b)$ *da*

$$0\leq f(x)\leq g(x);$$

bo'lsa, u holda $\int_a^b g(x)dx$ *integralning yaqinlashuvchiligidan* $\int_a^b f(x)dx$ *ning yaqinlashuvchiligi;* $\int_a^b f(x)dx$ *integralning uzoqlashuvchiligidan* $\int_a^b g(x)dx$ *ning uzoqlashuvchiligi kelib chiqadi.*

Natija. Agar $|f(x)| \leq c \cdot (b-x)^{-\lambda}$ bo'lib, $\lambda < 1$ bo'lsa (3)-xosmas integral yaqinlashadi. Agar $f(x) \geq \frac{c}{(b-x)^{\lambda}}$, $c > 0$, bo'lib, $\lambda \geq 1$ bo'lsa, u holda (3)-xosmas integral uzoqlashadi .

3-teorema. *Agar* $x \to b-0$ *da* $f(x) = O^{*}(g(x))$ *bo'lsa, unda* $\int_a^b f(x)dx$ *va* $\int_a^b g(x)dx$ *integrallar bir vaqtda yaqinlashadi yoki uzoqlashadi.*

4-teorema. $f(x)$ *va* $g(x)$ *funksiyalar* $[a,b)$ *da berilgan bo'lib, ular quyidagi shartlarni bajarsin:*

(Abel alomati) *a)* $\int_a^b f(x)dx$ *integral yaqinlashuvchi,*

b) $g(x)$ *funksiya* $[a,b)$ *da monoton va chegaralangan;*

(Dirixle alomati) *a)* $\forall K \quad \forall \delta > 0 \quad \left| \int_a^{b-\delta} f(x)dx \right| \leq K,$

b) $g(x)$ *funksiya* $[a, b)$ *da monoton va* $\lim_{x \to b-0} g(x) = 0$.

U holda $\int_a^b f(x)dx$ *xosmas integral yaqinlashuvchi bo'ladi.*

3⁰. Xosmas integralning bosh qiymati

1-ta'rif. *Aytaylik,* $f(x)$ *funksiya* $-\infty < x < +\infty$ *to'g'ri chiziqda aniqlangan bo'lib, undagi* $\forall$ *kesmada integrallanuvchi bo'lsin. Agar ushbu*

$$\lim_{A \to +\infty} \int_{-A}^{A} f(x)dx;$$

limit mavjud va chekli bo'lsa, $f(x)$ *funksiya* $(-\infty, +\infty)$ *oraliqda* ***Koshi ma'nosida integrallanuvchi*** *deyiladi. Bu limitning qiymatiga esa* $f(x)$ *funksiya xosmas integralining* ***Koshi ma'nosidagi bosh qiymati*** *deb ataladi va*

$$V.p \int_{-\infty}^{+\infty} f(x)dx;$$

kabi belgilanadi.

Demak,

$$V.p\int_{-\infty}^{+\infty} f(x)dx := \lim_{A\to+\infty}\int_{-A}^{A} f(x)dx;$$

Teorema. *Agar $f(x)$ funksiya toq bo'lsa, u holda u Koshi ma'nosida integrallanuvchi va uning bosh qiymati 0 ga teng bo'ladi. Agar $f(x)$ funksiya juft bo'lsa, u Koshi ma'nosida integrallanuvchi bo'lishi uchun*

$$\int_{0}^{+\infty} f(x)dx;$$

xosmas integralning yaqinlashuvchi bo'lishi zarur va yetarli.

2-ta'rif. *Faraz qilaylik, $f(x)$ funksiya $[a,b]$ kesmaning s $(a<c<b)$ nuqtasidan tashqari hamma nuqtalarida aniqlangan bo'lib, (a,c) va (c,b) ga qism bo'lgan $\forall$ kesmada integralanuvchi bo'lsin. U holda, agar*

$$\lim_{\alpha\to+0}\left[\int_{a}^{c-\alpha} f(x)dx + \int_{c+\alpha}^{b} f(x)dx\right];$$

limit mavjud va chekli bo'lsa, $f(x)$ funksiya $[a,b]$ kesmada ***Koshi ma'nosida integrallanuvchi*** *deyiladi va bu limitning qiymatiga integralning* ***Koshi ma'nosidagi bosh qiymati*** *deb ataladi hamda u*

$$V.p\int_{a}^{b} f(x)dx;$$

kabi belgilanadi.

Misol. $f(x)=\dfrac{1}{x-2}$ *funksiya $[1;\ 5]$ kesmada xosmas ma'noda integrallanuvchi emas, lekin Koshi ma'nosida integrallanuvchi ekanligi ko'rsatilsin.*

◁ Xosmas ma'noda integrallanuvchi emasligi ravshan. Koshi ma'nosida integrallanuvchi bo'lishini ko'rsatamiz.

$$V.p\int_{1}^{5}\frac{dx}{x-2} = \lim_{\alpha\to+0}\left[\int_{1}^{2-\alpha}\frac{dx}{x-2} + \int_{2+\alpha}^{5}\frac{dx}{x-2}\right] = \lim_{\alpha\to+0}\left[\ln|x-2|\Big|_{1}^{2-\alpha} + \ln|x-2|\Big|_{2+\alpha}^{5}\right] =$$

$$= \lim_{\alpha\to+0}(\ln\alpha + \ln 3 - \ln\alpha) = \ln 3. \triangleright$$

4^0. Parametrga bog'liq xos integrallar va ularning funksional xossalari

$f(x,y)$ funksiya R^2 fazodagi biror $D=\{(x,y)\in R^2 : a\le x\le b, y\in E\subset R\}$ aniqlangan va $\forall$ fiksirlangan $y\in E$ uchun $f(x,y)$ funktsiya x o'zgaruvchining funksiyasi sifatida $[a,b]$ oraliqda integrallanuvchi bo'lsin.

Quyidagi

$$\Phi(y)=\int_a^b f(x,y)dx; \qquad (4)$$

integralga **parametrga bog'liq integral,** u o'zgaruvchi esa **parametr** deyiladi.

Parametrga bog'liq integrallarda $\Phi(y)$ funksiyaning bir qator xossalari (limiti, uzluksizligi, differensiallanuvchiligi, integrallanuvchiligi va hokazo) o'rganiladi. Bu xossalarni o'rganishda $f(x,y)$ funksiyaning u bo'yicha limiti va unga intilish xarakteri muhim rol o'naydi.

$f(x,y)$ funksiya D to'plamda berilgan, y_0 esa **E** to'plamning limit nuqtasi bo'lsin.

1-ta'rif. *Agar $\forall\varepsilon>0$ olinganda ham ($\forall x\in[a,b]$ uchun) shunday $\delta=\delta(\varepsilon,x)>0$ topilsaki, $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun*

$$|f(x,y)-\varphi(x)|<\varepsilon, \quad x\in[a,b];$$

bo'lsa, u holda $\varphi(x)$ funksiya $f(x,y)$ funksiyaning $y\to y_0$ dagi ***limit funksiyasi*** *deyiladi.*

$f(x,y)$ funksiya D to'plamda berilgan bo'lib, ∞ nuqta Ye *to'plamning limit nuqtasi bo'lsin.*

2-ta'rif. *Agar $\forall\varepsilon>0$ olinganda ham ($\forall x\in[a,b]$ uchun) $\exists\Delta=\Delta(\varepsilon,x)>0$ topilsaki, $|y|>\Delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun*

$$|f(x,y)-\varphi(x)|<\varepsilon, \ x\in[a,b];$$

bo'lsa, u holda $\varphi(x)$ funksiya $f(x,y)$ funksiyaning $y\to\infty$ dagi ***limit funksiyasi*** *deyiladi.*

Limit funksiya ta'rifidagi $\delta=\delta(\varepsilon,x)>0$ ning faqat $\varepsilon>0$ gagina bog'liq qilib tanlanishi mumkin bo'lgan hol muhimdir.

3-ta'rif. *D to'plamda berilgan $f(x,y)$ funksiyaning $y\to y_0$ dagi limit funksiyasi $\varphi(x)$ bo'lsin. Agar $\forall\varepsilon>0$ uchun $\exists\delta=\delta(\varepsilon)>0$ topil-*

saki, $|y-y_0|<\delta$ *tengsizlikni qanoatlantiruvchi* $\forall y\in E$ ва $\forall x\in[a,b]$ *lar uchun*

$$|f(x,y)-\varphi(x)|<\varepsilon,$$

bo'lsa, $f(x,y)$ *funksiya o'z limit funksiyasi* $\varphi(x)$ *ga* $[a,b]$ *da **tekis yaqinlashadi** deyiladi.*

4-ta'rif. *D to'plamda berilgan* $f(x,y)$ *funksiyaning* $y\to y_0$ *dagi limit funksiyasi* $\varphi(x)$ *bo'lsin. Agar* $\exists\varepsilon_0>0$, $\forall\delta>0$ *olinganda ham* $\exists\ x_0\in[a,b]$ *va* $|y-y_0|<\delta$ *tengsizlikni qanoatlantiruvchi* $y_1\in E$ *topilsaki, ushbu*

$$|f(x_0,y_1)-\varphi(x_0)|\ge\varepsilon_0,$$

tengsizlik o'rinli bo'lsa, u holda $f(x,y)$ *funksiya* $\varphi(x)$ *ga **notekis yaqinlashadi** deyiladi.*

1-teorema. (Koshi kriteriyasi) $f(x,y)$ *funksiya* $y\to y_0$ *da limit funksiya* $\varphi(x)$ *ga ega bo'lib, unga tekis yaqinlashishi uchun quyidagi shartning bajarilishi zarur va yetarlidir:* $\forall\varepsilon>0$ *uchun* $\delta=\delta(\varepsilon)>0$ *topiladiki,* $|y''-y_0|<\delta$, $|y'-y_0|<\delta$ *tengsizliklarni qanoatlantiruvchi* $\forall y',y''\in E$ *hamda* $\forall x\in[a,b]$ *uchun*

$$|f(x,y'')-f(x,y')|<\varepsilon,$$

tengsizlik bajariladi.

Endi parametrga bog'liq integrallarning funksional xossalarini keltiramiz.

2-teorema. *Agar*

1) $\forall$ *fiksirlangan* $y\in E$ *uchun* $f(x,y)\in C[a,b]$,

2) $y\to y_0$ *da* $f(x,y)$ *funksiya* $\varphi(x)$ *ga tekis yaqinlashsa, u holda*

$$\lim_{y\to y_0}\int_a^b f(x,y)dx=\int_a^b\varphi(x)dx \qquad (5)$$

bo'ladi.

3-teorema. *Agar* $f(x,y)$ *funksiya*

$$D=\{(x,y)\in R^2: x\in[a,b],\quad y\in[c,d]\}$$

to'plamda uzluksiz bo'lsa, u holda

$$\Phi(y)=\int_a^b f(x,y)dx$$

funksiya $[c,d]$ *kesmada uzluksiz bo'ladi.*

4-teorema. *Aytaylik* $f(x,y)$ *funksiya*

$$D=\{(x,y)\in R^2 : x\in[a,b],\ y\in[c,d]\}$$

to'plamda aniqlangan va

1) $\forall$ *fiksirlangan* $y\in E$ *uchun* $f(x,y)\in C[a,b]$

2) $f'_y(x,y)-\exists$ *va* $\in C(D)$

bo'lsin. U holda $[c,d]$ *kesmada* $\Phi'(y)$ *mavjud va ushbu*

$$\Phi'(y)=\int_a^b f'_y(x,y)dx \tag{6}$$

tenglik o'rinli bo'ladi.

5-teorema. *Agar* $f(x,y)$ *funksiya 3-teorema shartlarini qanoatlantirsa, unda* $\int_c^d \Phi(y)dy$ *integral mavjud va*

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_a^b\left[\int_c^d f(x,y)dy\right]dx \tag{7}$$

munosabat o'rinlidir.

Endi umumiy ko'rinishda berilgan parametrga bog'liq integrallarni keltiramiz.

Faraz qilaylik, $x=\varphi(y), x=\psi(y)$ funksiyalar $[c,d]$ da aniqlangan bo'lib, $\forall y\in[c,d]$ uchun

$$a\le\varphi(y)\le\psi(y)\le b; \tag{8}$$

munosabat bajarilsin.

6-teorema. $f(x,y)$ *funksiya ushbu*

$$D=\{(x,y)\in R^2 : x\in[a,b],\ y\in[c,d]\}$$

to'plamda aniqlangan bo'lib,

1) $f(x,y)\in C(D)$;

2) $\varphi(y),\ \psi(y)\in C[c,d]$ *bo'lsin. U holda*

$$\tilde{\Phi}(y)=\int_{\varphi(y)}^{\psi(y)} f(x,y)dx \tag{9}$$

funksiya ham $[c,d]$ *oraliqda uzluksiz bo'ladi.*

7-teorema. (Leybnis formulasi). *Agar*

1) $f(x,y)\in C(D)$,

2) $f'_y(x,y)\in C(D)$,

3) $\varphi'(y)$ *va* $\psi'(y)\in C[c,d]$

bo'lsa, u holda $\bar{\Phi}(y)$ *funksiya ham* $[c,d]$ *oraliqda hosilaga ega va*

$$\bar{\Phi}'(y)=\int\limits_{\varphi(y)}^{\psi(y)} f'_y(x,y)dx+\psi'(y)\cdot f[\psi(y),y]-\varphi'(y)\cdot f[\varphi(y),y] \quad (10)$$

munosabat o'rinlidir.

6-teorema shartlari bajarilgan holda $\bar{\Phi}(y)$ funksiyaning $[c,d]$ oraliqda integrallanuvchi ekanligi kelib chiqadi va (9)-funksiya uchun ham (7)-tenglik kabi tenglik o'rinli bo'ladi.

5°. Parametrga bog'liq xosmas integrallar va ularning tekis yaqinlashishi

$f(x,y)$ funksiya

$$D=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to'plamda berilgan bo'lib, $\forall$ fiksirlangan $y\in E$ uchun

$$\int\limits_a^{+\infty} f(x,y)dx \quad (y\in E)$$

mavjud va chekli bo'lsin. Bu integral u *ning qiymatiga bog'liqdir.*

$$I(y)=\int\limits_a^{+\infty} f(x,y)dx \quad (11)$$

(11)-integralga **parametrga bog'liq I-tur xosmas integral** deyiladi.

Xuddi shu kabi

$$\int\limits_{-\infty}^{a} f(x,y)dx \text{ va } \int\limits_{-\infty}^{+\infty} f(x,y)dx$$

parametrga bog'liq bo'lgan I-tur xosmas integrallarning ta'rifini berish mumkin.

Endi $f(x,y)$ funksiya

$$D_1=\{(x,y)\in R^2 : x\in[a,b),\ y\in E\subset R\}$$

to'plamda berilgan bo'lib, $\forall$ fiksirlangan $y\in E$ da $x=b$ nuqta

$f(x,y)$ funksiyaning maxsus nuqtasi bo'lsin va bu funksiya $[a,b]$ oraliqda integrallanuvchi, ya'ni

$$\int_a^b f(x,y)dx \quad (y\in E)$$

xosmas integral mavjud bo'lsin. Unda

$$I_1(y)=\int_a^b f(x,y)dx \tag{12}$$

integralga **parametrga bog'liq bo'lgan II-tur xosmas integral** deyiladi.

Xuddi shunga o'xshash $x=a$ nuqta maxsus nuqta bo'lgan parametrga bog'liq bo'lgan II-tur xosmas integralga ta'rif berish mumkin.

Umumiy holda, parametrga bog'liq chegaralanmagan funksiyaning chegarasi cheksiz xosmas integrali tushunchasi ham yuqoridagidek kiritiladi.

Biz asosan, (11)-xosmas integralning xossalarini o'rganish bilan shug'ullanamiz.

Aytaylik, $f(x,y)$ funksiya D to'plamda aniqlangan bo'lib, $\forall$ fiksirlangan $y\in E$ uchun

$$\int_a^{+\infty} f(x,y)dx-\exists$$

bo'lsin. $\Rightarrow \forall[a,t]\subset[a,+\infty)$ da

$$F(t,y)=\int_a^t f(x,y)dx \tag{13}$$

integral mavjud va

$$I(y)=\int_a^{+\infty} f(x,y)dx=\lim_{t\to+\infty} F(t,y). \tag{14}$$

(14)-tenglikdan ko'rindiki $I(y)$ funksiya $F(t,y)$ funksiyaning $t\to+\infty$ dagi limit funksiyasi bo'ladi.

1-ta'rif. *Agar $t\to+\infty$ da $F(t,y)$ funksiya E to'plamda o'z limit funksiyasi $I(y)$ ga tekis yaqinlashsa u holda (11)-integral E to'plamda **tekis yaqinlashuvchi,** notekis yaqinlashganda esa **notekis yaqinlashuvchi** deyiladi.*

Shunday qilib, $\int_a^{+\infty} f(x,y)dx$ *integralning* Ye *to'plamda tekis yaqinlashuvchi bo'lishi quyidagini anglatadi:*

1) $\forall y \in E$ uchun $\int_a^{+\infty} f(x,y)dx$ xosmas integral yaqinlashuvchi;

2) $\forall \varepsilon > 0$ uchun $\exists \delta = \delta(\varepsilon) > 0 : \forall t > \delta$ va $\forall y \in E$ uchun

$$\left| \int_t^{+\infty} f(x,y)dx \right| < \varepsilon ;$$

tengsizlik bajariladi.

$\int_a^{+\infty} f(x,y)dx$ *integralning* E *to'plamda notekis yaqinlashuvchi ekanligi esa quyidagini anglatadi:*

1) $\forall y \in E$ uchun $\int_a^{+\infty} f(x,y)dx$ xosmas integral yaqinlashuvchi;

2) $\exists \varepsilon_0 > 0,\ \forall \delta > 0$ olinganda ham $\exists y_0 \in E$ va $\exists t_0 > \delta$, $t_0 \in [a, +\infty)$ topiladiki,

$$\left| \int_{t_0}^{+\infty} f(x,y_0)dx \right| \geq \varepsilon_0 ;$$

bo'ladi.

Misol. $I(y) = \int_0^{+\infty} ye^{-xy}dx$ **parametrga bog'liq integral a)** $E = (0, +\infty)$ **va b)** $E_1 = [2, +\infty) \subset E$ **oraliqlarda tekis yaqinlashishga tekshirilsin.**

◁a) $F(t,y) = \int_0^t ye^{-xy}dx = -\int_0^t e^{-xy}d(-xy) = 1 - e^{-ty}$

$(0 < t < +\infty) \Rightarrow \forall y \in (0, +\infty)$ uchun $I(y) = \lim\limits_{t \to +\infty} F(t,y) = \lim\limits_{t \to +\infty} (1 - e^{-ty}) =$
$1 = 1 \Rightarrow I(y) = \int_0^{+\infty} ye^{-xy}dx -$ yaqinlashuvchi.

Endi berilgan integralni tekis yaqinlashuvchanlikka tekshiramiz. $y \in E = (0, +\infty)$ bo'lsin. Agar $\forall \delta > 0$ uchun $\varepsilon_0 = \frac{1}{3}, t_0 > \delta$ va $y_0 = \frac{1}{t_0}$ deb olsak, u holda

$$\left|\int_{t_0}^{+\infty} y_0 e^{-xy_0}dx\right| = e^{-t_0 y_0} = e^{-1} = \frac{1}{e} > \frac{1}{3} = \varepsilon_0$$

bo'ladi. $\Rightarrow$ integral $E=(0,+\infty)$ da notekis yaqinlashadi.

b) Endi integralni $E_1=[2,+\infty)\subset E$ to'plamda tekis yaqinlashuvchanlikka tekshiramiz. $\forall\varepsilon>0$ olamiz.

$$\left|\int_t^{+\infty} ye^{-xy}dx\right| = \left|-e^{-xy}\right|_t^{+\infty} = e^{-ty} = \frac{1}{e^{ty}} = \left((y>2, t>\delta)\right) < \frac{1}{e^{2\delta}} = \varepsilon \Rightarrow \delta = \frac{1}{2}\ln\frac{1}{\varepsilon}$$

deb olsak, tekis yaqinlashish ta'rifidagi shartlar bajarilar ekan. $\Rightarrow I(y)=\int_0^{+\infty} ye^{-xy}dx$ integral $E_1=[2,+\infty)$ oraliqda tekis yaqinlashadi. ▷

2-ta'rif. *Agar* $\forall\varepsilon>0$ *uchun* $\exists\delta=\delta(\varepsilon)>0:\ t'>\delta,\ t''>\delta$ *ni qanoatlantiruvchi* $\forall t',t''$ *va* $\forall y\in E$ *uchun*

$$\left|\int_{t'}^{t''} f(x,y)dx\right| < \varepsilon;$$

tengsizlik bajarilsa, unda (11)-xosmas integral Ye to'plamda ***fundamental integral*** *deyiladi.*

1-teorema (Koshi). $I(y)=\int_a^{+\infty} f(x,y)dx$ *integralning Ye to'plamda tekis yaqinlashuvchi bo'lishi uchun uning Ye to'plamda fundamental bo'lishi zarur va yetarlidir.*

Bu teorema nazariy ahamiyatga ega bo'lib, undan amaliyotda foydalanish ancha qiyin.

2-teorema (Veyershtrass). *Agar* $\exists\varphi(x)\ge 0$ $(x\in[a,+\infty))$ *funksiya topilsaki*

1) $\forall x\in[a,+\infty)$ *va* $\forall y\in E$ *uchun* $|f(x,y)|\le\varphi(x)$,

2) $\int_a^{+\infty}\varphi(x)dx$ *yaqinlashuvchi*

bo'lsa, unda $I(y)=\int_a^{+\infty} f(x,y)dx$ *integral Ye to'plamda tekis yaqinlashuvchi bo'ladi.*

3-teorema (Abel alomati). $f(x,y)$ *va* $g(x,y)$ *funksiyalar*

$$D=\{(x,y)\in R^2 : x\in[a, +\infty),\ y\in E\};$$

to'plamda berilgan bo'lib,

1) $\forall$ *fiksirlangan* $y\in E$ *uchun* $g(x,y)$ *funksiya* $[a,+\infty)$ *da* x *o'zgaruvchi bo'yicha monoton va u* D *toplamda chegaralangan,*

2) $\int_a^{+\infty} f(x,y)dx$ *integral Ye da tekis yaqinlashuvchi bo'lsa, u holda*

$$\int_a^{+\infty} f(x,y)\cdot g(x,y)dx;$$

integral Ye to'plamda tekis yaqinlashuvchi bo'ladi.

4-teorema (Dirixle alomati). $f(x,y)$ *va* $g(x,y)$ *funksiyalar* D *to'plamda berilgan bo'lib,*

1) $\forall t\geq a$ *va* $\forall y\in E$ *uchun*

$$\left|\int_a^t f(x,y)dx\right|\leq c \quad (c=const),$$

2) $\forall$ *fiksirlangan* $y\in E$ *uchun* $g(x,y)$ *funksiya* $[a,+\infty)$ *da* x *o'zgaruvchi bo'yicha monoton va* $x\to+\infty$ *da* $g(x,y)$ *funksiya 0 ga tekis yaqinlashsa, u holda*

$$\int_a^{+\infty} f(x,y)\cdot g(x,y)dx;$$

integral E *to'plamda tekis yaqinlashuvchi bo'ladi.*

6^0. Parametrga bog'liq xosmas integrallarning funksional xossalari

$f(x,y)$ funksiya $D=\{(x,y)\in R^2 : x\in[a, +\infty),\ y\in E\}$ to'plamda berilgan bo'lib, y_0 nuqta Ye to'plamning limit nuqtasi bo'lsin.

1-teorema. *Agar*

1) $\forall$ *fiksirlangan* $y\in E$ *uchun* $f(x,y)\in C[a,+\infty)$,

2) $y\to y_0$ *da* $\forall\ [a,t]$ $(a<t<+\infty)$ *kesmada* $f(x,y)$ *funksiya* $\varphi(x)$ *ga tekis yaqinlashsa,*

3) $I(y)=\int_a^{+\infty} f(x,y)dx$ *integral Ye to'plamda tekis yaqinlashuvchi bo'lsa, u holda* $y\to y_0$ *da* $I(y)$ *funksiya limitga ega va*

$$\lim_{y\to y_0} I(y)=\lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\left[\lim_{y\to y_0} f(x,y)\right]dx=\int_a^{+\infty}\varphi(x)dx$$

bo'ladi.

2-teorema. *Agar* $f(x,y)$ *funksiya*

$$D=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in[c,d]\}$$

to'plamda berilgan bo'lib,

1) $f(x,y)\in C(D)$,

2) $I(y)=\int_a^{+\infty} f(x,y)dx$ *integral* $[c,d]$ *da tekis yaqinlashuvchi bo'lsa, u holda* $I(y)\in C[c,d]$ *bo'ladi.*

3-teorema. *Agar* $f(x,y)$ *funksiya*

$$D=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in[c,d]\}$$

to'plamda berilgan bo'lib,

1) $f(x,y)\in C(D)$, $f'_y(x,y)\in C(D)$,

2) $\forall$ *fiksirlangan* $y\in[c,d]$ *uchun* $I(y)=\int_a^{+\infty} f(x,y)dx$ *yaqinlashuvchi,*

3) $\int_a^{+\infty} f'_y(x,y)dx$ *integral* $[c,d]$ *da tekis yaqinlashuvchi bo'lsa, u holda* $I(y)$ *funksiya* $[c,d]$ *oraliqda* $I'(y)$ *hosilaga ega bo'ladi va*

$$I'(y)=\int_a^{+\infty} f'_y(x,y)dx;$$

tenglik bajariladi.

4-teorema. *Agar* $f(x,y)$ *funksiya*

$$D=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in[c,d]\}$$

to'plamda berilgan bo'lib,

1) $f(x,y)\in C(D)$,

2) $I(y)=\int_a^{+\infty} f(x,y)dx$ integral $[c,d]$ da tekis yaqinlashuvchi bo'lsa, u holda $I(y)$ funksiya $[c,d]$ da integrallanuvchi va

$$\int_c^d I(y)dy=\int_c^d\left[\int_a^{+\infty} f(x,y)dx\right]dy=\int_a^{+\infty}\left[\int_c^d f(x,y)dy\right]dx;$$

bo'ladi.

7^0. Eyler integrallari
(Beta va Gamma funksiyalar)

a) Beta funksiya (1-tur Eyler integrali) va uning xossalari

1-ta'rif. Quyidagi

$$B(p,q)=\int_0^1 x^{p-1}\cdot(1-x)^{q-1}\,dx \tag{15}$$

integralga ***Beta funksiya*** yoki ***1-tur Eyler integrali*** deyiladi.

Beta funksiya quyidagi xossalarga ega.

1) (15)-integral $M=\{(p,q)\in R^2:\ p\in(0,\ +\infty),\ \ q\in(0,\ +\infty)\}$ to'plamda yaqinlashuvchi, $(p_0>0,q_0>0)$ to'plamda esa tekis yaqinlashuvchi bo'ladi.

2) $B(p,q)\in C(M)$.

3) $B(p,q)=B(q,p)$.

4) $B(p,q)=\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^{p+q}}dt.$

Natija. Agar $q=1-p\ \ (0<p<1)$ bo'lsa,

$$B(p,1-p)=\int_0^{+\infty}\frac{t^{p-1}}{1+t}=\frac{\pi}{\sin p\pi}. \tag{16}$$

tenglik o'rinli bo'ladi.

(16) dan $\Rightarrow B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\pi}{\sin\frac{\pi}{2}}=\pi$;

5) $\forall p>0$ va $q>1$ uchun

$$B(p,q)=\frac{q-1}{p+q-1}B(p,q-1) \tag{17}$$

tenglik o'rinli.

Natija.

$$B(m\ n)=\frac{(n-1)!\cdot(m-1)!}{(m+n-1)!}$$

b) Gamma funksiya (2-tur Eyler integrali) va uning xossalari.

2-ta'rif. *Quyidagi*

$$\Gamma(p)=\int_0^{+\infty}x^{p-1}e^{-x}dx \tag{18}$$

*integralga **Gamma funksiya** yoki **2-tur Eyler integrali** deyiladi.*

Gamma funksiya quyidagi xossalarga ega.

1) $\Gamma(1)=\Gamma(2)=1$

2) (18)-integral $(0,+\infty)$ oraliqda yaqinlashuvchi, $\forall[a,b]\subset(0,+\infty)$ $(0<a<b<+\infty)$ kesmada esa tekis yaqinlashuvchi bo'ladi.

3) $\Gamma(p)\in C(0,+\infty)$ va $\forall n=1,2,...$ uchun

$$\Gamma^{(n)}(p)=\int_0^{+\infty} x^{p-1}e^{-x}(\ln x)^n\,dx\in C(0,+\infty)$$

4) $$\Gamma(p+1)=p\Gamma(p) \tag{19}$$

Natija. $\Gamma(n+1)=n!$

Beta va Gamma funksiyalar orasidagi bog'lanishni quyidagi teorema ifodalaydi.

Teorema. $\forall p>0,\ q>0$ *uchun*

$$B(p,q)=\frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} \tag{20}$$

tenglik o'rinli.

Natija. $\forall p\in(0,\ 1)$ uchun

$$\Gamma(p)\cdot\Gamma(1-p)=\frac{\pi}{\sin p\pi} \tag{21}$$

tenglik o'rinli bo'ladi.

Agar (21)-tenglikda $p=\frac{1}{2}$ desak

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \tag{22}$$

bo'ladi.

Eyler integrallari yordamida ko'pgina xosmas integrallarni hisoblash ancha osonlashadi.

Misollar.

1) $I=\int_0^{+\infty} e^{-x^2}dx$ – Eyler-Puasson integrali hisoblansin.

$$\triangleleft\ I=\int_0^{+\infty} e^{-x^2}dx=\left(\left(\begin{matrix} x^2=t\Rightarrow x=\sqrt{t} \\ dx=\frac{1}{2}t^{-\frac{1}{2}}dt\end{matrix}\right)\right)=\frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt=\frac{1}{2}\int_0^{+\infty} t^{\frac{1}{2}-1}\cdot e^{-t}dt=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}.\ \triangleright$$

2) $I=\int_0^{+\infty}\frac{x^2dx}{1+x^4}$ xosmas integral hisoblansin.

$$◁\ I=\int_0^{+\infty}\frac{x^2dx}{1+x^4}=\left(\left(\begin{array}{l}\frac{1}{1+x^4}=t;x=\left(\frac{1}{t}-1\right)^{\frac{1}{4}}\\ x=0\Rightarrow t=1\\ x=+\infty\Rightarrow t=0;dx=-\frac{1}{4t^2}\left(\frac{1}{t}-1\right)^{-\frac{3}{4}}dt\end{array}\right)\right)=\frac{1}{4}\int_0^1 t^{-\frac{3}{4}}\cdot(1-t)^{-\frac{1}{4}}dt=$$

$$=\frac{1}{4}\int_0^1 t^{\frac{1}{4}-1}\cdot(1-t)^{3/4-1}dt=\frac{1}{4}B\left(\frac{1}{4};\frac{3}{4}\right)=\frac{1}{2}B\left(\frac{1}{4};1-\frac{1}{4}\right)=\frac{1}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{2\sqrt{2}}.\ ▷$$

Nazorat savollari

1. 1-tur xosmas integral tushunchasi.
2. 1-tur xosmas integralning yaqinlashishi.
3. Koshi kriteriyasi.
4. Umumiy taqqoslash alomati.
5. Xususiy taqqoslash alomati.
6. Absolut va shartli yaqinlashuvchi 1-tur xosmas integrallar.
7. 1-tur xosmas integrallar uchun Abel alomati.
8. 1-tur xosmas integrallar uchun Dirixle alomati.
9. 2-tur xosmas integral tushunchasi.
10. 2-tur xosmas integralning yaqinlashishi.
11. 2-tur xosmas integral uchun Koshi kriteriyasi.
12. 2-tur xosmas integral taqqoslash alomatlari.
13. 2-tur xosmas integral Abel alomati.
14. 2-tur xosmas integral Dirixle alomati.
15. Xosmas integralning Koshi ma'nosidagi bosh qiymati.
16. Parametrga bog'liq xos integrallar.
17. Parametrga bog'liq xos integrallarning tekis yaqinlashishi.
18. Tekis yaqinlashishning inkori.
19. Koshi kriteriyasi.
20. Parametrga bog'liq xos integralning uzluksizligi.
21. Parametrga bog'liq xos integrallarni differensiallash.
22. Parametrga bog'liq xos integrallarni integrallash.
23. Parametrga bog'liq xosmas integrallar.
24. Parametrga bog'liq xosmas integrallarning tekis yaqinlashishi.
25. Koshi kriteriyasi.
26. Veyershtrass alomati.
27. Abel alomati.
28. Dirixle alomati.
29. Parametrga bog'liq xosmas integrallarning uzluksizligi.
30. Parametrga bog'liq xosmas integrallarni differensiallash.
31. Parametrga bog'liq xosmas integrallarni integrallash.
32. Beta funksiya (1-tur Eyler integrali) va uning xossalari.
33. Gamma funksiya (2-tur Eyler integrali) va uning xossalari.
34. Beta va Gamma funksiyalari orasidagi bog'lanish.
35. Eyler-Puasson integrali va uni hisoblash.

-B-

Mustaqil yechish uchun misol va masalalar

1-masala. Quyidagi xosmas integrallar hisoblansin.

1.1 $\int_{-\infty}^{-2} \frac{dx}{x\sqrt{x^2-1}}$.

1.2 $\int_{0}^{+\infty} \frac{dx}{e^x+\sqrt{e^x}}$.

1.3 $\int_{2}^{+\infty} \frac{xdx}{x^3-1}$.

1.4 $\int_{1}^{+\infty} \frac{x^2+1}{x^4+1}dx$.

1.5 $\int_{0}^{+\infty} \frac{dx}{(x^2+9)\sqrt{x^2+9}}$.

1.6 $\int_{0}^{+\infty} e^{-\sqrt{x}}dx$.

1.7 $\int_{1}^{+\infty} \frac{x\cdot e^{arctgx}}{(1+x^2)\cdot\sqrt{1+x^2}}dx$.

1.8 $\int_{2}^{+\infty} \frac{dx}{x\sqrt{x^2+x-1}}$.

1.9 $\int_{0}^{+\infty} \frac{dx}{\left(\sqrt{x^2+1}+x\right)^2}$.

1.10 $\int_{0}^{+\infty} \frac{dx}{(4x^2+1)\sqrt{x^2+1}}$.

1.11 $\int_{0}^{+\infty} \frac{x^2+12}{(x^2+1)^2}dx$.

1.12 $\int_{0}^{+\infty} e^{-ax}\cdot\sin^2 bxdx$.

1.13 $\int_{0}^{+\infty} x^n e^{-x}dx, n\in N$.

1.14 $\int_{\sqrt{2}}^{+\infty} \frac{dx}{(x-1)\sqrt{x^2-2}}$.

1.15 $\int_{1}^{+\infty} \frac{dx}{(2x-1)\sqrt{x^2-1}}$.

1.16 $\int_{0}^{+\infty} \frac{\ln x}{1+x^2}dx$.

1.17 $\int_{1}^{+\infty} \frac{dx}{(4x^2-1)\sqrt{x^2-1}}$.

1.18 $\int_{0}^{+\infty} \frac{x\ln x}{(1+x^2)^2}dx$.

1.19 $\int_{0}^{+\infty} \frac{arctg(1-x)}{\sqrt[3]{(x-1)^4}}dx$.

1.20 $\int_{1}^{+\infty} \frac{2-x}{x^3\cdot\sqrt{x^2-1}}dx$.

1.21 $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+x+1)^3}$.

2-masala. Quyidagi II-tur xosmas integrallar hisoblansin.

2.1 $\int\limits_0^{\pi/2} (\ln\cos x)\cdot\cos 2nx dx, n\in N.$

2.2 $\int\limits_0^{\pi} x\cdot(\ln\sin x)dx.$

2.3 $\int\limits_0^{\pi} \ln\cos x dx.$

2.4 $\int\limits_{-1}^{1} x^3\cdot\ln\frac{1+x}{1-x}\cdot\frac{dx}{\sqrt{1-x^2}}.$

2.5 $\int\limits_0^1 \frac{x^3\arcsin x}{\sqrt{1-x^2}}dx.$

2.6 $\int\limits_0^2 \left(x\sin\frac{\pi}{x^2}-\frac{\pi}{x}\cos\frac{\pi}{x^2}\right)dx.$

2.7 $\int\limits_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)\arccos x}}.$

2.8 $\int\limits_0^{\pi/2} \sqrt{tg x}dx.$

2.9 $\int\limits_0^{\pi/4} \sqrt{ctg x}dx.$

2.10 $\int\limits_a^b x\cdot\sqrt{\frac{x-a}{b-x}}dx, b>a.$

2.11 $\int\limits_{-a}^{a} \frac{dx}{\sqrt{a^2+b^2-2bx}}, a>0, b\ge 0.$

2.12 $\int\limits_{-a}^{a} \frac{x^4 dx}{(1+x^2)\cdot\sqrt{1-x^2}}$

2.13 $\int\limits_{-1}^{1} \frac{dx}{(16-x^2)\cdot\sqrt{1-x^2}}.$

2.14 $\int\limits_{-1}^{1} \frac{dx}{(4-x)\cdot\sqrt{1-x^2}}.$

2.15 $\int\limits_1^2 \frac{dx}{x\cdot\sqrt{3x^2-2x-1}}.$

2.16 $\int\limits_0^{\pi/2} \ln\sin x dx.$

2.17 $\int\limits_{\sqrt{2}}^{2} \frac{dx}{(x-1)\cdot\sqrt{x^2-2}}.$

2.18 $\int\limits_0^1 \frac{dx}{(2-x)\cdot\sqrt{1-x}}.$

2.19 $\int\limits_{-0,5}^{-0,25} \frac{dx}{x\cdot\sqrt{2x+1}}.$

2.20 $\int\limits_0^1 x\ln^3 x dx.$

2.21 $\int\limits_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}, b>a.$

3-masala. Quyidagi II-tur xosmas integrallarni yaqinlashishga tekshiring.

3.1 $\int\limits_0^1 \frac{\ln\left(1+\sqrt[3]{x^2}\right)}{\sqrt{x}\cdot\sin\sqrt{x}}dx.$

3.2 $\int\limits_0^{\pi} \sin\left(\frac{1}{\cos x}\right)\cdot\frac{dx}{\sqrt{x}}.$

3.3 $\int_0^1 \frac{|\ln x|}{x^\alpha}dx.$

3.4 $\int_{-1}^1 \frac{dx}{\ln(1+x)}.$

3.5 $\int_0^1 \frac{dx}{\sqrt[5]{1-x^{10}}}.$

3.6 $\int_0^2 \frac{\sqrt{x}dx}{e^{\sin x}-1}.$

3.7 $\int_0^\pi \frac{\sin x}{x^2}dx.$

3.8 $\int_0^1 \frac{dx}{\sqrt{x+arctgx}}.$

3.9 $\int_1^2 \frac{(x-2)dx}{x^3-3x^2+4}.$

3.10 $\int_0^\pi \frac{\ln x}{\sqrt{\sin x}}dx.$

3.11 $\int_0^\pi \frac{\ln \sin x}{\sqrt[3]{x}}dx.$

3.12 $\int_0^1 \frac{dx}{\arccos x}.$

3.13 $\int_0^\pi \frac{1-\cos x}{x^\alpha}dx.$

3.14 $\int_0^{\pi/2} \frac{e^{\alpha\cos x}-\sqrt{1+2\cos x}}{\sqrt{\cos^5 x}}dx.$

3.15 $\int_0^1 \frac{\sqrt{e^2+x^2}-e^{\cos x}}{x^\alpha}dx$

3.16 $\int_0^1 \frac{\ln(1+2x)-xe^{-x}}{1-\cos^\alpha x}dx.$

3.17 $\int_0^{0.5} \frac{\ln^\alpha(1/x)}{tg^\beta x}dx.$

3.18 $\int_0^1 x^\alpha\cdot(1-x)^\beta\cdot\ln x dx.$

3.19 $\int_0^1 x^\alpha\cdot\ln^\beta\frac{1}{x}dx.$

3.20 $\int_0^1 x^\alpha\cdot(1-x)^\beta dx.$

3.21 $\int_0^{\pi/2} \sin^\alpha x\cdot\cos^\beta x dx.$

4-masala. Quyidagi xosmas integrallarni yaqinlashishga tekshiring.

4.1 $\int_1^{+\infty} \frac{\ln x dx}{x^\alpha}.$

4.2 $\int_e^{+\infty} \frac{dx}{x^\alpha\cdot\ln x}.$

4.3 $\int_e^{+\infty} \frac{dx}{x\cdot\ln^\alpha x}.$

4.4 $\int_2^{+\infty} \frac{e^{\alpha x}dx}{(x-1)^\alpha\cdot\ln x}.$

4.5 $\int_0^{+\infty} \frac{arctg 2x}{x^\alpha}dx.$

4.6 $\int_0^{+\infty} \frac{\ln(1+x^{-2\alpha})}{\sqrt{x^\alpha+x^{-\alpha}}}dx.$

4.7 $\int_{0}^{+\infty}\frac{dx}{1+x^{\alpha}\cdot\sin^2 x}$.

4.8 $\int_{0}^{+\infty}\frac{\ln\left(1+x^2\right)}{(x+\alpha)^2}dx$.

4.9 $\int_{0}^{+\infty} arctg\frac{x^{\alpha}}{1+x^2}\cdot\frac{dx}{x}(\alpha>0)$.

4.10 $\int_{0}^{+\infty}\frac{\ln\left(1+x+x^{\alpha}\right)}{\sqrt{x^3}}dx(\alpha>0)$.

4.11 $\int_{0}^{+\infty}\frac{\ln\left(x^{\alpha}+e^{x}\right)}{\sqrt{x^3+x^5}}dx(\alpha>0)$.

4.12 $\int_{1}^{+\infty}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}$.

4.13 $\int_{0}^{+\infty}\frac{x^{\alpha}dx}{x^{\beta}+1}, \beta\geq 0$.

4.14 $\int_{0}^{+\infty}\frac{dx}{x^{\alpha}+x^{\beta}}$.

4.15 $\int_{0}^{+\infty}\frac{\sin^2 x}{x^2}dx$.

4.16 $\int_{2}^{+\infty}\left(\cos\frac{2}{x}-1\right)dx$.

4.17 $\int_{1}^{+\infty}\frac{\ln x dx}{x\cdot\sqrt{x^2-1}}$.

4.18 $\int_{0}^{+\infty}\frac{\sin\frac{1}{x}}{\left(x-\cos\frac{\pi}{x}\right)^2}dx$.

4.19 $\int_{0}^{+\infty}\frac{1}{\sqrt{x}} arctg\frac{x}{2+\sqrt{x}}dx$.

4.20 $\int_{0}^{+\infty}\frac{xdx}{1+x^2\cdot\sin^2 x}$.

4.21 $\int_{2}^{+\infty}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}$.

5-masala. Quyidagi xosmas integrallar absolut va shartli yaqinlashishga tekshirilsin.

5.1 $\int_{0}^{1}\frac{1}{x\sqrt{x}}\cos\frac{\sqrt{x}-1}{\sqrt{x}}dx$.

5.2 $\int_{0}^{\pi/4}\sin\left(\frac{1}{\sin x}\right)\cdot\frac{dx}{\sin^{\alpha}x}$.

5.3 $\int_{0}^{0.5}\frac{\cos^3(\ln x)}{x\ln x}dx$.

5.4 $\int_{-1}^{1}\sin\frac{1+x}{1-x}\cdot\frac{dx}{\left(1-x^2\right)^{\alpha}}$.

5.5 $\int_{0}^{1}(1-x)^{\alpha}\sin\frac{\pi}{1-x}dx$.

5.6 $\int_{0}^{1}\frac{x^{\alpha}}{e^{x}-1}\sin\frac{1}{x}dx$.

5.7 $\int_0^1 \frac{x^\alpha}{x^2+1}\sin\frac{1}{x}dx.$

5.8 $\int_0^1 \frac{\sin x^\alpha}{x^2}dx.$

5.9 $\int_0^1 \cos\left(\frac{1}{\sqrt{x}}-1\right)\frac{dx}{x^\alpha}.$

5.10 $\int_0^1 \frac{\sin\frac{1}{x}}{\left(\sqrt{x}-x\right)^\alpha}dx.$

5.11 $\int_0^{0.5} \left(\frac{x}{1-x}\right)^\alpha \cdot\cos\frac{1}{x^2}dx.$

5.12 $\int_0^1 \frac{(1-x)^\alpha}{x}\sin\frac{1}{x}dx.$

5.13 $\int_1^{+\infty} \frac{\sin x}{x^\alpha}dx.$

5.14 $\int_1^{+\infty} \frac{x^\alpha\cdot\sin x}{x^3+1}dx.$

5.15 $\int_2^{+\infty} \frac{(x+1)^\alpha\cdot\sin x}{\ln x}dx.$

5.16 $\int_2^{+\infty} \frac{\cos x dx}{x^\alpha+\ln x}.$

5.17 $\int_1^{+\infty} \frac{\sin(\ln x)}{x^\alpha}\cdot\sin x dx.$

5.18 $\int_2^{+\infty} \frac{\cos\sqrt{x}}{x^\alpha\cdot\ln x}dx.$

5.19 $\int_0^{+\infty} \frac{x^\alpha\cdot\sin x}{1+x^\beta}dx, \beta\geq 0.$

5.20 $\int_0^{+\infty} x^\alpha\cdot\sin x^\beta dx.$

5.21 $\int_0^1 \sin\left(\frac{1}{1-x}\right)\cdot\frac{dx}{1-x}.$

6-masala. Quyidagi xosmas integrallarning Koshi ma'nosidagi bosh qiymati topilsin.

6.1 $V.p.\int_{-\infty}^{+\infty} \sin x dx.$

6.2 $V.p.\int_2^5 \frac{dx}{(x-3)^2}.$

6.3 $V.p.\int_{0.5}^4 \frac{dx}{x\ln x}.$

6.4 $V.p.\int_{-\infty}^{+\infty} \cos x dx.$

6.5 $V.p.\int_{-\infty}^{+\infty} arctgx dx.$

6.6 $V.p.\int_0^\pi x tgx dx.$

6.7 $V.p.\int_0^{\pi/2} \frac{dx}{3-5\sin x}.$

6.8 $V.p.\int_{-\infty}^{+\infty}\left(arctgx+\frac{1}{1+x^2}-\frac{\pi}{2}\right)dx.$

6.9 $V.p.\int\limits_0^{\pi/2}\frac{dx}{\frac{1}{2}-\sin x}.$

6.10 $V.p.\int\limits_1^7\frac{dx}{5-x}.$

6.11 $V.p.\int\limits_{-\infty}^{+\infty}\frac{13+x}{17+x^2}dx.$

6.12 $V.p.\int\limits_0^{+\infty}\frac{dx}{x^2+x-6}.$

6.13 $V.p.\int\limits_0^{\pi/2}\frac{dx}{1-2\sin x}.$

6.14 $V.p.\int\limits_0^{\pi/2}\frac{dx}{\frac{1}{2}-\cos x}.$

6.15 $V.p.\int\limits_0^{+\infty}\frac{dx}{x^2+x-2}.$

6.16 $V.p.\int\limits_{-\infty}^{+\infty}\frac{dx}{x}.$

6.17 $V.p.\int\limits_0^{+\infty}\frac{dx}{1-x^2}.$

6.18 $V.p.\int\limits_0^{10}\frac{dx}{7-x}.$

6.19 $V.p.\int\limits_{-1}^{1}\frac{dx}{x}.$

6.20 $V.p.\int\limits_{-1}^{7}\frac{dx}{(x-1)^3}.$

6.21 $V.p.\int\limits_0^{+\infty}\frac{dx}{x^2-3x+2}.$

7-masala. Quyidagi funksiyalarning berilgan to'plamda limit funksiyalarini toping va tekis yaqinlashishga tekshiring.

7.1 $f(x,y)=\sqrt{y}\cdot\sin\frac{x}{y\sqrt{y}};D=\{(x,y)\in R^2: x\in R, 0<y<+\infty\},\ y_0=+\infty.$

7.2 $f(x,n)=x^{2n};D=\left\{(x,y)\in R^2: 0\le x\le\frac{1}{2}, n\in N\right\},\ n_0=\infty.$

7.3 $f(x,n)=\frac{n\,x}{1+n^3x^2};D=\{(x,n)\in R^2: 1\le x<+\infty, n\in N\},\ n_0=\infty.$

7.4 $f(x,n)=\frac{n^2x^2}{1+n^2x^4}\cdot\sin\frac{x^2}{\sqrt{n}};D=\{(x,n)\in R^2: 1\le x<+\infty,\ n\in N\},\ n_0=\infty.$

7.5 $f(x,n)=\sin(ne^{-nx});D=\{(x,n)\in R^2: 1\le x<+\infty, n\in N\}, n_0=\infty.$

7.6 $f(x,n)=\dfrac{\ln nx}{nx^2}; D=\{(x,n)\in R^2: 1\le x<+\infty, n\in N\}, n_0=\infty.$

7.7 $f(x,n)=n^{3/2}\left(1-\cos\dfrac{\sqrt[4]{x}}{n}\right); D=\{(x,n)\in R^2: 0\le x<+\infty,\ n\in N\},\ n_0=\infty.$

7.8 $f(x,y)=\dfrac{1}{x^3}\cdot\cos\dfrac{x}{y}; D=\{(x,y)\in R^2: 0<x<1, 0<y<+\infty\},\ y_0=\infty.$

7.9 $f(x,y)=(x-1)\,arctgx^y; D=\{(x,y)\in R^2: 0<x<+\infty, 0<y<+\infty\},\ y_0=+\infty.$

7.10 $f(x,y)=\sqrt{x^2+\dfrac{1}{\sqrt{y}}}; D=\{(x,y)\in R^2: x\in R, 0<y<+\infty\},\ y_0=+\infty.$

7.11 $f(x,y)=x^y; D=\{(x,y)\in R^2: 0<x\le 1, 0<y\le 1\}, y_0=0.$

7.12 $f(x,n)=\dfrac{\cos\sqrt{nx}}{\sqrt{n+2x}}; D=\{(x,n)\in R^2: 0\le x<+\infty, n\in N\}, n_0=+\infty.$

7.13 $f(x,n)=\sqrt[n]{1+x^n}; D=\{(x,n)\in R^2: 0\le x\le 2, n\in N\}, n_0=\infty.$

7.14 $f(x,n)=n\,arctg\dfrac{1}{nx^2}; D=\{(x,n)\in R^2: 0\le x<+\infty, n\in N\}, n_0=\infty.$

7.15 $f(x,n)=n^3x^2e^{-nx}; D=\{(x,n)\in R^2: 0\le x<+\infty, n\in N\}, n_0=\infty.$

7.16 $f(x,n)=\sqrt{n}\sin\dfrac{x}{n\sqrt{n}}; D=\{(x,n)\in R^2: 0\le x<+\infty, n\in N\}, n_0=\infty.$

7.17 $f(x,n)=\ln\left(1+\dfrac{\cos nx}{\sqrt{n+x}}\right); D=\{(x,n)\in R^2: 0<x<+\infty, n\in N\}, n_0=\infty.$

7.18 $f(x,y)=\dfrac{xy}{1+x^2y^2}; D=\{(x,y)\in R^2: 0\le x\le 1, y\in R\}, y_0=\infty.$

7.19 $f(x,y)=x\sin y; D=\{(x,y)\in R^2: 0\le x\le 3, y\in R\}, y_0=\dfrac{\pi}{2}.$

7.20 $f(x,y)=\sin\dfrac{x}{y}; D=\{(x,y)\in R^2: x\in R, 0<y<+\infty\}, y_0=+\infty.$

7.21 $f(x,y)=x^2\sin y; D=\{(x,y)\in R^2: 0\le x\le 5, 0<y<\pi\}, y_0=\dfrac{\pi}{3}.$

8-masala. Quyidagi funksiyalarning hosilalarini toping.

8.1 $F(\alpha)=\int\limits_{\sin\alpha}^{\cos\alpha} e^{\alpha\sqrt{1-x^2}}dx.$

8.2 $F(\alpha)=\int\limits_0^{\alpha}\frac{\ln(1+\alpha x)}{x}dx.$

8.3 $F(\alpha)=\int\limits_{a+\alpha}^{b+\alpha}\frac{\sin\alpha x}{x}dx.$

8.4 $F(\alpha)=\int\limits_0^{\alpha}f(x+\alpha,x-\alpha)dx.$

8.5 $F(\alpha)=\int\limits_0^{\alpha^2}dx\int\limits_{x-\alpha}^{x+\alpha}\sin(x^2+y^2-\alpha^2)dy.$

8.6 $F(\alpha)=\int\limits_0^{x}(x+y)f(y)dy.$

8.7 $F(\alpha)=\int\limits_0^{x}(x+y)f(y)dy, f(y)-$ differensiallanuvchi funksiya; $F''(x)-?$

8.8 $F(\alpha)=\int\limits_a^{b}f(y)\cdot|x-y|dy,\quad a<b$ ва $f(y)\in C[a,b]: F''(x)-?$

8.9 $F(\alpha)=\int\limits_0^{x}f(y)\cdot(x-y)^{n-1}dy,\ F^{(n)}(x)-?$

8.10 $F(\alpha)=\int\limits_x^{x^2}e^{-xy^2}dy.$

8.11 $F(\alpha)=\int\limits_{\frac{1}{x}}^{x}e^{-xy^2}dy.$

8.12 $F(y)=\int\limits_{\cos y}^{\sin y}e^{y\sqrt{1-x^2}}dx.$

8.13 $F(y)=\int\limits_{-y}^{y}f(x+y,x-y)dx.$

8.14 $F(y)=\int\limits_{y^2}^{y^4}\frac{\ln(1+xy)}{x}dx.$

8.15 $F(y)=\int\limits_{1+y}^{2+y}\frac{\sin xy}{x}dx.$

8.16 $F(y)=\int\limits_{y^2}^{y^3}2^{-x^2y}dx.$

8.17 $F(y)=\int\limits_{\frac{\pi}{2}}^{y}\left(x^2\sin y+\frac{x}{y}\right)dx.$

8.18 $F(y)=\int\limits_0^{x}f(y)\cdot(x-y)^5dy, F^{IV}(x)-?$

8.19 $F(y)=\int\limits_{1/x}^{2}f(y)\cdot|x-y|dy, f(y)\in C[1,2]\quad F'(x)-?$

8.20 $F(y)=\int\limits_{-x}^{x}(x+y)f(y)dy, F''(x)-?,\ f(y)-$ differensiallanuvchi funksiya;

8.21 $F(x,y)=\int_{x/y}^{xy}(x-yz)f(z)dz, F''_{xy}(x,y)-?,\ f(z)-$ differensiallanuvchi funksiya.

9-masala. Quyidagi integrallarni ko'rsatilgan oraliqda tekis yaqinlashishga tekshiring.

9.1 $\int_0^{+\infty} e^{-\alpha x}\sin x dx;\ 1\le\alpha<+\infty.$

9.2 $\int_0^{+\infty} x^{\alpha}e^{-x}dx;\ 2\le\alpha\le3.$

9.3 $\int_{-\infty}^{+\infty}\frac{\cos\alpha x}{1+x^2}dx;\ -\infty<\alpha<+\infty.$

9.4 $\int_0^{+\infty}\frac{dx}{(x-\alpha)^2+1};\ 0\le\alpha<+\infty.$

9.5 $\int_0^{+\infty}\frac{\sin x}{x}e^{-\alpha x}dx;\ 0\le\alpha<+\infty.$

9.6 $\int_1^{+\infty}\frac{\ln^p x}{x\sqrt{x}}dx;\ 0\le p\le10.$

9.7 $\int_1^{+\infty} e^{-\alpha x}\frac{\cos x}{x}dx;\ 0\le\alpha<+\infty.$

9.8 $\int_0^{+\infty}\sqrt{\alpha}e^{-\alpha x^2}dx;\ 0\le\alpha<+\infty.$

9.9 $\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}dx;\ 2\le\alpha\le3.$

9.10 $\int_{-\infty}^{+\infty} e^{-(x-\alpha)^2}dx;\ -\infty<\alpha<+\infty.$

9.11 $\int_0^{+\infty} e^{-x^2(1+y^2)}\sin xdy,\ -\infty<x<+\infty.$

9.12 $\int_1^{+\infty} e^{-\alpha x}\frac{\cos x}{x^2}dx; 0\le\alpha<+\infty.$

9.13 $\int_0^{+\infty}\frac{\sin x^2}{1+x^p}dx;\ p\ge0.$

9.14 $\int_0^{1}\frac{x^n}{\sqrt{1-x^2}}dx;\ 0\le n<+\infty.$

9.15 $\int_0^{1}\sin\frac{1}{x}\cdot\frac{dx}{x^n};\ 0<n<2.$

9.16 $\int_0^{1}\frac{x^{\alpha}dx}{\sqrt[3]{(x-1)(x-2)^2}}dx; |\alpha|<\frac{1}{2}.$

9.17 $\int_2^{+\infty}\sqrt{\alpha}\cdot e^{-\alpha x^2}dx;\ 0\le\alpha<+\infty.$

9.18 $\int_0^{1}\frac{\sin\alpha x}{\sqrt{|x-\alpha|}}dx;\ 0\le\alpha\le1.$

9.19 $\int_0^{+\infty} e^{-x^2(1+y^2)}\cos ydy;\ x\in R.$

9.20 $\int_0^{1} x^{p-1}\cdot\ln\frac{1}{x}dx,\ p>0.$

9.21 $\int_1^{+\infty} e^{-\alpha x}\frac{\cos x}{x^p}dx;\ 0\le\alpha<+\infty, p>0-$ fiksirlangan.

10-masala.

10.1 Agar $f(x)\in C(0,+\infty)$ va $\forall A>0$ uchun $\int\limits_A^{+\infty}\frac{f(x)}{x}dx$ integral mavjud bo'lsa, unda ushbu

$$\int\limits_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=f(0)\cdot\ln\frac{b}{a}\qquad(a>0,\ b>0).$$

Frullani formulasini isbotlang.

10.2 $I(\alpha)=\int\limits_0^{+\infty}e^{-\alpha x}\frac{\sin\beta x}{x}dx\quad(\alpha\ge 0)$ integraldan foydalanib, ushbu

$$\int\limits_0^{+\infty}\frac{\sin\beta x}{x}dx=\frac{\pi}{2}sign\beta.$$

Dirixle formulasini isbotlang.

Quyidagi integrallarni hisoblang.

10.3 $\int\limits_0^{+\infty}\frac{\sin ax-\sin bx}{x}dx\ (a>0,b>0).$ **10.4** $\int\limits_0^{+\infty}\frac{arctgax-arctgbx}{x}dx\ (a>0,b>0).$

10.5 $\int\limits_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}dx\ (\alpha>0,\beta>0).$ **10.6** $\int\limits_0^{+\infty}\frac{e^{-\alpha x}-e^{-\beta x}}{x}\cos mxdx\ (\alpha>0,\beta>0).$

10.7 $\int\limits_0^{1}\frac{\ln(1-\alpha^2x^2)}{x^2\cdot\sqrt{1-x^2}}dx\ (|\alpha|\le 1).$ **10.8** $\int\limits_0^{1}\frac{\ln(1-\alpha^2x^2)}{\sqrt{1-x^2}}dx\ (|\alpha|\le 1).$

10.9 $\int\limits_0^{+\infty}\frac{\ln(\alpha^2+x^2)}{\beta^2+x^2}dx.$ **10.10** $\int\limits_0^{+\infty}\frac{arctg\alpha x\cdot arctg\beta x}{x^2}dx.$

10.11 $\int\limits_0^{+\infty}\frac{\ln(1+\alpha^2x^2)\cdot\ln(1+\beta^2x^2)}{x^4}dx.$ **10.12** $\int\limits_0^{+\infty}e^{-\left(x^2+\frac{a^2}{x^2}\right)}dx\ (a>0).$

10.13 $\int\limits_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x^2}dx\ (\alpha>0,\beta>0).$ **10.14** $\int\limits_0^{+\infty}e^{-ax^2}\cos bxdx\ (a>0).$

10.15 $\int\limits_0^{+\infty} xe^{-\alpha x^2}\sin bxdx\,(a>0).$

10.16 $\int\limits_0^{+\infty}\left(\frac{\sin\alpha x}{x}\right)^2 dx.$

10.17 $\int\limits_0^{+\infty}\frac{e^{-\alpha x^2}-\cos\beta x}{x^2}dx.$

10.18 $\int\limits_0^{+\infty}\frac{\sin^3 dx}{x}dx.$

10.19 $\int\limits_0^{+\infty}\left(\frac{\sin\alpha x}{x}\right)^3 dx.$

10.20 $\int\limits_0^{+\infty}\frac{\sin^4 xdx}{x^2}dx.$

10.21 $\int\limits_1^{+\infty}\frac{arctg\alpha x}{x^2\cdot\sqrt{x^2-1}}dx.$

11-masala. Quyidagi integrallarni hisoblang.

11.1 $\int\limits_0^{\pi/2}\ln\left(a^2\sin^2 x+b^2\cos^2 x\right)dx.$

11.2 $\int\limits_0^{\pi}\ln\left(1-2a\cos x+a^2\right)dx.$

11.3 $\int\limits_0^{1}\frac{arctgx}{x}\cdot\frac{dx}{\sqrt{1-x^2}}.$

11.4 $\int\limits_0^{1}\cos\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}dx, a>0, b>0.$

11.5 $\int\limits_0^{+\infty}\frac{\sin^4\alpha x-\sin^4\beta x}{x}dx.$

11.6 $\int\limits_0^{+\infty}\frac{\cos\alpha x}{1+x^2}dx.$

11.7 $\int\limits_0^{+\infty}\frac{x\sin\alpha x}{1+x^2}dx.$

11.8 $\int\limits_0^{+\infty}\frac{\cos\alpha x}{\left(1+x^2\right)^2}dx.$

11.9 $\int\limits_0^{+\infty}\frac{\sin^2 x}{1+x^2}dx.$

11.10 $\int\limits_{-\infty}^{+\infty}\sin\left(ax^2+bx+c\right)dx\,(a\neq 0).$

11.11 $\int\limits_{-\infty}^{+\infty}\sin x^2\cdot\cos 2axdx.$

11.12 $\int\limits_{-\infty}^{+\infty}\cos x^2\cdot\cos 2axdx.$

11.13 $\int\limits_0^{+\infty}\frac{\cos xy}{a^2-x^2}dx.$

11.14 $\int\limits_0^{+\infty}\frac{x\sin xy}{a^2-x^2}dx.$

11.15 $\int\limits_0^{+\infty}\frac{x^{a-1}-x^{b-1}}{1-x}dx\,(a>0, b>0).$

11.16 $\int\limits_0^{+\infty}e^{a\cos x}\cdot\sin\left(a\sin x\right)\frac{dx}{x}.$

11.17 $\int\limits_0^{+\infty} \frac{e^{-ax}\cos bx - e^{-cx}\cdot\cos lxdx}{x}dx \quad (a,c>0).$

11.18 $\int\limits_0^{+\infty} e^{-x^2}\cdot\cos\frac{\alpha^2}{x^2}dx.$

11.19 $\int\limits_0^{+\infty} e^{-x^2}\cdot\sin\frac{\alpha^2}{x^2}dx.$

11.20 $\int\limits_0^{\pi/2} \ln\frac{1+a\cos x}{1-a\cos x}\cdot\frac{dx}{\cos x}(|a|<1).$

11.21 $\int\limits_0^{1} \sin\left(\ln\frac{1}{x}\right)\cdot\frac{x^b-x^a}{\ln x}dx, a>0, b>0.$

Ko'rsatma. 10 va 11-masalalarni yechishda xosmas integrallarni parametr bo'yicha differensiallash yoki integrallash hamda Frullani va Dirixle integrallaridan foydalanish yaxshi natija beradi.

12-masala. Eyler integrallaridan foydalanib, quyidagi integrallarni hisoblang.

12.1 $\int\limits_0^{+\infty} \frac{x^{p-1}\cdot\ln x}{1+x}dx.$

12.2 $\int\limits_0^{1} \frac{x^{\alpha-1}\cdot(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}}dx(\alpha>0,\beta>0).$

12.3 $\int\limits_0^{1} \frac{x^{a-1}\cdot x^{-a}}{1-x}dx(0<a<1).$

12.4 $\int\limits_0^{+\infty} \frac{dx}{(1+x^2)^n}.$

12.5 $\int\limits_0^{1} \frac{x^{2a-1}}{1+x^2}dx(0<a<1).$

12.6 $\int\limits_0^{1} \frac{x^{2n}dx}{\sqrt[3]{x\cdot(1-x^2)}}.$

12.7 $\int\limits_0^{+\infty} \frac{\sqrt[4]{x}}{(1+x)^2}dx.$

12.8 $\int\limits_0^{+\infty} \frac{dx}{(1+x^3)^2}.$

12.9 $\int\limits_0^{\pi/2} \sin^6 x\cdot\cos^4 xdx.$

12.10 $\int\limits_0^{1} \frac{dx}{\sqrt[n]{1-x^n}}(n>1).$

12.11 $\int\limits_0^{+\infty} x^{2n}e^{-x^2}dx(n-butun\ \ son)$

12.12 $\int\limits_0^{+\infty} \frac{x^{m-1}}{1+x^n}dx(n>0).$

12.13 $\int\limits_0^{+\infty} \frac{x^{m-1}}{(1+x)^n}dx.$

12.14 $\int\limits_0^{+\infty} \frac{x^m dx}{(a+bx^n)^p}(a>0,b>0,n>0).$

12.15 $\int_0^1 \frac{dx}{\sqrt[n]{1-x^m}} \quad (m>0).$

12.16 $\int_0^{+\infty} e^{-x^n}dx \quad (n>0).$

12.17 $\int_0^{\pi/2} \sin^m x\cdot\cos^n x dx.$

12.18 $\int_0^{+\infty} x^m e^{-x^n}dx$

12.19 $\int_0^{\pi/2} tg^n x dx.$

12.20 $\int_0^1 \left(\ln\frac{1}{x}\right)^p dx.$

12.21 $\int_a^b \frac{(x-a)^m\cdot(b-x)^n}{(x+c)^{m+n+2}}dx \quad (0<a<b,\ c>0).$

-D-

Namunaviy variant yechimi

1.21-masala. Quyidagi

$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+x+1)^3}.$$

xosmas integral hisoblansin.

$$\triangleleft \quad I=\int_{-\infty}^{+\infty}\frac{dx}{(x^2+x+1)^3}=\int_{-\infty}^{+\infty}\frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]^3}=\left(\begin{pmatrix} x+\frac{1}{2}=t \\ almashtirish \\ bajaramiz\end{pmatrix}\right)=\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^3}.$$

Bu integralni hisoblash uchun xosmas integralda bo‘laklab integrallash usulidan foydalanib, quyidagi ishlarni bajaramiz.

$$\int_{-\infty}^{+\infty}\frac{dt}{t^2+\frac{3}{4}}=\frac{2}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}\Bigg|_{-\infty}^{+\infty}=\frac{2\pi}{\sqrt{3}}.\Rightarrow\frac{2\pi}{\sqrt{3}}=\int_{-\infty}^{+\infty}\frac{dt}{t^2+\frac{3}{4}}=\left(\begin{pmatrix} u=\frac{1}{t^2+\frac{3}{4}}\Rightarrow du=\frac{-2tdt}{\left(t^2+\frac{3}{4}\right)^2} \\ v=dt, v=t\end{pmatrix}\right)=$$

$$\overset{Б.И.}{=}\frac{t}{t^2+\frac{3}{4}}\Bigg|_{-\infty}^{+\infty}+2\int_{-\infty}^{+\infty}\frac{t^2}{\left(t^2+\frac{3}{4}\right)^2}dt=2\int_{-\infty}^{+\infty}\frac{dt}{t^2+\frac{3}{4}}-\frac{3}{2}\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}=\frac{4\pi}{\sqrt{3}}-\frac{3}{2}\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}\Rightarrow$$

$$\Rightarrow\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}=\frac{4\pi}{3\sqrt{3}}.$$

Demak,

$$\frac{4\pi}{\sqrt{3}}=\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}=\left(\begin{array}{l}u=\frac{1}{\left(t^2+\frac{3}{4}\right)^2}\Rightarrow du=\frac{-4tdt}{\left(t^2+\frac{3}{4}\right)^3}\\ dv=dt, v=t\end{array}\right)\overset{Б.И}{=}\left.\frac{t}{\left(t^2+\frac{3}{4}\right)^2}\right|_{-\infty}^{+\infty}+4\int_{-\infty}^{+\infty}\frac{t^2}{\left(t^2+\frac{3}{4}\right)^3}dt=$$

$$=4\cdot\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}-3\cdot\int_{-\infty}^{+\infty}\frac{dt}{\left(t^2+\frac{3}{4}\right)^3}=\frac{16\pi}{3\sqrt{3}}-3I.$$

Shunday qilib, berilgan integral I *ga nisbatan ushbu*

$$\frac{4\pi}{3}=\frac{16\pi}{3\sqrt{3}}-3I;$$

tenglamaga keldik. Bu tenglamadan

$I=\int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+x+1\right)^3}=\frac{4\pi}{3\sqrt{3}}$ ekanligini hosil qilamiz. ▷

2.21-masala. Quyidagi

$$\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}, b>a.$$

II-tur xosmas integral hisoblansin.

◁ $x=a$ va $x=b$ nuqtalar integral ostidagi funksiyaning maxsus nuqtalari bo‘ladi. Agar integralda

$$x=a\cos^2 t+b\sin^2 t;$$

almashtirish bajarsak, berilgan xosmas integral oddiy xos aniq integralga kelib qoladi. Darhaqiqat,

$$\left.\begin{array}{l}x=a\Rightarrow t=0\\ x=b\Rightarrow t=\frac{\pi}{2}\end{array}\right\} va \left\{\begin{array}{l}x-a=(b-a)\sin^2 t\\ b-x=(b-a)\cos^2 t\end{array}\right.\Rightarrow dx=2(b-a)\sin t\cos t dt.$$

Bu ifodalarni berilgan integrallarga olib borib qo‘yib topamiz:

$$\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}=2\int_0^{\pi/2}dt=\pi. \triangleright.$$

3.21-masala. Quyidagi

$$\int_0^{\pi/2} \sin^\alpha x \cdot \cos^\beta x dx.$$

integralni yaqinlashishga tekshiring.

◁ Integral ostidagi funksiya uchun $\alpha < 0$ bo'lganda $x = 0$ nuqta, $\beta < 0$ bo'lganda esa $x = \frac{\pi}{2}$ nuqta maxsus nuqta bo'ladi. Shu sababli integrallash oralig'ini ikkiga ajratamiz:

$$I = \int_0^{\pi/2} \sin^\alpha x \cdot \cos^\beta x dx = \int_0^{\pi/4} \sin^\alpha x \cdot \cos^\beta x dx + \int_{\frac{\pi}{4}}^{\pi/2} \sin^\alpha x \cdot \cos^\beta x dx = I_1 + I_2.$$

$x \to 0$ da $\sin^\alpha x \cdot \cos^\beta x = 0^*(\sin^\alpha x) = 0^*(x^\alpha)$, $x \to \frac{\pi}{2}$ da $\sin^\alpha x \cdot \cos^\beta x = 0^*(\cos^\beta x) = 0^*\left(\left(\frac{\pi}{2} - x\right)^\beta\right)$ bo'lganligi va $\int_0^{\pi/4} x^\alpha dx = \int_0^{\pi/4} \frac{dx}{x^{-\alpha}}$ integral $\alpha > -1$ da $\int_{\frac{\pi}{4}}^{\pi/2} \left(\frac{\pi}{2} - x\right)^\beta dx = \int_{\frac{\pi}{4}}^{\pi/2} \frac{dx}{\left(\frac{\pi}{2} - x\right)^{-\beta}}$ integral $\beta > -1$ da yaqinlashishini e'tiborga olsak, taqqoslash alomatiga ko'ra I_1 integral $\alpha > -1, \beta - \forall$ va I_2 integral $\beta > -1$, $\alpha - \forall$ bo'lganda yaqinlashishini hosil qilamiz $\Rightarrow$ Berilgan integral $\alpha > -1$, $\beta > -1$ da yaqinlashadi. ▷

4.21-masala. Quyidagi

$$\int_2^{+\infty} \frac{dx}{x^\alpha \cdot \ln^\beta x}.$$

integralni yaqinlashishga tekshiring.

◁ 1) **Faraz qilaylik** $\alpha > 1$ **bo'lsin.** $\Rightarrow \varepsilon = \alpha - 1$ deb belgilasak, $\varepsilon > 0$ bo'ladi. Unda

$$f(x) = \frac{1}{x^\alpha \cdot \ln^\beta x} = \frac{1}{x^{1+\varepsilon} \cdot \ln^\beta x} = \frac{1}{x^{\varepsilon/2} \cdot \ln^\beta x} \cdot \frac{1}{x^{1+\varepsilon/2}}$$

$$\lim_{x\to+\infty}\frac{1}{x^{\varepsilon/2}\cdot\ln^{\beta}x}=0\Rightarrow\exists A>2:\forall x\geq A \text{ uchun } \frac{1}{x^{\varepsilon/2}\cdot\ln^{\beta}x}<1 \text{ bo'ladi}$$

$$\Rightarrow\forall x\geq A \text{ da } f(x)\leq\frac{1}{x^{1+\varepsilon/2}}=\varphi(x) \text{ bo'ladi}$$

$$\int_2^{+\infty}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}=\int_2^{A}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}+\int_A^{+\infty}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}=I_1+I_2$$

desak, $I_1=\int_2^{A}\frac{dx}{x^{\alpha}\ln^{\beta}x}$ integral oddiy uzluksiz funksiyaning integrali bo'lgani uchun yaqinlashuvchi.

$I_2=\int_A^{+\infty}\frac{dx}{x^{\alpha}\ln^{\beta}x}$ integral esa taqqoslash alomatiga ko'ra yaqinlashuvchi, chunki $\int_A^{+\infty}\varphi(x)dx=\int_A^{+\infty}\frac{dx}{x^{1+\varepsilon/2}}$ yaqinlashuvchi.

Shunday qilib, $\int_2^{+\infty}\frac{dx}{x^{\alpha}\ln^{\beta}x}$ integral $\alpha>1$ bo'lganda $\forall\beta$ uchun yaqinlashuvchi.

2) **Endi $\alpha=1$ bo'lsin.**

$$\int_2^{+\infty}\frac{dx}{x^{\alpha}\cdot\ln^{\beta}x}=\int_2^{+\infty}\frac{dx}{x\ln^{\beta}x}=\int_2^{+\infty}\frac{d(\ln x)}{\ln^{\beta}x}=\begin{cases}\text{yaqinlashadi, } \beta>1,\\ \text{uzoqlashadi, } \beta\leq1\end{cases}$$

3) $\alpha<1$ **bo'lsin.** Bunda $\varepsilon=1-\alpha$ deb belgilab, 1)-holda bajargan ishlarni bajarsak, berilgan integralning uzoqlashuvchi ekanligiga ishonch hosil qilamiz.

Demak, berilgan integral $\alpha>1$ bo'lganda $\forall\ \beta$ va $\alpha=1$ bo'lganda, $\beta>1$ lar uchun yaqinlashadi. Qolgan barcha hollarda esa uzoqlashadi. ▷

5.21-masala. Quyidagi

$$\int_0^1\sin\left(\frac{1}{1-x}\right)\cdot\frac{dx}{1-x}.$$

integral absolut va shartli yaqinlashishga tekshirilsin.

◁ Berilgan integralning yaqinlashishini Dirixle alomatidan foydalanib, ko'rsatamiz.

$$f(x)=\frac{1}{(1-x)^2}\sin\left(\frac{1}{1-x}\right) \text{ va } g(x)=1-x$$

deb belgilaymiz va Dirixle alomatining shartlarini tekshiramiz:

1) $f(x)\in C[0,1)$ va $f(x)$ ning boshlang'ich funksiyasi $F(x)=-\cos\left(\frac{1}{1-x}\right)$-chegaralangan;

2) $g(x)=1-x$ funksiya $[0,1)$ da $\downarrow$ va $\lim\limits_{x\to1-0} g(x)=0$.

3) $g'(x)=-1\in C[0,1)$.

Dirixle alomatining shartlari bajarilayapti $\Rightarrow$

$$\Rightarrow \int_0^1 f(x)\cdot g(x)dx=\int_0^1 \sin\left(\frac{1}{1-x}\right)\cdot\frac{dx}{1-x} \text{ yaqinlashuvchi.}$$

Berilgan integral absolut yaqinlashuvchi emas. Bu tasdiq

$$\left|\frac{1}{1-x}\cdot\sin\frac{1}{1-x}\right|\ge\frac{1}{1-x}\cdot\sin^2\frac{1}{1-x};$$

tengsizlikdan va

$$\int_0^1 \sin^2\left(\frac{1}{1-x}\right)\cdot\frac{dx}{1-x};$$

integralning uzoqlashishidan kelib chiqdi. Oxirgi integralning uzoqlashishini 1^0-punktda keltirilgan 2)-misoldan foydalanib, ko'rsatish qiyin emas. Shunday qilib, berilgan integral shartli yaqinlashuvchi. $\rhd$

6.21-masala. Xosmas integralning Koshi ma'nosidagi bosh qiymati topilsin:

$$V.p\int_0^{+\infty}\frac{dx}{x^2-3x+2}.$$

$$\lhd\ V.p\int_0^{+\infty}\frac{dx}{x^2-3x+2}=V.p\int_0^{+\infty}\frac{dx}{(x-1)(x-2)}=V.p\int_0^{1,5}\frac{dx}{(x-1)(x-2)}+$$

$$+V.p\int_{1,5}^{3}\frac{dx}{(x-1)(x-2)}+\int_3^{+\infty}\frac{dx}{(x-1)(x-2)}=\lim_{\alpha\to+0}\left[\int_0^{1-\alpha}\frac{dx}{(x-1)(x-2)}+\right.$$

$$+\int_{1+\varepsilon}^{1,5}\frac{dx}{(x-1)(x-2)}\Bigg]+\lim_{\beta\to+0}\left[\int_{1,5}^{2-\beta}\frac{dx}{(x-1)(x-2)}+\int_{2+\beta}^{3}\frac{dx}{(x-1)(x-2)}\right]+$$

$$+\lim_{A\to\infty}\int_{3}^{A}\frac{dx}{(x-1)(x-2)}.$$

Agar $\int\frac{dx}{(x-1)(x-2)}=\int\left[\frac{1}{x-2}-\frac{1}{x-1}\right]dx=\ln|x-2|-\ln|x-1|$ ekanligidan foydalanib, yuqoridagi limitlarni hisoblasak, $V.p\int_0^{+\infty}\frac{dx}{x^2-3x+2}=-\ln 2$ tenglikni hosil qilamiz. ▷

7.21-masala. $D=\{(x,y)\in R^2: 0\le x\le 5,\ 0<y<\pi\}$, **to'plamda berilgan** $f(x,y)=x^2\sin y$ **funksiyaning** $y_0=\frac{\pi}{3}$ **nuqtadagi limit funksiyasini toping va tekis yaqinlashishga tekshiring.**

◁ $\varphi(x)=\lim_{y\to y_0}f(x,y)=\lim_{y\to\frac{\pi}{3}}x^2\sin y=\frac{\sqrt{3}}{2}x^2$ – limit funksiya. $f(x,y)$ funksiya $\varphi(x)$ ga tekis yaqinlashuvchi ekanligini 4⁰-punktdagi 3-ta'rifdan foydalanib, ko'rsatamiz. $\forall\varepsilon>0$ va quyidagi ayirmani olamiz.

$$|f(x,y)-\varphi(x)|=\left|x^2\cdot\sin y-\frac{\sqrt{3}}{2}x^2\right|=x^2\left|\sin y-\sin\frac{\pi}{3}\right|=x^2\cdot\left|2\sin\frac{y-\pi/3}{2}\cdot\cos\frac{y+\pi/3}{2}\right|<$$

$$<x^2\cdot 2\cdot\frac{|y-\pi/3|}{2}<x^2\cdot\delta\le 25\delta=\varepsilon\Rightarrow\delta=\frac{\varepsilon}{25}.$$

Demak, $\forall\varepsilon>0$ olingarda ham $\delta=\frac{\varepsilon}{25}$ deb olsak, $\left|y-\frac{\pi}{3}\right|<\delta$ tengsizlikni qanoatlantiruvchi $\forall y\in(0,\pi)$ va $\forall x\in[0,5]$ lar uchun $|f(x,y)-\varphi(x)|<\varepsilon$ tengsizlik bajariladi. Bu esa $y\to\frac{\pi}{3}$ da $f(x,y)$ funksiya $\varphi(x)$ ga tekis yaqinlashuvchi ekanligini anglatadi. ▷

8.21-masala. Agar

$$F(x,y)=\int\limits_{\frac{x}{y}}^{xy}(x-yz)f(z)dz$$ **bo'lib,** $f(z)$ **-differensiallanuvchi funksiya bo'lsa,** $F_{xy}''(xy)$ **ni toping.**

◁ Bu masalani 4^0-punktdagi 7-teorema va (10)-tenglikdan foydalanib, yechamiz. Teoremaning shartlari bajarilishi ko'rinib turibdi. (10)-formuladan ikki marta foydalanish natijasida talab qilingan hosilani topamiz:

$$F_x'(x,y)=\int\limits_{\frac{x}{y}}^{xy}f(z)dz+y\cdot(x-y\cdot xy)f(xy)-\frac{1}{y}\cdot\left(x-y\cdot\frac{x}{y}\right)f\left(\frac{x}{y}\right)=$$

$$=\int\limits_{\frac{x}{y}}^{xy}f(z)dz+\left(xy-xy^3\right)\cdot f(xy);$$

$$F_{xy}''(x,y)=\int\limits_{\frac{x}{y}}^{xy}0\cdot dz+x\cdot f(xy)-\left(-\frac{x}{y^2}\right)f\left(\frac{x}{y}\right)+\left(x-3xy^2\right)f(xy)+$$

$$+\left(xy-xy^3\right)\cdot f'(xy)\cdot x=x\left(2-3y^2\right)f(xy)+\frac{x}{y^2}f\left(\frac{x}{y}\right)+x^2y\left(1-y^2\right)f'(xy).\ \triangleright$$

9.21-masala. Quyidagi

$$\int\limits_1^{+\infty}e^{-\alpha x}\frac{\cos x}{x^p}dx;$$

integralni $0\le\alpha<+\infty$, $p>0$ **-fiksirlangan bo'lganda tekis yaqinlashishga tekshiring.**

◁ Berilgan integralning tekis yaqinlashishini Abel alomatidan (5^0-punktdagi 3-teorema) foydalanib, ko'rsatamiz. $f(x,\alpha)=e^{-\alpha x}$ va $g(x,\alpha)=\frac{\cos x}{x^p}$ deb belgilab, Abel alomatining shartlarini tekshiramiz.

1) $f(x,\alpha)=e^{-\alpha x}$ funksiya har bir fiksirlangan $\alpha\in[0,+\infty)$ uchun monoton va $D=\{(x,\alpha)\in R^2 : x\in[1,+\infty),\alpha\in[0,+\infty)\}$, to'plamda chegaralangan $|f(x,\alpha)|\le 1$.

2) $\int_1^{+\infty}\frac{\cos x}{x^p}dx-$ integral Dirixle alomatiga ko'ra $0\le\alpha<+\infty$ to'plamda tekis yaqinlashuvchi. Abel teoremasining shartlari bajarildi. $\Rightarrow$ berilgan intengral $0\le\alpha<+\infty$ to'plamda tekis yaqinlashadi.$\triangleright$

10.21-masala. Quyidagi

$$\int_1^{+\infty}\frac{arctg\alpha x}{x^2\cdot\sqrt{x^2-1}}dx.$$

integral hisoblang.

$\triangleleft$ $I(\alpha)=\int_1^{+\infty}\frac{arctg\alpha x}{x^2\cdot\sqrt{x^2-1}}dx.$ deb belgilab olib, bu integralni parametr bo'yicha differensiallash amalidan foydalanib, hisoblaymiz. Buning uchun avval xosmas integrallarda parametr bo'yicha differensiallash mumkinligi haqidagi 6^0-punktda keltirilgan 3-teoremaning shartlari bajarilishini ko'rsatamiz.

$f(x,\alpha)=\frac{arctg\alpha x}{x^2\cdot\sqrt{x^2-1}}$ va $D=\{(x,\alpha)\in R^2:1<x<+\infty,-\infty<\alpha<+\infty\}$ deb belgilaymiz.

$$|f(x,\alpha)|=\frac{|arctg\alpha x|}{x^2\cdot\sqrt{x^2-1}}\le\frac{\pi}{2x^2\cdot\sqrt{x^2-1}};$$

$$|f_\alpha'(x,\alpha)|=\frac{1}{x(1+\alpha^2x^2)\sqrt{x^2-1}}\le\frac{1}{x\sqrt{x^2-1}};$$

tengsizliklar va $\int_1^{+\infty}\frac{\pi dx}{2x^2\sqrt{x^2-1}}dx$, $\int_1^{+\infty}\frac{1}{x\sqrt{x^2-1}}dx$ integrallar yaqinlashuvchi ekanligidan Veyershtrass alomatiga ko'ra $\int_1^{+\infty}f(x,\alpha)dx$ va $\int_1^{+\infty}f_\alpha'(x,\alpha)dx$ integrallarning $-\infty<\alpha<+\infty$ to'plamda tekis yaqinlashishini hosil qilamiz. Demak, berilgan integraldan parametr α bo'yicha xosila olish mumkin:

$I'(\alpha)=\int_1^{+\infty}\frac{dx}{x(1+\alpha^2x^2)\cdot\sqrt{x^2-1}}$. Bu integralda $x=cht$ almashtirish bajarib, $I'(\alpha)=\frac{\pi}{2}\left(1-\frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)$ bo'lishini topamiz. Bu tenglikdan $I(\alpha)$ ni topamiz. $\alpha\geq 0$ bo'lganda

$$I(\alpha)=\int\frac{\pi}{2}\left(1-\frac{\alpha}{\sqrt{1+\alpha^2}}\right)d\alpha+c=\frac{\pi}{2}\left(\alpha-\sqrt{1+\alpha^2}\right)+c,\alpha\geq 0$$

$$I(0)=0\Rightarrow c=\frac{\pi}{2}\Rightarrow I(\alpha)=\frac{\pi}{2}\left(1+\alpha-\sqrt{1+\alpha^2}\right),\alpha\geq 0.$$

Xuddi shu kabi $\alpha\leq 0$ bo'lganda $I(\alpha)=-\frac{\pi}{2}\left(1-\alpha-\sqrt{1+\alpha^2}\right)$ ekanligini topamiz. Ikkala javobni umumlashtirsak, $I(\alpha)=\frac{\pi}{2}\left(1+|\alpha|-\sqrt{1+\alpha^2}\right)\operatorname{sgn}\alpha,\ |\alpha|<\infty$ tenglikni hosil qilamiz. ▷

11.21-masala. Quyidagi

$$\int_0^1\sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}dx,\ a>0,\ b>0;$$

integralni hisoblang.

◁ Bu integralni ushbu $\frac{x^b-x^a}{\ln x}=\int_a^b x^y dy$ tenglik va parametrga bog'liq integrallarni parametr bo'yicha integrallash haqidagi teoremadan foydalanib, hisoblaymiz:

$$\int_0^1\sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}dx=\int_0^1\left[\sin\left(\ln\frac{1}{x}\right)\int_a^b x^y dy\right]dx=\int_0^1\left[\int_a^b x^y\cdot\sin\left(\ln\frac{1}{x}\right)dy\right]dx=$$

$$=\int_a^b\left[\int_0^1 x^y\cdot\sin\left(\ln\frac{1}{x}\right)dx\right]dy=\left(\left(\begin{array}{l}x=e^{-t}\Rightarrow dx=-e^{-t}dt\\ x=0\Rightarrow t=+\infty;x=1\Rightarrow t=0\end{array}\right)\right)=$$

$$=\int_a^b\left[\int_0^{+\infty}e^{-(1+y)t}\cdot\sin t dt\right]dy=$$

$=\Bigg(\Bigg(I=\int_0^{+\infty}e^{-(1+y)t}\cdot\sin tdt$ *integralda ikki marta bo'laklab integralla-*

sak, I *ga nisbatan chiziqli tenglama hosil qilamiz va* $I=\frac{1}{(y+1)^2+1}$ ekanligini topamiz $\Big)\Big)=\int_a^b\frac{dy}{1+(y+1)^2}=arctg(y+1)\Big|_a^b=arctg(b+1)-$

$$-arctg(a+1)=arctg\frac{b-a}{1+(a+1)\cdot(b+1)}.\triangleright$$

12.21-masala. Eyler integrallaridan foydalanib, quyidagi

$$I=\int_a^b\frac{(x-a)^m\cdot(b-x)^n}{(x+c)^{m+n+2}}dx\left(0<a<b,c>0\right);$$

integralni hisoblang.

$\triangleleft$ Berilgan integralni Eyler integraliga keltirish uchun shunday almashtirish bajarishimiz kerakki, natijada $[a,b]$ kesma $[0,1]$ kesmaga o'tsin. Buning uchun $\frac{x-a}{x+c}=\frac{b-a}{b+c}\cdot t$ almashtirish bajarish kifoya.

Agar berilgan integralda shu almashtirishni bajarsak,

$$\left(\frac{x-a}{x+c}\right)^m=\left(\frac{b-a}{b+c}\right)^m\cdot t^m,\quad\left(\frac{b-x}{x+c}\right)^n=\left(\frac{b-a}{a+c}\right)^n\cdot(1-t)^n \text{ va}$$

$$\frac{dx}{(x+c)^2}=\frac{b-a}{(b+c)\cdot(a+c)}dt$$

bo'lib, u quyidagi ko'rinishga keladi va oson hisoblanadi:

$$I=\int_a^b\frac{(x-a)^m\cdot(b-x)^n}{(x+c)^{m+n+2}}dx=\frac{(b-a)^{m+n+1}}{(b+c)^{m+1}\cdot(a+c)^{n+1}}\int_0^1 t^m\cdot(1-t)^n\,dt=$$

$$=\frac{(b-a)^{m+n+1}}{(b+c)^{m+1}\cdot(a+c)^{n+1}}B(m+1,n+1).$$

Natija. Agar **berilgan integrallarda** m va n lar **natural sonlar** bo'lsa, unda

$$I=\frac{(b-a)^{m+n+1}}{(b+c)^{m+1}\cdot(a+c)^{n+1}}\cdot\frac{m!\cdot n!}{(m+n+1)!}\ \textbf{bo'ladi.}\triangleright$$

9-§. 8-MUSTAQIL ISH

Karrali va egri chiziqli integrallar. Sirt integrallari va maydonlar nazariyasi elementlari. Furye qatorlari

Ikki karrali integrallar va ularning asosiy xossalari.
Ikki karrali integrallarni hisoblash.
Ikki karrali integrallarda o'zgaruvchilarni almashtirish.
Silindrik va sferik koordinatalar.
Ikki karrali integrallarning ba'zi tatbiqlari.
1-tur egri chiziqli integral va uni hisoblash.
2-tur egri chiziqli integral va uni hisoblash.
Grin formulasi va uning tatbiqlari.
1-tur sirt integrali va uni hisoblash.
2-tur sirt integrali va uni hisoblash.
Stoks va Gauss-Ostrogradskiy formulalari.
Maydonlar nazariyasi elementlari.
Furye qatorlari.

-A-

Asosiy tushunsha va teoremalar

1^0. Ikki karrali integralning ta'rifi va uning asosiy xossalari

Rimanning karrali integrallar nazariyasi R^n fazodagi Jordan o'lchoviga asoslangan. Jordan bo'yicha o'lchovli to'plamlarning asosiy xossalaridan biri, uning chegaralangan bo'lishidir. To'plam chegarasining Jordan o'lchovi 0 ga teng bo'lishi zarur va etarlidir. R^2 (R^3) fazoda Jordan bo'yicha o'lchovga ega bo'lgan to'plamga kvadratlanuvchi (kublanuvchi) soha deyiladi. $n \geq 3$ bo'lganda karrali integrallar nazariyasi ikki karrali integrallar nazariyasidan prinsipial jihatdan farq qilmaganligi va ikki karrali integrallarni tasavvur qilish osonroq bo'lganligi sababli biz asosan ikki karrali integrallar nazariyasini keltirish bilan kifoyalanamiz. Butun paragraf davomida biz qaralayotgan sohani kvadratlanuvchi deb faraz qilamiz.

Aytaylik $D \subset R^2$ sohada $f(x,y)$ funksiya aniqlangan bo'lsin. D sohani $\forall$ egri chiziqlar to'ri yordamida n ta $D_1, D_2, \ldots, D_n$ sohachalarga bo'lamiz.

D_κ sohada $\forall(\xi_\kappa, \eta_k)$ nuqta olib, $f(\xi_\kappa, \eta_k)$ ni hisoblaymiz hamda quyidagi

$$\sigma = \sum_{k=1}^{\infty} f(\xi_k, \eta_k) \cdot S_k \tag{1}$$

$f(x,y)$ funksiyaning D soha uchun **integral yig'indisini tuzamiz.** Bu yerda S_k – D_κ sohaning yuzasi.

Ta'rif. *Agar (1)-integral yig'indining* $\lambda = \max\limits_{k=\overline{1,n}} diam\, D_\kappa$ *0 ga intilgandagi limiti mavjud bo'lib, u chekli songa teng bo'lsa hamda uning qiymati sohaning bo'linish usuliga va* (ξ_κ, η_k) *nuqtalarning tanlanishiga bog'liq bo'lmasa, u holda o'sha son* $f(x,y)$ *funksiyaning D soha bo'yicha* ***ikki karrali integrali*** *(Riman ma'nosidagi integrali) deyiladi va u*

$$I = \iint_D f(x,y)ds \quad yoki \quad I = \iint_D f(x,y)dxdy;$$

kabi belgilanadi. $f(x,y)$ *funksiya D sohada* ***integrallanuvchi*** *deyiladi. Aks holda* $f(x,y)$ *funksiya D sohada integrallanuvchi emas, deyiladi.*

Shunday qilib,

$$I = \iint_D f(x,y)dxdy := \lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot S_k \tag{2}$$

Izoh. Karrali integrallar uchun integrallanuvchi funksiya chegaralangan bo'lishi shart emas. Lekin, biz tasdiqlarning sodda bo'lishi uchun paragraf davomida integrallanuvchi funksiyalardan ularning chegaralangan bo'lishini talab qilamiz.

Ikki karrali integralni ham bir o'zgaruvchili funksiyaning aniq integralidagi kabi Darbu yig'indilari yordamida ham aniqlash mumkin.

Aytaylik, $M_k = Sup\{f(x,y):(x,y) \in D_\kappa\}$ va $m_k = \inf\{f(x,y):(x,y) \in D_\kappa\}$ bo'lib, $\omega_k = M_k - m_k - f(x,y)$ funksiyaning D_κ sohadagi tebranishi bo'lsin.

1-teorema. $f(x,y)$ funksiya D sohada integrallanuvchi bo'lishi uchun

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} \omega_k S_k = 0 \tag{3}$$

tenglikning bajarilishi zarur va etarlidir.

2-ta'rif. *Agar* $\forall \varepsilon > 0$ *uchun* $E \subset R^2$ *to'plamni yuzalarining yig'indisi* ε *dan kichik bo'lgan sanoqli sondagi to'g'ri to'rtburchaklar bilan qoplash mumkin bo'lsa, u holda E to'plamning* ***Lebeg o'lchovi*** *0*

***ga teng** deyiladi. Agar E to'plamni yuzalarining yig'indisi etarlicha kichik bo'lgan chekli sondagi to'g'ri to'rtburchaklar bilan qoplash mumkin bo'lsa, unda E to'plamning **Jordan o'lchovi 0 ga teng** deyiladi.*

Ta'rifdan ko'rinadiki, Jordan o'lchovi 0 ga teng to'plamning Lebeg o'lchovi ham 0 ga teng bo'ladi. Teskarisi o'rinli emas lekin Lebeg o'lchovi 0 ga teng kompakt to'plamning Jordan o'lchovi ham 0 ga teng bo'ladi. Jordan o'lchovi 0 ga teng bo'lgan to'plamlarning chekli sondagi yig'indisining Jordan o'lchovi, Lebeg o'lchovi 0 ga teng bo'lgan to'plamlarning sanoqli sondagi yig'indisining Lebeg o'lchovi 0 ga teng bo'ladi.

2-teorema. *(Lebeg teoremasi). Agar $f(x,y)$ funksiya o'lchovga ega bo'lgan yopiq D sohada chegaralangan va bu sohadagi Lebeg o'lchovi* 0 *ga teng bo'lgan E sohada uzilishga ega bo'lib, qolgan barcha nuqtalarda uzluksiz bo'lsa, u holda $f(x,y)$ funksiya D sohada integrallanuvchi bo'ladi.*

Natija. Agar $f(x,y)$ funksiya o'lchovga ega bo'lgan chegaralangan yopiq D sohada uzluksiz bo'lsa, u holda $f(x,y)$ funksiya D sohada integrallanuvchi bo'ladi.

Ikki karrali integrallar ham oddiy bir o'zgaruvchili funksiyaning aniq integrali uchun o'rinli bo'lgan qator xossalarga ega. Biz ularning barchasini takrorlamay, o'rta qiymat haqidagi teoremalarga to'xtalamiz, xolos.

$f(x,y)$ funksiya D sohada aniqlangan bo'lib, shu sohada chegaralangan bo'lsin, ya'ni $\exists m$ va M sonlar: $\forall(x,y)\in D$ uchun

$$m \le f(x,y) \le M;$$

bo'ladi.

3-teorema. *$f(x,y)$ funksiya D sohada integrallanuvchi bo'lsa, u holda $\exists$ o'zgarmas μ $(m \le \mu \le M)$ son mavjudki,*

$$I = \iint_D f(x,y)\,dxdy = \mu \cdot S;$$

bo'ladi. Bu yerda S — D sohaning yuzasi.

Natija. Agar $f(x,y) \in C(D)$ bo'lib, D yopiq bo'lsa, unda $\exists(a,b)\in D$ nuqta topiladiki

$$I = \iint_D f(x,y)\,dxdy = f(a,b)\cdot S;$$

bo'ladi.

4-teorema. *Agar* $g(x,y)$ *funksiya* D *sohada integrallanuvchi bo'lib, u shu sohada o'z ishorasini o'zgartirmasa va* $f(x,y)\in C(D)$ *bo'lsa, u holda* $\exists(a,b)\in D$ *nuqta topiladiki,*

$$\iint\limits_D f(x,y)g(x,y)dxdy = f(a,b)\cdot\iint\limits_D g(x,y)dxdy;$$

bo'ladi.

2⁰. Ikki karrali integrallarni hisoblash

Ikki karrali integrallar amaliyotda takroriy integralga keltirish yordamida hisoblanadi. Biz D soha to'g'ri to'rtburchakli va egri chiziqli trapetsiya bo'lgan 2 ta holda ikki karrali integralni takroriy integralga keltirish haqidagi teoremalarni keltiramiz.

1-teorema. $f(x,y)$ *funksiya* $D=\{(x,y)\in R^2 : a\le x\le b,\ c\le y\le d\}$ *sohada berilgan va integrallanuvchi bo'lsin. Agar har bir fiksirlangan* $x\in[a,b]$ *da*

$$I(x)=\int_c^d f(x,y)dy;$$

integral mavjud bo'lsa, u holda ushbu

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx;$$

takroriy integral mavjud bo'lib,

$$\iint\limits_D f(x,y)dxdy=\int_a^b\left[\int_c^d f(x,y)dy\right]dx; \qquad (4)$$

tenglik o'rinli bo'ladi.

2-teorema. $f(x,y)$ funksiya D sohada integrallanuvchi bo'lib, $\forall$ fiksirlangan $y\in[c,d]$da

$$I(x)=\int_a^b f(x,y)dy;$$

mavjud bo'lsa, u holda

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy;$$

integral ham mavjud bo'ladi va

$$\iint_D f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx \right] dy \qquad (5)$$

tenglik bajariladi.

Endi soha

$$D = \{(x,y) \in R^2 : a \le x \le b, \quad \varphi_1(x) \le y \le \varphi_2(x)\}$$

egri chiziqli trapesiya ko'rinishida berilgan bo'lib, $\varphi_1(x)$ va $\varphi_2(x) \in C[a,b]$ bo'lsin.

3-teorema. *$f(x,y)$ funksiya D sohada berilgan va integrallanuvchi bo'lsin. Agar $\forall$ fiksirlangan $x \in [a,b]$ uchun*

$$I(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

integral mavjud bo'lsa, u holda

$$\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right] dx$$

mavjud bo'ladi va

$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right] dx; \qquad (6)$$

tenglik bajariladi.

Agar D soha

$$D = \{(x,y) \in R^2 : \psi_1(y) \le x \le \psi_2(y), \quad c \le y \le d\}$$

ko'rinishda bo'lib, $\psi_1(y)$ va $\psi_2(y) \in C[c,d]$ bo'lsa, unda quyidagi teorema o'rinli bo'ladi.

4-teorema. $f(x,y)$ funksiya D sohada integrallanuvchi bo'lib, $\forall$ fiksirlangan $y \in [c,d]$ uchun

$$I(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx;$$

mavjud bo'lsa, unda

$$\int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right] dy;$$

mavjud va

$$\iint_D f(x,y)dxdy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right] dy \tag{7}$$

bo'ladi.

3^0. Ikki karrali integrallarda o'zgaruvchilarni almashtirish. Silindrik va sferik koordinatalar sistemasi

$D \subset R^2$ soha berilgan bo'lib, $f(x,y)$ funksiya D da integrallanuvchi bo'lsin. $\Rightarrow \iint_D f(x,y)dxdy - \exists$

$$I = \iint_D f(x,y)dxdy \tag{7}$$

deb belgilaymiz. Bizdan (7) ni hisoblash talab qilinsin. Ravshanki, $f(x,y)$ funksiya hamda D soha murakkab bo'lsa, (7)-integralni hisoblash qiyin bo'ladi.

Ko'p hollarda x va u o'zgaruvchilarni boshqa o'zgaruvchilarga almashtirish natijasida funksiya ham, soha ham soddalashib, ikki karrali integralni hisoblash osonlashadi.

Aytaylik, 2 ta xOu va uOv tekisliklar berilgan bo'lsin. xOu tekisligida chegaralangan, chegarasi ∂D sodda, bo'lakli silliq chiziqdan iborat bo'lgan D sohani qaraylik. Ikkinchi uov tekisligida ham xuddi shunga o'xshash Δ sohani olamiz.

$\varphi(u,v)$ va $\psi(u,v)$ funksiyalar Δ da berilgan shunday funksiyalar bo'lsinki, ular Δ sohadagi $\forall(u,v)$ nuqtani D sohadagi (x,y) nuqtaga akslantirsin, ya'ni

$$\begin{cases} x = \varphi(u,v) \\ y = \psi(u,v) \end{cases} \tag{8}$$

funksiyalar Δ sohani D sohaga akslantiradi.

Faraz qilaylik, bu akslantirish quyidagi shartlarni qanoatlantirsin:

1) (8)-akslantirish o'zaro bir qiymatli,

2) $\varphi(x,y), \psi(x,y) \in C(D)$ bo'lib, bu funksiyalarga teskari bo'lgan funksiyalar $\varphi_1(u,v), \psi_2(u,v) \in C(\Delta)$ va ularning barcha birinchi tartibli xususiy hosilalari $\exists$ bo'lib, ular ham mos sohalarda uzluksiz bo'lsin,

3) (8)-sistemadagi funksiyalarning xususiy hosilalaridan tuzilgan determinant **(yakobian)** uchun

$$I(u,v)=\frac{D(x,y)}{D(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0 \qquad (9)$$

shart bajarilsin.

Teorema. *Faraz qilaylik, (8)-sistema yordamida aniqlangan funksiyalar* Δ *sohani* D *sohaga akslantirsin va yuqoridagi 1)-3)-shartlarni qanoatlantirsin. U holda*

$$I=\iint\limits_{D} f(x,y)\,dxdy=\iint\limits_{\Delta} f\left[\varphi(u,v),\psi(u,v)\right]\cdot\left|I(u,v)\right|dudv \qquad (10)$$

bo'ladi.

(10)-formulaga **ikki karrali integrallarda o'zgaruvchilarni almashtirish formulasi** deyiladi.

Uch karrali integrallarda o'zgaruvchilarni almashtirish formulalari ham shu kabi bo'ladi. Masalan,

$$\begin{cases} x=\varphi(u,v,w) \\ y=\psi(u,v,w) \\ z=\chi(u,v,w) \end{cases}$$

funksiyalar $\Delta\subset R^3$ sohani $D\subset R^3$ sohaga akslantirib, yuqoridagi 1)-3) shartlarni qanoatlantirsin. Agar D sohada integrallanuvchi $f(x,y,z)$ funksiya berilgan bo'lsa, u holda

$$\iiint\limits_{D} f(x,y,z)\,dxdydz=\iiint\limits_{\Delta} f\left[\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w)\right]\cdot\left|I(u,v,w)\right|\cdot dudvdw$$

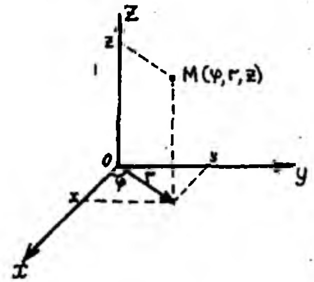
tenglik o'rinli bo'ladi. Bu yerda

$$I(u,v,w)=\frac{D(x,y,z)}{D(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} -$$

berilgan akslantirishning yakobiani.

Ikki karrali integrallarni hisoblashda qutb koordinatalar sistemasiga o'tish ($x=r\cos\varphi$, $y=r\sin\varphi$, $I=r$), uch karrali integrallarni hisoblashda esa silindrik yoki sferik koordinatalar sistemasiga o'tish ko'p hollarda yaxshi natija beradi.

Silindrik koordinatalar sistemasida $\forall M \in R^3$ nuqta $M(\varphi, r, z)$ kabi beriladi (10-chizma).



10-chizma.

Silindrik koordinatalar sistemasini Dekart koordinatalar sistemasi bilan bog'lovchi formulalar (11) va (12) tengliklarda keltirilgan.

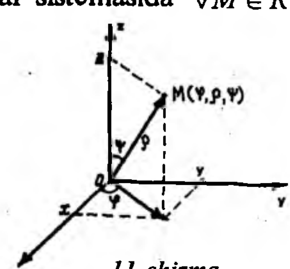
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z, \qquad 0 \le \varphi \le 2\pi, \quad 0 \le r < +\infty \end{cases} \qquad (11)$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \varphi = arctg\dfrac{y}{x} \\ z = z \end{cases} \qquad (12)$$

(11)-sistema uchun yakobian

$$\left|I(\varphi, r, z)\right| = \left|\frac{D(x, y, z)}{D(\varphi, r, z)}\right| = r.$$

Sferik koordinatalar sistemasida $\forall M \in R^3$ nuqta $M(\varphi, \rho, \psi)$ kabi



11-chizma.

beriladi (11-chizma). Sferik koordinatalar sistemasini Dekart koordinatalar sistemasi bilan bog'lovchi formulalar (13)-tenglikda keltirilgan.

$$\begin{cases} x = \rho\cos\varphi\cdot\sin\psi \\ y = \rho\sin\varphi\cdot\sin\psi \\ z = \rho\cos\psi \end{cases} \quad 0\le\varphi\le 2\pi,\ \ 0\le\rho<+\infty,\ \ 0\le\psi\le\pi \tag{13}$$

(13)-sistema uchun yakobian

$$\left|I(\varphi,\rho,\psi)\right| = \left|\frac{D(x,y,z)}{D(\varphi,\rho,\psi)}\right| = \rho^2\cdot\sin\psi\,.$$

4⁰. Ikki karrali integrallarning ba'zi bir tatbiqlari

a) Jismning hajmi. R^3 fazoda yuqoridan $z=f(x,y)$ sirt bilan, yon tomonlaridan yasovchilari $0Z$ o'qiga parallel bo'lgan silindrik sirt hamda pastdan $0XY$ tekisligidagi D soha bilan chegaralangan (V) jismning hajmi V ushbu

$$V = \iint\limits_D f(x,y)\,dxdy \tag{14}$$

formula yordamida hisoblanadi.

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$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1;$$

ellipsoidning hajmi topilsin.

◁ Agar $\{z\ge 0\}$ yarim fazodagi ellipsoid bo'lagining hajmini V_1 desak, unda

$$V = 2V_1 = 2c\iint\limits_D\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy\,;$$

bo'ladi. Bu yerda

$$D=\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases}$ almashtirish bajaramiz, unda D soha

$\Delta=\{(r,\varphi):0\le r\le 1, 0\le\varphi\le 2\pi\}$ to'g'ri to'rtburchakka **akslanadi va ya-** kobian

$$I(r,\varphi)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix} a\cos\varphi, -ar\sin\varphi\\ b\sin\varphi, br\cos\varphi\end{vmatrix}=abr;$$

bo'ladi. Unda

$$V=2c\iint_{\Delta}\sqrt{1-r^2}\cdot abrdrd\varphi=2abc\int_0^1\left[r\sqrt{1-r^2}\int_0^{2\pi}d\varphi\right]dr=4\pi abc\int_0^1 r\sqrt{1-r^2}\,dr=\frac{4\pi}{3}abc.\triangleright$$

b) Yassi shaklning yuzasi.

Ikki karrali integralning ta'rifiga ko'ra, D sohaning yuzasi

$$S=\iint_D dxdy \tag{15}$$

formula yordamida hisoblanadi.

Xususan, soha $D=\{(x,y)\in R^2: a\le x\le b,\ 0\le y\le f(x)\}$ egri chiziqli trapetsiya bo'lsa, u holda

$$S=\iint_D dxdy=\int_a^b\left[\int_0^{f(x)}dy\right]dx=\int_a^b f(x)dx;$$

bizga ma'lum bo'lgan formulaga kelamiz.

d) Sirtning yuzasi. Aytaylik, $z=f(x,y)\in C^1(D)$ bo'lib, bu funksiyaning grafigi R^3 fazoda (S) sirtdan iborat bo'lsin. U holda bu sirt yuzasi

$$S=\iint_D\sqrt{1+\left[f_x'(x,y)\right]^2+\left[f_y'(x,y)\right]^2}\,dxdy \tag{16}$$

formula yordamida hisoblanadi.

e) Ikki karrali integrallar yordamida mexanikaga oid masalalarni yechish.

Aytaylik, D – xOu tekisligida berilgan zichligi $\rho(x,y)$ ga teng bo'lgan bir jinsli plastinka bo'lsin. Unda quyidagi formulalar o'rinli bo'ladi.

$$M=\iint_D\rho(x,y)\,dxdy \tag{17}$$

(17)-plastinkaning massasi.

$$\begin{cases} M_x = \iint\limits_D y \cdot \rho(x,y)dxdy, \\ M_y = \iint\limits_D x \cdot \rho(x,y)dxdy \end{cases} \tag{18}$$

(18)-plastinkaning 0X va 0Y o'qlariga nisbatan statik momentlari.

$$\begin{cases} x_0 = \frac{M_y}{M} \\ y_0 = \frac{M_x}{M} \end{cases} \tag{19}$$

(19)-plastinka og'irlik markazining koordinatalari.

$$\begin{cases} I_x = \iint\limits_D y^2 \cdot \rho(x,y)dxdy; \\ I_y = \iint\limits_D x^2 \rho(x,y)dxdy \end{cases} \tag{20}$$

(20)-plastinkaning 0X va 0Y o'qlariga nisbatan inersiya momentlari.

$$I_0 = I_x + I_y = \iint\limits_D (x^2 + y^2) \cdot \rho(x,y)dxdy \tag{21}$$

(21)-plastinkaning koordinata boshiga nisbatan inersiya momenti.

Eslatma. Ikki karrali integral oddiy bir o'zgaruvchili funksiya aniq integralining qanday umumlashmasi bo'lsa, uch karrali integral ham ikki karrali integralning shunday umumlashmasi bo'ladi va prinsipial jihatdan undan farq qilmaydi. Shu munosabat bilan uch karrali integralning ta'rifi, uni hisoblash usullari va ularning tatbiqlarini o'qib, o'rganib olishni o'quvchining o'ziga havola qilamiz.

5⁰. Birinchi tur egri chiziqli integrallar va ularni hisoblash

Ushbu $x = \varphi(t)$, $y = \psi(t)$ funksiyalar $[\alpha, \beta]$ kesmada aniqlangan va uzluksiz bo'lib, ular t ning turli qiymatlariga R^2 da turli nuqtalarni mos qo'ysin. Bu holda $[\alpha, \beta]$ kesmaning

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

funksiyalar yordamida R^2 da hosil bo'ladigan aksi γ ga sodda egri chiziq deyiladi:

$$\gamma = \{(x,y) \in R^2 : x = \varphi(t), y = \psi(t), t \in [\alpha, \beta]\}.$$

$A=\gamma(\alpha)$ ga egri chiziqning boshlang'ich nuqtasi $B=\gamma(\beta)$ nuqtaga esa egri chiziqning oxirgi nuqtasi deb ataladi. Biz qaralayotgan egri chiziq to'g'rilanuvchi, ya'ni chekli uzunlikka ega bo'lsin deb faraz qilamiz.

Aytaylik, xOy tekisligida biror sodda $\breve{AB}$ egri chiziq yoyi va bu yoyda $f(x,y)$ funksiya berilgan bo'lsin. $\breve{AB}$ egri chiziqni A dan V ga qarab $A_0=A,\ A_1,\ A_2,\ \dots,A_n=B$ nuqtalar yordamida n ta A_kA_{k+1} $(k=\overline{0,n-1})$ yoyga ajratamiz. A_kA_{k+1} yoyning uzunligini ΔS_k va $\lambda=\max\limits_{k=\overline{0,n-1}}\Delta S_k$ deb belgilaymiz. Endi $\forall(\xi_k,\eta_k)\in A_kA_{k+1}$ $(k=\overline{0,n-1})$ nuqtalar olamiz va quyidagi

$$\sigma=\sum_{k=0}^{n-1}f(\xi_k,\eta_k)\cdot\Delta S_k\ ;$$

yig'indini tuzamiz.

Ta'rif. *Agar* $\lim\limits_{\lambda\to 0}\sigma-\exists$ *bo'lib, u chekli I soniga teng bo'lsa va I ning qiymati* $\breve{AB}$ *ning bo'linish usuliga hamda* (ξ_k,η_k) *nuqtalarning tanlanishiga bog'liq bo'lmasa, u holda shu I soniga* $f(x,y)$ *funksiyaning* $\breve{AB}$ *egri chiziq bo'yicha* ***birinchi tur egri chiziqli integrali*** *deb ataladi va u*

$$\int\limits_{\breve{AB}}f(x,y)ds$$

kabi belgilanadi.

Shunday qilib,

$$\int\limits_{\breve{AB}}f(x,y)ds=:\lim_{\lambda\to 0}\sigma=\lim_{\lambda\to 0}\sum_{k=0}^{n-1}f(\xi_k,\eta_k)\cdot\Delta S_k \qquad (22)$$

ekan.

Birinchi tur egri chiziqli integrallar quyidagi xossalarga ega.

1) $\int\limits_{\breve{AB}}f(x,y)ds=\int\limits_{\breve{BA}}f(x,y)ds$.

2) $\breve{AB}=\breve{AC}\cup\breve{CB}\Rightarrow\int\limits_{\breve{AB}}f(x,y)ds=\int\limits_{\breve{AC}}f(x,y)ds+\int\limits_{\breve{CB}}f(x,y)ds$.

3) $\int\limits_{\breve{AB}}cf(x,y)ds=c\cdot\int\limits_{\breve{AB}}f(x,y)ds(c=const)$.

4) $\int\limits_{\breve{AB}}[f(x,y)\pm g(x,y)]ds=\int\limits_{\breve{AB}}f(x,y)ds\pm\int\limits_{\breve{AB}}g(x,y)ds$.

5) Agar $\forall(x,y)\in \overline{AB}$ da $f(x,y)\geq 0$ bo'lsa, u holda

$$\int_{\breve{A}\breve{B}} f(x,y)ds\geq 0.$$

6) $\left|\int_{\breve{A}\breve{B}} f(x,y)ds\right|\leq \int_{\breve{A}\breve{B}}|f(x,y)|ds.$

7) $\exists(c_1,c_2)\in \breve{A}\breve{B}$ nuqta topiladiki, $\int_{\breve{A}\breve{B}} f(x,y)ds=f(c_1,c_2)\cdot S$ bo'ladi.

Izoh. Yuqoridagi xossalarning hammasida $f(x,y)\in C(\breve{A}\breve{B})$ deb faraz qilinadi.

Teorema. Agar $\breve{A}\breve{B}=\{(x,y)\in R^2: x=\varphi(t), y=\psi(t), t\in[\alpha,\beta]\}$ sodda egri chiziq va $f(x,y)\in C(\breve{A}\breve{B})$ bo'lsa,

$$\int_{\breve{A}\breve{B}} f(x,y)ds=\int_{\alpha}^{\beta} f[\varphi(t),\psi(t)]\cdot\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt \qquad (23)$$

bo'ladi.

Bu teoremadan quyidagi muhim natijalar kelib chiqadi.

1-natija. Agar $\breve{A}\breve{B}=\{(x,y)\in R^2: y=y(x), c\leq x\leq b\}$ $(y(a)=A, y(b)=B)$ bo'lib, $y'(x)\in C[\alpha,\beta]$ bo'lsa, u holda

$$\int_{\breve{A}\breve{B}} f(x,y)ds=\int_{a}^{b} f[x,y(x)]\cdot\sqrt{1+[y'(x)]^2}dx \qquad (24)$$

bo'ladi.

2-natija. Agar $\breve{A}\breve{B}=\{(r,\varphi): r=r(\varphi), \varphi_1\leq\varphi\leq\varphi_2\}$ bo'lib, $r'(\varphi)\in C[\varphi_1,\varphi_2]$ bo'lsa, u holda

$$\int_{\breve{A}\breve{B}} f(x,y)ds=\int_{\varphi_1}^{\varphi_2} f(r\cos\varphi, r\sin\varphi)\cdot\sqrt{r^2+[r'(\varphi)]^2}d\varphi \qquad (25)$$

bo'ladi.

Eslatma. Agar 1-tur egri chiziqli integralda $f(x,y)\equiv 1$ desak,

$$\int_{\breve{A}\breve{B}} ds=\lim_{\lambda\to 0}\sum_{k=0}^{n-1}\Delta S_k=\ell$$ bo'ladi, ya'ni

$\ell=\int_{\breve{A}\breve{B}} ds$ (26)- $A\breve{B}$ yoyning uzunligini hisoblash formulasi.

6^0. Ikkinchi tur egri chiziqli integrallar va ularni hisoblash

Tekislikda biror yopiq bo'lmagan sodda $\breve{AB}$ egri chiziq berilgan bo'lib, $f(x,y)$ funksiya shu chiziqda aniqlangan bo'lsin. $\breve{AB}$ egri chiziqni A_k $(k=0,n)$ nuqtalar yordamida n ta $\breve{A_k A}_{k+1}$ $(k=0,n-1)$ bo'lakka ajratamiz va $\forall(\xi_n,\eta_k)\in \breve{A_k A}_{k+1}$ nuqtalar olib, quyidagi yig'indini tuzamiz:

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\cdot\Delta x_k .$$

Bu yerda $\Delta x_k = x_{k+1}-x_k$ – $\breve{A_k A}_{k+1}$ yoyning $0X$ o'qidagi proeksiyasi, $\lambda=\max\limits_{k=0,\ n-1}\Delta S_k$, $\Delta S_k - \breve{A_k A}_{k+1}$ ning uzunligi, deb belgilaymiz.

Ta'rif. *Agar* $\lim\limits_{\lambda\to 0}\sigma = I$ *mavjud va chekli bo'lib, I ning qiymati* $\breve{AB}$ *ning bo'linish usuliga va* (ξ_k,η_k) *nuqtalarning tanlanishiga bog'liq bo'lmasa, u holda I soniga* $f(x,y)$ *funksiyadan* $\breve{AB}$ *egri chiziq bo'yicha olingan* ***ikkinchi tur egri chiziqli integral*** *deb ataladi hamda u*

$$I=\int\limits_{\breve{AB}} f(x,y)dx .$$

kabi belgilanadi.

Shunday qilib,

$$I=\int\limits_{\breve{AB}} f(x,y)dx=\lim_{\lambda\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta x_k \qquad (27)$$

ekan.

Xuddi shunga o'xshash $f(\xi_k,\eta_k)$ larni Δx_k ga emas, Δy_k larga ko'paytirib,

$$I^*=\int\limits_{\breve{AB}} f(x,y)dy=\lim_{\lambda\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta y_k \qquad (28)$$

ni hosil qilamiz.

2-tur egri chiziqli integral ta'rifidan quyidagi xossalar kelib chiqadi.

1) $\int\limits_{\breve{AB}} f(x,y)dx=-\int\limits_{\breve{BA}} f(x,y)dx$ va $\int\limits_{\breve{AB}} f(x,y)dy=-\int\limits_{\breve{BA}} f(x,y)dy$.

2) Agar $\breve{AB}$ yoy 0X o'qiga (0Y o'qiga) perpendikular bo'lgan to'g'ri chiziq kesmasidan iborat bo'lsa, u holda

$$\int\limits_{\breve{AB}} f(x,y)dx=0 \qquad \left(\int\limits_{\breve{AB}} f(x,y)dy=0\right)$$

bo'ladi.

Endi faraz qilaylik, $\breve{AB}$ egri chiziqda 2 ta $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo‘lib,

$$\int\limits_{\breve{AB}} P(x,y)dx \quad \text{va} \quad \int\limits_{\breve{AB}} Q(x,y)dy;$$

2-tur egri chiziqli integrallar mavjud bo‘lsin. Ushbu

$$\int\limits_{\breve{AB}} P(x,y)dx + \int\limits_{\breve{AB}} Q(x,y)dy;$$

yig‘indi **2-tur egri chiziqli integralning umumiy ko‘rinishi** deb ataladi va

$$\int\limits_{\breve{AB}} P(x,y)dx + Q(x,y)dy;$$

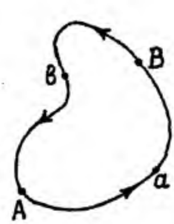
kabi yoziladi. Demak,

$$\int\limits_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int\limits_{\breve{AB}} P(x,y)dx + \int\limits_{\breve{AB}} Q(x,y)dy \qquad (29)$$

Aytaylik, $\breve{AB}$ egri chiziq, sodda yopiq egri chiziq bo‘lsin, ya’ni A va V nuqtalar ustma-ust tushsin. Bu yopiq chiziqni γ deb belgilaymiz. Bu yopiq chiziqda ikkita yo‘nalish bo‘ladi. Ularning birini musbat (soat strelkasiga qarama-qarshi yo‘nalganini), ikkinchisini manfiy yo‘nalish deb qabul qilamiz.

Faraz qilaylik, γ da $f(x,y)$ funksiya berilgan bo‘lsin. Bu γ chiziqda 2 ta $\forall A \neq B$ nuqtalar olamiz. Natijada γ yopiq chiziq 2 ta $A\breve{a}B$ va $B\breve{b}A$ egri chiziqlarga ajraladi (12-chizma).

Agar $\int\limits_{A\breve{a}B} f(x,y)dx + \int\limits_{B\breve{b}A} f(x,y)dx$ integral mavjud bo‘lsa, u $f(x,y)$ funksiyaning γ **yopiq chiziq bo‘yicha 2-tur egri chiziqli integrali** deb ataladi va $\oint\limits_{\gamma} f(x,y)dx$ kabi belgilanadi.



12-chizma.

$\oint_{\gamma} P(x,y)dx + \oint_{\gamma} Q(x,y)dy$ bo'lgan umumiy hol ham xuddi shunga o'xshash ta'riflanadi.

Agar $\breve{A}B$ egri chiziq fazoviy chiziq bo'lib, $f(x,y,z)$ funksiya shu chiziqda aniqlangan bo'lsa, u holda

$$\int_{\breve{A}B} f(x,y,z)dx, \quad \int_{\breve{A}B} f(x,y,z)dy, \quad \int_{\breve{A}B} f(x,y,z)dz, \text{ lar va}$$

$$\int_{\breve{A}B} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int_{\breve{A}B} P(x,y,z)dx + \int_{\breve{A}B} Q(x,y,z)dy + \int_{\breve{A}B} R(x,y,z)dz$$

lar ham yuqoridagidek aniqlanadi.

Endi ikkinchi tur egri chiziqli integrallarni hisoblashni o'rganamiz.

Faraz qilaylik, $\breve{A}B = \{(x,y): x = \varphi(t), y = \psi(t), t \in [\alpha,\beta]\}$ bo'lib, $\varphi(t), \psi(t) \in C[\alpha,\beta], (\varphi(\alpha),\psi(\alpha)) = A, \ (\varphi(\beta),\psi(\beta)) = B$ bo'lsin, hamda t parametr α dan β ga qarab o'zgarganda $(x,y) = (\varphi(t),\psi(t))$ nuqta A dan V ga qarab o'zgarsin.

1-teorema. *Agar* $\varphi'(t) \in C[\alpha,\beta]$ *bo'lib,* $f(x,y) \in C(\breve{A}B)$ *bo'lsa, u holda*

$$\int_{\breve{A}B} f(x,y)dx = \int_{\alpha}^{\beta} f[\varphi(t),\psi(t)]\varphi'(t)dt \tag{30}$$

bo'ladi.

2-teorema. *Agar* $\psi'(t) \in C[\alpha,\beta]$ *bo'lib,* $f(x,y) \in C(\breve{A}B)$ *bo'lsa, u holda*

$$\int_{\breve{A}B} f(x,y)dy = \int_{\alpha}^{\beta} f[\varphi(t),\psi(t)]\psi'(t)dt \tag{31}$$

bo'ladi.

1-natija. Agar $\varphi'(t), \psi'(t) \in C[\alpha,\beta]$ *bo'lib,*

$P(x,y), Q(x,y) \in C(\breve{A}B)$ *bo'lsa, u holda*

$$\int_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int_{\alpha}^{\beta} [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t))]dt \tag{32}$$

bo'ladi.

2-natija. Agar $\breve{A}B=\{(x,y): y=y(x), a\le x\le b\}$, $y'(x)\in C[a,b]$ bo‘lib, $P(x,y), Q(x,y)\in C(\breve{A}B)$ bo‘lsa, u holda

$$\int_{\breve{A}B} P(x,y)dx+Q(x,y)dy=\int_a^b\left[P(x,y(x))+Q(x,y(x))\cdot y'(x)\right]dx. \tag{33}$$

bo‘ladi.

Misol. Agar $\breve{O}\breve{A}$ egri chiziq $O(0,0)$ va $A(1,1)$ nuqtalarni tutashtiruvchi

a) to‘g‘ri chiziq kesmasi.

b) ORA siniq chiziq, $P=(1,0)$ nuqta;

d) OQA siniq chiziq, $Q=(0,1)$ nuqta bo‘lsa,

$$I=\int_{\breve{O}A}\left(x-y^2\right)dx+2xydy\ ;$$

hisoblansin.

◁ **a)** $\breve{O}A: y=x, 0\le x\le 1 \Rightarrow I=\int_0^1\left[\left(x-x^2\right)+2x^2\right]dx=\frac{5}{6}$

b) $I=\int_{\breve{O}PA}\left(x-y^2\right)dx+2xydy=\int_{\breve{O}P}\left(x-y^2\right)dx+2xydy+\int_{\breve{P}A}\left(x-y^2\right)dx+2xydy=$

$$=\left(\begin{pmatrix} OP: y=0, 0\le x\le 1\Rightarrow dy=0\\ PA: x=1, 0\le y\le 1\Rightarrow dx=0\end{pmatrix}\right)=\int_0^1 xdx+\int_0^1 2ydy=\frac{3}{2}.$$

d) $I=\int_{OQA}\left(x-y^2\right)dx+2xydy=\int_{\breve{O}Q}\left(x-y^2\right)dx+2xydy+\int_{\breve{Q}A}\left(x-y^2\right)dx+2xydy=$

$$=\left(\begin{pmatrix} OQ: x=0, 0\le y\le 1\Rightarrow dx=0\\ QA: y=1, 0\le x\le 1\Rightarrow dy=0\end{pmatrix}\right)=\int_0^1 0dy+\int_0^1(x-1)dx=-\frac{1}{2}. \ ▷$$

Izoh. Bu misoldan ko‘rinib turibdiki, 2-tur egri chiziqli integralning qiymati umuman olganda, A va V nuqtalarni tutashtiruvchi integrallash yo‘liga bog‘liq ekan.

Qanday shartlar bajarilganda uning qiymati integrallash yo‘liga bog‘liq bo‘lmaydi, degan savolga keyingi punktda javob beramiz.

7⁰. Grin formulasi va uning ba’zi bir tatbiqlari

a) Grin formulasi.

1-teorema. (Grin). *$D\subset R^2$ soha berilgan bo‘lib, uning chegarasi ∂D bo‘lakli-silliq chiziqdan iborat bo‘lsin. Agar $P(x,y)$ va $Q(x,y)$*

funksiyalar $\bar{D}$ *da berilgan va* $P(x,y)$, $Q(x,y)$,

$\frac{\partial P(x,y)}{\partial x}, \frac{\partial Q(x,y)}{\partial y} \in C(\bar{D})$ *bo'lsa, u holda*

$$\oint_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \qquad (34)$$

tenglik o'rinli bo'ladi.

(34)-formulaga **Grin formulasi** deyiladi.

Grin formulasidan D sohaning yuzasini hisoblash uchun ushbu

$$S = -\oint_{\partial D} ydx, \qquad (35)$$

$$S = \oint_{\partial D} xdy, \text{ va} \qquad (36)$$

$$S = \frac{1}{2}\oint_{\partial D} xdy - ydx, \qquad (37)$$

formulalar kelib chiqadi.

2-teorema. *Agar* $P(x,y)$ *va* $Q(x,y)$ *funksiyalar D sohada Grin teoremasining shartlarini bajarsa, unda quyidagi 4 ta shart bir-biriga ekvivalent bo'ladi.*

1) D da $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (38) tenglik bajariladi.

2) D sohadagi $\forall$ yopiq kontur γ uchun $\oint_\gamma Pdx + Qdy = 0$.

3) $\forall A, B \in D$ nuqtalar va bu nuqtalarni tutashtiruvchi $\breve{AB}$ yoy uchun

$$\int_{\breve{AB}} Pdx + Qdy;$$

integralning qiymati integrallash yo'liga bog'liq emas;

4) $P(x,y)dx + Q(x,y)dy$ ifoda to'liq differensial bo'ladi, ya'ni D sohada shunday $u(x,y)$ funksiya topiladiki $du = Pdx + Qdy$ tenglik bajariladi va unda

$$\int_{\breve{AB}} Pdx + Qdy = u(B) - u(A) \text{ bo'ladi.}$$

Agar $du = Pdx + Qdy$ bo‘lsa, unda $u(x,y)$ funksiya ushbu

$$u(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy + c \tag{39}$$

formula yordamida topiladi. Bu yerda $(x_0,y_0) \in D$-ixtiyoriy nuqta.

Misol. Agar γ chiziq koordinata boshidan o‘tmaydigan va yo‘nalishi musbat bo‘lgan $\forall$ yopiq chiziq bo‘lsa,

$$I = \oint_{\gamma} \frac{xdy - ydx}{x^2 + y^2};$$

integralni hisoblang.

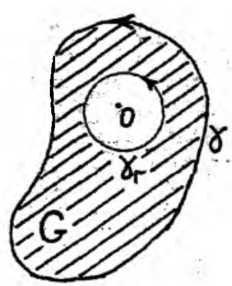
$\triangleleft$ γ chiziq bilan chegaralangan sohani D deb belgilaymiz.

1-hol. $0 \notin D$ bo‘lsin.

$$P = -\frac{y}{x^2+y^2}, Q = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} \Rightarrow$$

Grin formulasiga ko‘ra $I = 0$.

2-hol. $0 \in D$ bo‘lsin. Bu holda Grin formulasidan foydalana olmaymiz, chunki, $P(x,y)$ va $Q(x,y)$ funksiyalar $0(0,0)$ nuqtada aniqlanmagan.



13-chizma.

D sohaning ichida yotuvchi $\forall\ \gamma_r = \{(x,y): x^2 + y^2 = r^2\}$ *aylana olamiz. Endi G soha sifatida* γ *va* γ_r *chiziqlari bilan chegaralangan sohani olamiz. G da* $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ va $0 \notin G$. *Unda Grin formulasiga ko‘ra*

$$\oint_{\partial G} Pdx + Qdy = 0 \Rightarrow 0 = \oint_{\gamma} Pdx + Qdy + \oint_{\gamma_r^-} Pdx + Qdy = \oint_{\gamma} Pdx + Qdy - \oint_{\gamma_r} Pdx + Qdy \Rightarrow$$

$$I=\oint_{\gamma_r}\frac{xdy-ydx}{x^2+y^2}=\left(\begin{pmatrix}x=r\cos t, 0\le t\le 2\pi, dx=-r\sin tdt\\ y=r\sin t, dy=r\cos tdt\end{pmatrix}\right)=\int_0^{2\pi}dt=2\pi. \triangleright$$

8⁰. Birinchi tur sirt integrallari va ularni hisoblash

Birinchi tur egri chiziqli integrallar oddiy aniq integrallarning qanday umumlashtirilishi bo'lsa, birinchi tur sirt integrallari ham ikki karrali integrallarining shunday tabiiy umumlashtirilishidir.

Bizga bo'lakli silliq kontur bilan chegaralangan ikki tomonli silliq (yoki bo'lakli silliq) $(S)\subset R^3$ sirt berilgan bo'lib, $f(x,y,z)$ funksiya shu sirtda aniqlangan bo'lsin. (S) sirtni $\forall$ tarzda o'tkazilgan egri chiziqlar to'ri yordamida $(S_1),(S_2),...,(S_n)$ qismlarga ajratamiz. (S_k) ning yuzasini S_k deb belgilaymiz $(k=\overline{1,n})$. Har bir (S_k) da $\forall(\xi_k,\eta_k,\zeta_k)$ nuqta olib

$$\sigma=\sum_{k=1}^{n}f(\xi_k,\eta_k,\zeta_k)S_k;$$

integral yig'indini tuzamiz va $\lambda=\max\limits_{k=\overline{1,n}} diam(S_k)$ deb belgilaymiz.

Ta'rif. Agar $\lim\limits_{\lambda\to 0}\sigma=I$ mavjud va chekli bo'lib, I ning qiymati (S) sirtning bo'linish usuli hamda (ξ_k,η_k,ζ_k) *nuqtalarning* tanlanishiga bog'liq bo'lmasa, u holda I ga $f(x,y,z)$ funksiyadan (S) sirt bo'yicha olingan **1-tur sirt integrali** deyiladi va

$$\iint_{(S)}f(x,y,z)ds;$$

kabi belgilanadi.

Teorema. *Agar sirt ushbu* $(S)=\{(x,y,z)\in R^3: z=z(x,y),(x,y)\in D\}$ *ko'rinishda berilgan bo'lib,* $z(x,y),z'_x(x,y),z'_y(x,y)\in C(D)$ *va* $f(x,y,z)\in C[(S)]$ *bo'lsa, u holda*

$$\iint_{(S)}f(x,y,z)dS=\iint_{D}f[x,y,z(x,y)]\sqrt{1+[z'_x(x,y)]^2+[z'_y(x,y)]^2}dxdy \quad (40)$$

bo'ladi.

9^0. Ikkinchi tur sirt integrallari va ularni hisoblash

Bizga ikki tomonli silliq (yoki bo‘lakli silliq) $(S)\subset R^3$ sirt berilgan bo‘lib, $f(x,y,z)$ funksiya shu sirtda aniqlangan bo‘lsin. (S) sirtni $\forall$ tarzda o‘tkazilgan egri chiziqlar to‘ri yordamida $(S_1),(S_2),...,(S_n)$ qismlarga ajratamiz. (S_k) $(k=\overline{1,n})$ ning 0XY tekislikdagi proeksiyasini D_k, D_k ning yuzasini esa S_{D_k} deb belgilaymiz. Har bir (S_k) da $\forall$ (ξ_k,η_k,ζ_k) nuqta olib quyidagi

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)S_{D_k};$$

yig‘indini tuzamiz va $\lambda=\max\limits_{k=\overline{1,n}} diam(S_k)$ deb belgilaymiz

Ta’rif. *Agar* $\lim\limits_{\lambda\to 0}\sigma=I$ *mavjud va chekli bo‘lib, I ning qiymati (S) sirtning bo‘linish usuliga hamda* (ξ_k,η_k,ζ_k) *nuqtalarning tanlanishiga bog‘liq bo‘lmasa, u holda I ga* $f(x,y,z)$ *funksiyadan (S) sirtning tanlangan tomoni bo‘yicha olingan* ***ikkinchi tur sirt integrali*** *deyiladi va*

$$\iint\limits_{(S)} f(x,y,z)\,dxdy;$$

kabi belgilanadi.

Shunday qilib,

$$I=\iint\limits_{(S)} f(x,y,z)\,dxdy=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)S_{Д_k} \qquad (41)$$

Shuni aytib o‘tish kerakki, funksiyadan (S) sirtning bir tomoni bo‘yicha olingan ikkinchi tur sirt integrali, funksiyadan shu sirtning ikkinchi tomoni bo‘yicha olingan ikkinchi tur sirt integralidan faqat ishorasi bilangina farq qiladi.

Ushbu $\iint\limits_{(S)} f(x,y,z)\,dydz$ va $\iint\limits_{(S)} f(x,y,z)\,dzdx$ ikkinchi tur sirt integrallari ham yuqoridagidek ta’riflanadi.

Ikkinchi tur sirt integralining umumiy ko‘rinishini keltiramiz. Faraz qilaylik, (S) sirtda $P(x,y,z)$, $Q(x,y,z)$ va $R(x,y,z)$ funksiyalar berilgan bo‘lib,

Ushbu

$$\iint\limits_{(S)} P(x,y,z)\,dxdy, \quad \iint\limits_{(S)} Q(x,y,z)\,dydz, \quad \iint\limits_{(S)} R(x,y,z)\,dzdx;$$

integrallar mavjud bo'lsa, u holda ularning **yig'indisi 2-tur sirt integralining umumiy ko'rinishi** deb ataladi va

$$\iint_{(S)} P(x,y,z)\,dxdy + Q(x,y,z)\,dydz + R(x,y,z)\,dzdx\,;$$

kabi belgilanadi.

Endi R^3 fazoda biror (V) jism berilgan bo'lsin. Bu jismni o'rab turgan yopiq sirt silliq sirt bo'lib, uni (S) deylik. $f(x,y,z)$ funksiya (S) da berilgan bo'lsin. OXY tekislikka parallel bo'lgan tekislik bilan (V) ni 2 qismga ajratamiz: $(V)=(V_1)\cup(V_2)$. Natijada, uni o'rab turgan (S) sirt (S_1) va (S_2) sirtlarga ajraladi. Ushbu

$$\iint_{(S_1)} f(x,y,z)\,dxdy + \iint_{(S_2)} f(x,y,z)\,dxdy \tag{42}$$

integral (agar u mavjud bo'lsa) $f(x,y,z)$ funksiyaning yopiq sirt bo'yicha 2-tur sirt integrali deb ataladi va

$$\oiint_{(S)} f(x,y,z)\,dxdy\,;$$

kabi belgilanadi. Bu yerda (42)-munosabatdagi birinchi integral (S_1) sirtning ustki tomoni, ikkinchi integral esa (S_2) sirtning pastki tomoni bo'yicha olingan. Xuddi shunga o'xshash

$$\oiint_{(S)} f(x,y,z)\,dydz\,, \qquad \oiint_{(S)} f(x,y,z)\,dzdx\,;$$

hamda umumiy holda

$$\oiint_{(S)} P(x,y,z)\,dxdy + Q(x,y,z)\,dydz + R(x,y,z)\,dzdx\,;$$

integrallar aniqlanadi.

Teorema. *Agar sirt ushbu*

$$(S)=\{(x,y,z)\in R^3 : z=z(x,y),(x,y)\in D\}$$

ko'rinishda berilgan bo'lib, $z(x,y)$, $z'_x(x,y)$, $z'_y(x,y)\in C(D)$ *va* $f(x,y,z)\in C[(S)]$ *bo'lsa, u holda*

$$\iint_{(S)} f(x,y,z)\,dxdy = \iint_{D} f[x,y,z(x,y)]\,dxdy \tag{43}$$

bo'ladi.

Izoh. Agar (S) sirtning pastki tomoni qaralsa, unda barcha S_{D_k} lar manfiy bo'lib,

$$\iint\limits_{(S)} f(x,y,z)dxdy = -\iint\limits_{D} f[x,y,z(x,y)]dxdy;$$

bo'ladi.

Natija. Agar (S) sirt yasovchilari OZ o'qiga parallel bo'lgan silindrik sirt bo'lsa, unda

$$\iint\limits_{(S)} f(x,y,z)dxdy = 0;$$

bo'ladi.

Demak ikkinchi tur sirt integrallari ikki karrali Riman integrallariga keltirilib hisoblanar ekan.

Agar (S)-ikki tomonli silliq sirt bo'lib, $P(x,y,z)$, $Q(x,y,z)$ $R(x,y,z) \in C[(S)]$ bo'lsa va (S) sirt normalining yo'naltiruvchi kosinusilarini $\cos\alpha$, $\cos\beta$, $\cos\gamma$ desak, u holda 1 va 2-tur sirt integrallari orasida quyidagi munosabat o'rinli.

$$\iint\limits_{(S)} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy =$$

$$= \iint\limits_{(S)} [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]ds \qquad (44)$$

Izoh. Agar (S) sirt $z = z(x,y)$ tenglama yordamida berilgan bo'lsa, sirtning (x_0, y_0, z_0) nuqtadagi normalining *OX, OY, OZ* o'qlarining musbat yo'nalishlari bilan tashkil qilgan burchaklarini mos ravishda α, β, γ orqali belgilasak,

$$\begin{cases} \cos\alpha = \dfrac{-z_x'}{\pm\sqrt{1+(z_x')^2+(z_y')^2}}, \\ \cos\beta = \dfrac{-z_y'}{\pm\sqrt{1+(z_x')^2+(z_y')^2}}, \\ \cos\gamma = \dfrac{1}{\pm\sqrt{1+(z_x')^2+(z_y')^2}} \end{cases} \qquad (45)$$

bo'la.di va ular **normalning yo'naltiruvchi kosinuslari** deyiladi.

Ildiz oldida ma'lum bir ishorani tanlab olish bilan biz sirtning aniq bir tomonini tanlab olgan bo'lamiz. Masalan, ildiz uchun musbat ishorani olsak, $\cos\gamma > 0$, ya'ni normal OZ o'qi bilan γ o'tkir burchak tashkil etadi va bu holda (S) sirtning "yuqori" tomonini tanlab olgan bo'lamiz.

10⁰. Stoks va Gauss-Ostrogradskiy formulalari

$(S)=\{(x,y,z)\in R^3 : z=z(x,y),(x,y)\in D\}$ bo'lib, $\partial(S)$ – bo'lakli silliq egri chiziq va $\partial(S)$ – ning OXY tekisligiga proyeksiyasi ∂D bo'lsin.

Faraz qilaylik, (S) sirtda uzluksiz $P(x,y,z)$, $Q(x,y,z)$ $R(x,y,z)$ funksiyalar aniqlangan bo'lib, bu funksiyalarning barcha birinchi tartibli xususiy hosilalari (S) sirtda uzluksiz bo'lsin.

1-teorema. (Stoks). *Agar yuqoridagi shartlar bajarilsa, u holda ushbu*

$$\int_{\partial(S)} P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=\iint_{(S)}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy+$$

$$+\left[\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right]dydz+\left[\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right]dzdx. \tag{46}$$

Stoks formulasi *o'rinli bo'ladi.*

Shunday qilib, Stoks formulasi (S) sirt bo'yicha olingan 2-tur sirt integrali bilan shu sirtning chegarasi bo'yicha olingan egri chiziqli integralni bog'lovchi formuladir.

Endi Ostrogradskiy formulasini keltiramiz. R^3 fazoda pastdan $z=\varphi_1(x,y)$ tenglama bilan aniqlangan silliq (S_1) sirt bilan, yuqoridan $z=\varphi_2(x,y)$ tenglama yordamida aniqlangan (S_2) sirt bilan, yon tomondan esa yasovchilari OZ o'qiga parallel bo'lgan silindrik (S_3) sirt bilan chegaralangan (V) jismni qaraylik. Bu jismning OXY tekisligidagi proeksiyasini D deb belgilaymiz. Faraz qilaylik, (V) da uzluksiz $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar berilgan bo'lib, $\frac{\partial P}{\partial x},\frac{\partial Q}{\partial y},\frac{\partial R}{\partial z}\in C[(V)]$ shartlar bajarilsin.

2-teorema. (Ostrogradskiy). *Agar yuqoridagi shartlar bajarilsa, u holda ushbu*

$$\iiint_{(V)}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz=\iint_{(S)} Pdydz+Qdzdx+Rdxdy \tag{47}$$

Gauss-Ostrogradskiy formulasi *o'rinli bo'ladi.*

11^0. Maydonlar nazariyasi elementlari

Uch o'lchovli Yevklid fazosi R^3 ni olib, undagi nuqtalarni (x, y, z), koordinata o'qlari bo'ylab yo'nalgan birlik vektorlarni esa e_1, e_2, e_3 kabi belgilaymiz. Aytaylik, $D \subset R^3$ sohadagi har bir (x, y, z) nuqtaga $A(x, y, z)$ vektor mos qo'yilgan bo'lib, tanlangan koordinatalar sistemasida u $A_1(x, y, z)$, $A_2(x, y, z)$, $A_3(x, y, z)$ ko'rinishga ega bo'lsin. U holda D sohada vektor funksiya aniqlangan yoki D da vektorlar maydoni berilgan deyiladi. Agar har bir $A_k(x, y, z)$, $k=1, 2, 3$, funksiya uzluksiz, differensiallanuvchi va hakozo bo'lsa, unda A vektor maydon uzluksiz, differensiallanuvchi va hokazo deb ataladi. Agar D sohada $U(x, y, z)$ funksiya aniqlangan bo'lsa, u holda D da U skalyar maydon berilgan deyiladi. Tanlangan koordinatalar sistemasida qaralayotgan skalyar va vektorlar maydoni kerakli darajada silliq bo'lsin deb faraz qilamiz.

Ushbu

$$gradU := \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = e_1\frac{\partial U}{\partial x} + e_2\frac{\partial U}{\partial y} + e_3\frac{\partial U}{\partial z} : U \to A$$

operatorni (gradiyent) aniqlaymiz. U bilan bir qatorda A vektorni U skalyar maydonga akslantiruvchi

$$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} : A \to U$$

operatorni (divergensiya) qaraymiz. Vektor maydon vektor maydonga quyidagi

$$rotA = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} : A \to B$$

formula yordamida aniqlanadigan **rotor** operatori yordamida akslanadi.

Agar simvolik nabla differensial operatorini **(Gamilton operatori)**

$$\nabla = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z};$$

tenglik yordamida aniqlasak, u holda

$$gradU = \nabla U,$$

$$rotA = \nabla \times A,$$

$$divA = \nabla \cdot A$$

bo‘ladi.

Kiritilgan operatorlardan foydalanib, quyidagi tengliklarning o‘rinli bo‘lishini ko‘rsatish qiyin emas:

1) $rotgradU = \nabla \times \nabla U = 0$,
2) $divrotA = \nabla \cdot (\nabla \times A) = 0$,
3) $graddivA = \nabla(\nabla \cdot A)$,
4) $rotrotA = \nabla \times (\nabla \times A)$,
5) $divgradU = \nabla \cdot \nabla U$

(44)-formuladan foydalangan holda Stoks formulasini quyidagi ko‘rinishda yozish mumkin:

$$\int\limits_{\partial(S)} P(x,y,z)dx + Q(x;y,z)dy + R(x,y,z) =$$

$$= \iint\limits_{(S)} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS.$$

Agar bu tenglikdagi P, Q, R funksiyalar o‘rniga A vektor maydonning komponentalarini olsak va $dS = (dx, dy, dz)$ deb belgilasak, tenglikning chap tomonidagi integralostidagi funksiyani $A \cdot dS$ deb yozish mumkin. Agar $n = (\cos\alpha,\ \cos\beta,\ \cos\gamma)$-birlik normal vektor bo‘lsa, tenglikning o‘ng tomonidagi integral ostidagi funksiyani $rotA \cdot n$ deb yozish mumkin bo‘lib, Stoks formulasining vektor ko‘rinishi quyidagicha bo‘ladi:

$$\oint\limits_{\partial(S)} A \cdot dS = \iint\limits_{(S)} n \cdot rotAdS \qquad (48)$$

(48)-tenglik maydonlar nazariyasi tilida quyidagicha aytiladi: A **vektor maydonnning sirtning chegarasi bo‘yicha olingan sirkulyatsiyasi** $rotA$ **maydonning** (S) **sirt bo‘yicha olingan oqimiga teng.**

(44) formuladan foydalanib, Gauss-Ostrogradskiy formulasi vektor ko‘rinishida quyidagicha yoziladi:

$$\iint\limits_{\partial V} A \cdot ndS = \iiint\limits_{V} divAdV, \qquad (49)$$

bu yerda $dV = dxdydz$ hajm elementi.

(48) va (49) formulalar vektorlar maydonidagi rotor va divergensiya operatorlarining invariant ekanligini ko'rsatish imkoniyatini beradi.

1-ta'rif. *Agar A vektor maydon uchun shunday skalyar* U *maydon topilib,* $gradU = A$ *tenglik bajarilsa, unda A vektor maydon* ***potensial maydon*** *deyiladi.* U *funksiya esa A maydonning* ***skalyar potensiali*** *deb ataladi.*

2-ta'rif. *Agar A vektor maydon uchun shunday B vektor maydon topilib,* $rotB = A$ *tenglik bajarilsa, u holda A maydon* ***solenoidal maydon*** *deyiladi. V vektor maydon esa A maydonning* ***vektor potensiali*** *deb ataladi.*

Punktning oxirigacha biz maydonlar uch o'lchovli fazoda qaralayapti, deb faraz qilamiz.

1-teorema. *A maydon potensial maydon bo'lishi uchun* $rotA = 0$ *bo'lishi zarur va yetarli.*

2-teorema. *A maydon solenoidal bo'lishi uchun* $divA = 0$ *bo'lishi zarur va yetarli.*

(48)-Stoks formulasidan va 1-teoremadan quyidagi tasdiq kelib chiqadi: **Agar A potensial maydon bo'lsa, u holda maydonning yopiq egri chiziq bo'yicha olingan sirkulyatsiyasi 0 ga teng bo'ladi.**

(49)-Gauss-Ostrogradskiy va 2-teoremadan quyidagi tasdiq kelib chiqadi: **agar A solenoidal maydon bo'lsa, u holda maydonning biror jismni o'rovchi yopiq sirt bo'yicha olingan oqimi 0 ga teng bo'ladi.**

Shuni ta'kidlash lozimki, ixtiyoriy vektor maydonni potensial va solenoidal maydonlarning yig'indisi ko'rinishida ifodalash mumkin.

11°. Furye qatorlari

1-ta'rif. $f(x)$ *funksiya* $[-\pi;\pi]$ *kesmada absolut integrallanuvchi bo'lsin. Koeffisientlari*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0,\ 1,\ 2,...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1,\ 2,...$$

formulalar yordamida aniqlangan ushbu

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{50}$$

trigonometrik qator $f(x)$ *funksiyaning Furye qatori,* a_n, b_n *sonlar esa-Furye koeffisientlari deyiladi.*

Absolut integrallanuvchi funksiyaning Furye koeffisientlari $n \to \infty$ da 0 ga intiladi. Agar funksiya juft bo'lsa, Furye qatori faqat kosinuslarni, toq bo'lsa faqat sinuslarni o'z ichida saqlaydi.

1-teorema. (Rimanning lokallashtirish prinsipi). *$f(x)$ funksiya Furye qatorining x_0 nuqtada yaqinlashishi, ixtiyoriy kichik $\delta > 0$ sonl uchun $f(x)$ funksiyaning $[x_0 - \delta; x_0 + \delta]$ kesmadagi qiymatlarigagina bog'liq bo'lib, bu kesmadan tashqaridagi qiymatlariga bog'liq emas.*

2-teorema. Agar $f(x)$ funksiya $[-\pi; \pi]$ kesmada bo'lakli-uzluksiz bo'lib, har x nuqtada chekli bir tomonli

$$f'_+(x) = \lim_{\Delta x \to +0} \frac{f(x+\Delta x) - f(x+0)}{\Delta x},$$

$$f'_-(x) = \lim_{\Delta x \to -0} \frac{f(x+\Delta x) - f(x-0)}{\Delta x};$$

hosilalarga ega bo'lsa, u holda $f(x)$ funksiyaning Furye qatori har bir x nuqtada yaqinlashadi va uning yig'indisi $\dfrac{f(x+0) - f(x-0)}{2}$ ga teng bo'ladi. Xususan, funksiya uzluksiz bo'lgan nuqtada Furye qatori funksiyaning shu nuqtadagi qiymatiga yaqinlashadi.

Yaqinlashuvchi Furye qatorining yig'indisi davri 2π ga teng bo'lgan davriy funksiya bo'ladi.

3-teorema. *Agar $f(x)$ funksiya $[-\pi; \pi]$ kesmada kvadrati bilan integrallanuvchi funksiya bo'lsa, u holda quyidagi **Bessel tengsizligi** o'rinli:*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx. \qquad (51)$$

*Agar funksiya $[-\pi; \pi]$ da uzluksiz va $f(-\pi) = f(\pi)$ bo'lsa, unda ushbu **Parseval tengligi** o'rinli:*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx. \qquad (52)$$

Agar $f(x)$ funksiya $[a, b]$ kesmada berilgan bo'lib, ma'lum shartlarni qanoatlantirsa, unda uni umumiyroq ko'rinishdagi trigonometrik qatorga yoyish mumkin. Buning uchun $t = -\pi + 2\pi \dfrac{x-a}{b-a}$ akslantirish yordamida $[a, b]$ kesmani $[-\pi; \pi]$ kesmaga akslantiramiz. Natijada,

$$f(x) \mapsto \varphi(t) = f\left(a + \frac{t+\pi}{2\pi}(b-a)\right)$$

bo'lib, $\varphi(t)$ funksiya $[-\pi; \pi]$ kesmada aniqlangan. $\varphi(t)$ funksiya $[-\pi; \pi]$ kesmada Furye qatoriga yoyiladi va t o'zgaruvchidan x o'zgaruvchiga qaytsak, $f(x)$ funksiyaning $[a, b]$ kesmadagi Furye qatorini hosil qilamiz. Masalan, $f(x)$ funksiya $[-l; l]$ kesmada 2-teoremaning shartlarini qanoatlantirsa va u shu kesmada uzluksiz bo'lsa, u holda

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right);$$

tenglik o'rinli bo'lib,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n = 0,\ 1,\ 2,\ldots,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1,\ 2,\ldots,$$

bo'ladi.

Agar $f(x)$ funksiya $[0; 2l]$ oraliqda berilgan holda ham yuqoridagi tengliklar o'rinli bo'ladi, faqat koeffitsientlarni hisoblashda integrallarni $[0; 2l]$ oraliq bo'yicha olish kerak.

Nazorat savollari

1. Ikki karrali integralning ta'rifi.
2. Darbuning yuqori va quyi yig'indilari hamda ularning xossalari.
3. Lebeg teoremasi.
4. Ikki karrali integralning asosiy xossalari.
5. O'rta qiymat haqidagi teoremalar.
6. Ikki karrali integrallarni hisoblash.
7. Ikki karrali integrallarda o'zgaruvchilarni almashtirish.
8. Silindrik koordinatalar sistemasi.
9. Sferik koordinatalar sistemasi.
10. Ikki karrali integral yordamida hajm hisoblash.
11. Tekis shaklning yuzasini hisoblash.
12. Sitr yuzasini hisoblash.
13. Ikki karrali integrallarning mexanika masalalariga tatbiqlari.
14. 1-tur egri chiziqli integral tushunchasi.
15. 1-tur egri chiziqli integrallarning xossalari.
16. 1-tur egri chiziqli integrallarni hisoblash.

17. 2-tur egri chiziqli integral tushunchasi.
18. 2-tur egri chiziqli integrallarning xossalari.
19. 2-tur egri chiziqli integrallarni hisoblash.
20. Grin formulasi.
21. Grin formulasining tatbiqlari.
22. 1-tur sirt integrali tushunchasi.
23. 1-tur sirt integralini hisoblash.
24. 2-tur sirt integrali tushunchasi.
25. 2-tur sirt integralini hisoblash.
26. Stoks formulasi.
27. Gauss-Ostrogradskiy formulasi.
28. Maydonlar nazariyasi elementlari.
29. Furye qatorining ta'rifi.
30. Rimanning lokallashtirish prinsipi.
31. Furye qatorining yaqinlashishi.
32. Bessel tengsizligi.
33. Parseval tengligi.

-B-

Mustaqil echish uchun misol va masalalar

1-masala. Berilgan egri chiziqlar bilan chegaralangan D soha uchun $\iint\limits_D f(x,y)dxdy$ ikki karrali integral takroriy integralga keltirilsin va integrallash chegaralari ikki xil tartibda qo'yilsin.

1.1 $y=0,\ y=3,\ y=x,\ y=x-6.$ **1.2** $y=1,\ 2y=x,\ 2y=8-x,\ y=0.$

1.3 $y=x, y=x+3, y=2x, y=2x-3.$ **1.4** $y=x^2,\ x-y+2=0.$

1.5 $x^2+y^2\ge 2a^2,\ x^2+y^2\le 2ax.$ **1.6** $y=2x-x^2,\ y=-x.$

1.7 $y=x^2-4x,\ y=x.$ **1.8** $xy=4,\ y\ge\frac{1}{2}x^2,\ y\le 6.$

1.9 $y\le 9-x^2,\ y\ge 2x^2.$ **1.10** $y=x,\ y=4x,\ xy\ge 4,\ y\le 8.$

1.11 $y=x,\ y=4x,\ xy\ge 4,\ y\le 6.$

1.12 $y=\sqrt{2ax},\ x^2+y^2\ge 2ax,\ x=0,\ x=2a,\ y=0.$

1.13 $y=x^2-4x,\ 2x-y=5.$ **1.14** $y=\frac{1}{2}x^2,\ y\sqrt{3-x^2},\ 0\le x\le 1.$

1.15 $xy=9,\ x+y=10,\ 1\le y\le 3.$

1.16 $y^2+8x=16,\ y^2-24x=48.$

1.17 $y\ge x^2+4x, y=x+4.$

1.18 $y^2\ge x^2-4x,\ y\le x,\ x\ge 1.$

1.19 $y^2-3x=4,\ y^2+4x=11.$

1.20 $y^2\le 6+3x,\ y^2\le 8-4x,\ |y|\le\sqrt{2}.$

1.21 $y\ge x^2+2x,\ y=x+2.$

2-masala. Integrallash tartibini o'zgartiring.

2.1 $\int\limits_{-2}^{-1}dy\int\limits_{-\sqrt{2+y}}^{0}fdx+\int\limits_{-1}^{0}dy\int\limits_{-\sqrt{-y}}^{0}fdx.$

2.2 $\int\limits_{0}^{1}dy\int\limits_{-\sqrt{y}}^{0}fdx+\int\limits_{1}^{\sqrt{2}}dy\int\limits_{-\sqrt{2-y^2}}^{0}fdx.$

2.3 $\int\limits_{0}^{1}dy\int\limits_{0}^{y}fdx+\int\limits_{1}^{\sqrt{2}}dy\int\limits_{0}^{\sqrt{2-y^2}}fdx.$

2.4 $\int\limits_{0}^{1}dy\int\limits_{0}^{\sqrt{y}}fdx+\int\limits_{1}^{2}dy\int\limits_{0}^{\sqrt{2-y}}fdx.$

2.5 $\int\limits_{-\sqrt{2}}^{-1}dx\int\limits_{-\sqrt{2-x^2}}^{0}fdy+\int\limits_{-1}^{0}dx\int\limits_{x}^{0}fdy.$

2.6 $\int\limits_{0}^{1/\sqrt{2}}dy\int\limits_{0}^{\arcsin y}fdx+\int\limits_{1/\sqrt{2}}^{1}dy\int\limits_{0}^{\arccos y}fdx.$

2.7 $\int\limits_{-2}^{-1}dy\int\limits_{0}^{\sqrt{2+y}}fdx+\int\limits_{-1}^{0}dy\int\limits_{0}^{\sqrt{-y}}fdx.$

2.8 $\int\limits_{0}^{1}dy\int\limits_{-\sqrt{y}}^{0}fdx+\int\limits_{1}^{e-\ln y}dy\int\limits_{-1}^{-\ln y}fdx.$

2.9 $\int\limits_{\sqrt{2}}^{-1}dx\int\limits_{0}^{\sqrt{2-x^2}}fdy+\int\limits_{-1}^{0}dx\int\limits_{0}^{x^2}fdy.$

2.10 $\int\limits_{-2}^{-\sqrt{3}}dx\int\limits_{-\sqrt{4-x^2}}^{0}fdy+\int\limits_{-\sqrt{3}}^{0}dx\int\limits_{\sqrt{4-x^2}-2}^{0}fdy.$

2.11 $\int\limits_{0}^{1}dx\int\limits_{1-x^2}^{1}fdy+\int\limits_{1}^{e}dx\int\limits_{\ln x}^{1}fdy.$

2.12 $\int\limits_{0}^{1}dy\int\limits_{0}^{\sqrt[3]{y}}fdx+\int\limits_{1}^{2}dy\int\limits_{0}^{2-y}fdx.$

2.13 $\int\limits_{0}^{\pi/4}dy\int\limits_{0}^{\sin y}fdx+\int\limits_{\pi/4}^{\pi/2}dy\int\limits_{0}^{\cos y}fdx.$

2.14 $\int\limits_{-2}^{-1}dx\int\limits_{-(2+x)}^{0}fdy+\int\limits_{-1}^{0}dx\int\limits_{\sqrt[3]{x}}^{0}fdy.$

2.15 $\int\limits_{0}^{1}dy\int\limits_{0}^{\sqrt{y}}fdx+\int\limits_{1}^{e}dy\int\limits_{\ln y}^{1}fdx.$

2.16 $\int\limits_{0}^{1}dy\int\limits_{-\sqrt{y}}^{0}fdx+\int\limits_{1}^{2}dy\int\limits_{-\sqrt{2-y}}^{0}fdx.$

2.17 $\int\limits_{0}^{1}dy\int\limits_{-y}^{0}fdx+\int\limits_{1}^{\sqrt{2}}dy\int\limits_{-\sqrt{2-y^2}}^{0}fdx.$

2.18 $\int\limits_{0}^{1}dy\int\limits_{0}^{y^3}fdx+\int\limits_{1}^{2}dy\int\limits_{0}^{2-y}fdx.$

2.19 $\int_{0}^{\sqrt{3}} dx \int_{\sqrt{4-x^2}-2}^{0} f dy + \int_{\sqrt{3}}^{2} dx \int_{-\sqrt{4-x^2}}^{0} f dy$ 2.20 $\int_{-2}^{-1} dy \int_{-(2+y)}^{0} f dx + \int_{-1}^{0} dy \int_{\sqrt[3]{y}}^{0} f dx.$

2.21 $\int_{0}^{1} dx \int_{0}^{x^2} f dy + \int_{1}^{\sqrt{2}} dx \int_{0}^{\sqrt{2-x^2}} f dy.$

3-masala. Ko'rsatilgan D soha uchun $\iint_D f(x,y)\,dxdy$ integralda qutb koordinatalariga $(x = r\cos\varphi, y = r\sin\varphi)$ o'tib, integrallash chegaralari ikki xil tartibda qo'yilsin.

3.1 $D = \{(x,y): x^2 + y^2 \le 2y\}.$

3.2 $D = \{(x,y): a^2 \le x^2 + y^2 \le b^2, a > 0, b > 0\}.$

3.3 $D = \{(x,y): (x^2 + y^2)^2 = a^2(x^2 - y^2), x \le 0\}.$

3.4 ***D*** **soha** $x = 0,\ y = 0,\ y = 1 - x$ **chiziqlar bilan chegaralangan.**

3.5 ***D*** **soha** $x^2 = ay,\ y = a\ (a > 0)$ **chiziqlar bilan chegaralangan.**

3.6 $D = \{(x,y): 0 \le x \le 1, x^2 \le y \le x\}.$

3.7 $D = \{(x,y): 0 \le y \le 2, y \le x \le \sqrt{3}y\}.$

3.8 $D = \{(x,y): 0 \le x \le 2, 0 \le y \le \sqrt{3}x\}.$

3.9 $D = \{(r,\varphi): r \ge 2\cos\varphi, r \le 4\cos\varphi\}.$

3.10 $D = \{(x,y): x^2 + y^2 \le 4, x + y \ge 2\}.$

3.11 $D = \{(x,y): x^2 + y^2 \ge 8, x^2 + y^2 \le 4x\}.$

3.12 $D = \{(x,y): x^2 + y^2 \ge 18, x^2 + y^2 \le 6y\}.$

3.13 $D = \{(x,y): x^2 + y^2 + 4x \ge 0, x^2 + y^2 + 8x \le 0\}.$

3.14 $D = \{(x,y): x \ge y,\ x + y \le 6,\ y \ge 0\}.$

3.15 $D = \{(x,y): x^2 + y^2 \le 4x, |y| \le |x|\}.$

3.16 $D = \left\{(r,\varphi): r \ge 2\sin\varphi, r \le 5\sin\varphi, 0 \le \varphi \le \frac{\pi}{2}\right\}.$

3.17 $D=\{(x,y): x^2+y^2\le 16, x^2+y^2\ge 4x\}$.

3.18 $D=\{(r,\varphi): r\le 2\cos 3\varphi, r\ge 1(I$ va IV chorakdagi qismi$),\}$.

3.19 $D=\{(x,y): x^2+y^2\ge x, x^2+y^2\le 2x\}$.

3.20 $D=\{(x,y): x^2+y^2\le 4x, x^2+y^2\ge 2y\}$.

3.21 $D=\{(x,y): 0\le x\le 1, 2x\le y\le 3x\}$.

4-masala. Hisoblang.

4.1 $\iint_D (12x^2y^2+16x^3y^3)dxdy;\quad D: x=1,\ y=x^2,\ y=-\sqrt{x}$.

4.2 $\iint_D (9x^2y^2+48x^3y^3)dxdy;\quad D: x=1, y=\sqrt{x},\ y=-x^2$.

4.3 $\iint_D (36x^2y^2-96x^3y^3)dxdy;\quad D: x=1,\ y=\sqrt[3]{x}, y=-x^3$.

4.4 $\iint_D (18x^2y^2+32x^3y^3)dxdy;\quad D: x=1, y=x^3, y=-\sqrt[3]{x}$.

4.5 $\iint_D (27x^2y^2+48x^3y^3)dxdy;\quad D: x=1, y=x^2, y=-\sqrt[3]{x}$.

4.6 $\iint_D (18x^2y^2+32x^3y^3)dxdy;\quad D: x=1,\ y=\sqrt[3]{x},\ y=-x^2$.

4.7 $\iint_D (18x^2y^2+32x^3y^3)dxdy;\quad D: x=1, y=x^3,\ y=-\sqrt{x}$.

4.8 $\iint_D (27x^2y^2+48x^3y^3)dxdy;\quad D: x=1,\ y=\sqrt{x},\ y=-x^3$.

4.9 $\iint_D (4xy+3x^2y^2)dxdy;\quad D: x=1,\ y=x^2,\ y=-\sqrt{x}$.

4.10 $\iint_D (12xy+9x^2y^2)dxdy;\quad D: x=1, y=\sqrt{x},\ y=-x^2$.

4.11 $\iint_D (8xy+9x^2y^2)dxdy;\quad D: x=1, y=\sqrt[3]{x},\ y=-x^3$.

4.12 $\iint_D (24xy+18x^2y^2)dxdy;\quad D: x=1,\ y=x^3,\ y=-\sqrt[3]{x}$.

4.13 $\iint_D (12xy+27x^2y^2)dxdy;\quad D: x=1,\ y=x^2,\ y=-\sqrt[3]{x}$.

4.14 $\iint\limits_D (8xy+18x^2y^2)dxdy;\quad D: x=1,\ y=\sqrt[3]{x},\ y=-x^2.$

4.15 $\iint\limits_D \left(\frac{4}{5}xy+\frac{9}{11}x^2y^2\right)dxdy;\quad D: x=1,\ y=x^3,\ y=-\sqrt{x}.$

4.16 $\iint\limits_D \left(\frac{4}{5}xy+9x^2y^2\right)dxdy;\quad D: x=1,\ y=\sqrt{x},\ y=-x^3.$

4.17 $\iint\limits_D (24xy-48x^3y^3)dxdy;\quad D: x=1,\ y=x^2,\ y=-\sqrt{x}.$

4.18 $\iint\limits_D (6xy+24x^3y^3)dxdy;\quad D: x=1,\ y=\sqrt{x},\ y=-x^2.$

4.19 $\iint\limits_D (4xy+16x^3y^3)dxdy;\quad D: x=1,\ y=\sqrt[3]{x},\ y=-x^2.$

4.20 $\iint\limits_D (4xy+16x^3y^3)dxdy;\quad D: x=1, y=x^3, y=-\sqrt[3]{x}.$

4.21 $\iint\limits_D (xy-4x^3y^3)dxdy;\quad D: x=1,\ y=x^3,\ y=-\sqrt{x}.$

5-masala. Hisoblang.

5.1 $\iint\limits_D ye^{\frac{xy}{2}}dxdy;\quad D: y=\ln 2,\ y=\ln 3,\ x=2,\ x=4.$

5.2 $\iint\limits_D y^2\sin\frac{xy}{2}dxdy;\quad D: x=0,\ y=\sqrt{\pi},\ y=\frac{x}{2}.$

5.3 $\iint\limits_D y\cos xy dxdy;\quad D: y=\frac{\pi}{2},\ y=\pi,\ x=1,\ x=2.$

5.4 $\iint\limits_D y^2e^{-\frac{xy}{4}}dxdy;\quad D: x=0,\ y=2,\ y=x.$

5.5 $\iint\limits_D y\sin xy dxdy;\quad D: y=\frac{\pi}{2},\ y=\pi,\ x=1,\ x=2.$

5.6 $\iint\limits_D 12y\sin 2xy dxdy;\quad D: y=\frac{\pi}{4},\ y=\frac{\pi}{2},\ x=2,\ x=3.$

5.7 $\iint\limits_D y^2\cos\frac{xy}{2}dxdy;\quad D: x=0,\ y=\sqrt{\frac{\pi}{2}},\ y=\frac{x}{2}.$

5.8 $\iint\limits_D y^2\cos xy dxdy;\quad D: x=0,\ y=\sqrt{\pi},\ y=x.$

5.9 $\iint\limits_D 4ye^{2xy}dxdy;\quad D: y=\ln 3,\ y=\ln 4,\ x=\frac{1}{2},\ x=1.$

5.10 $\iint\limits_D ye^{\frac{xy}{4}}dxdy;\quad D: y=\ln 2,\ y=\ln 3,\ x=4,\ x=8.$

5.11 $\iint\limits_D 4y^2\sin xydxdy;\quad D: x=0,\ y=\sqrt{\frac{\pi}{2}},\ y=x.$

5.12 $\iint\limits_D 4y^2\sin 2xydxdy;\quad D: x=0,\ y=\sqrt{2\pi},\ y=2x.$

5.13 $\iint\limits_D y\cos 2xydxdy;\quad D: y=\frac{\pi}{2},\ y=\pi,\ x=\frac{1}{2},\ x=1.$

5.14 $\iint\limits_D 2y\cos 2xydxdy;\quad D: x=\frac{\pi}{4},\ y=\frac{\pi}{2},\ x=1,\ x=2.$

5.15 $\iint\limits_D y^2\cdot e^{-\frac{xy}{8}}dxdy;\quad D: x=0,\ y=2,\ y=\frac{x}{2}.$

5.16 $\iint\limits_D y^2\cdot e^{-\frac{xy}{2}}dxdy;\quad D: x=0,\ y=\sqrt{2},\ y=x.$

5.17 $\iint\limits_D y\sin xydxdy;\quad D: y=\pi,\ y=2\pi,\ x=\frac{1}{2},\ x=1.$

5.18 $\iint\limits_D y^2\cos 2xydxdy;\quad D: x=0,\ y=\sqrt{\frac{\pi}{2}},\ y=\frac{x}{2}.$

5.19 $\iint\limits_D 8ye^{4xy}dxdy;\quad D: y=\ln 3,\ y=\ln 4,\ x=\frac{1}{4},\ x=\frac{1}{2}.$

5.20 $\iint\limits_D 3y^2\sin\frac{xy}{2}dxdy;\ D: x=0,\ y=\sqrt{\frac{4\pi}{3}},\ y=\frac{2}{3}x.$

5.21 $\iint\limits_D y\cos xydxdy;\ D: y=\pi, y=3\pi, x=\frac{1}{2}, x=1.$

6-masala. Hisoblang.

6.1 $\iiint\limits_{(V)}\frac{dxdydz}{\left(1+\frac{x}{3}+\frac{y}{4}+\frac{z}{8}\right)};\quad (V):\begin{cases}\frac{x}{3}+\frac{y}{4}+\frac{z}{8}=1;\\ x=0, y=0, z=0.\end{cases}$

6.2 $\iiint\limits_{(V)}15\left(y^2+z^2\right)dxdydz;\quad (V):\begin{cases}z=x+y, x+y=1;\\ x=0, y=0, z=0.\end{cases}$

6.3 $\iiint\limits_{(V)}(3x+4y)dxdydz;\quad (V):\begin{cases}y=x, y=0, x=1;\\ z=5\left(x^2+y^2\right), z=0.\end{cases}$

6.4 $\iiint\limits_{(V)}\left(27x+54y^3\right)dxdydz;\quad (V):\begin{cases} y=x, y=0, x=1; \\ z=\sqrt{xy}, z=0. \end{cases}$

6.5 $\iiint\limits_{(V)}\dfrac{dxdydz}{\left(1+\dfrac{x}{16}+\dfrac{y}{8}+\dfrac{z}{3}\right)^2};\quad (V):\begin{cases} \dfrac{x}{16}+\dfrac{y}{8}+\dfrac{z}{3}=1; \\ x=0, y=0, z=0. \end{cases}$

6.6 $\iiint\limits_{(V)}\left(3x^2+y^2\right)dxdydz;\quad (V):\begin{cases} z=10y, x+y=1, \\ x=0, y=0, z=0. \end{cases}$

6.7 $\iiint\limits_{(V)}\left(15x+30z\right)dxdydz;\quad (V):\begin{cases} z=x^2+3y^2, z=0, \\ y=x, y=0, x=1. \end{cases}$

6.8 $\iiint\limits_{(V)}\left(4+8z^3\right)dxdydz;\quad (V):\begin{cases} y=x, y=0, x=1; \\ z=\sqrt{xy}, z=0. \end{cases}$

6.9 $\iiint\limits_{(V)}\left(1+2x^3\right)dxdydz;\quad (V):\begin{cases} y=36x, y=0, x=1; \\ z=\sqrt{xy}, z=0. \end{cases}$

6.10 $\iiint\limits_{(V)}21xzdxdydz;\quad (V):\begin{cases} y=x, y=0, x=2; \\ z=xy, z=0. \end{cases}$

6.11 $\iiint\limits_{(V)}\dfrac{dxdydz}{\left(1+\dfrac{x}{10}+\dfrac{y}{8}+\dfrac{z}{3}\right)^6};\quad (V):\begin{cases} \dfrac{x}{10}+\dfrac{y}{8}+\dfrac{z}{3}=1; \\ x=0, y=0, z=0. \end{cases}$

6.12 $\iiint\limits_{(V)}\left(60y+90z\right)dxdydz;\quad (V):\begin{cases} y=x, y=0, x=1; \\ z=x^2+y^2, z=0. \end{cases}$

6.13 $\iiint\limits_{(V)}\left(\dfrac{10}{3}x+\dfrac{5}{3}\right)dxdydz;\quad (V):\begin{cases} y=9x, y=0, x=1; \\ z=\sqrt{xy}, z=0. \end{cases}$

6.14 $\iiint\limits_{(V)}\left(9+18z\right)dxdydz;\quad (V):\begin{cases} y=4x, y=0, x=1; \\ z=\sqrt{xy}, z=0. \end{cases}$

6.15 $\iiint\limits_{(V)} \dfrac{dxdydz}{\left(1+\dfrac{x}{2}+\dfrac{y}{4}+\dfrac{z}{6}\right)^4}$; $(V):\begin{cases}\dfrac{x}{2}+\dfrac{y}{4}+\dfrac{z}{6}=1;\\ x=0, y=0, z=0.\end{cases}$

6.16 $\iiint\limits_{(V)} (8y+12z)\,dxdydz$; $(V):\begin{cases}y=x, y=0, x=1;\\ z=3x^2+2y^2, z=0.\end{cases}$

6.17 $\iiint\limits_{(V)} (x+yz)\,dxdydz$; $(V):\begin{cases}y=x, y=0, x=1;\\ z=30x^2+60y^2, z=0.\end{cases}$

6.18 $\iiint\limits_{(V)} \dfrac{dxdydz}{\left(1+\dfrac{x}{6}+\dfrac{y}{4}+\dfrac{z}{16}\right)^5}$; $(V):\begin{cases}\dfrac{x}{6}+\dfrac{y}{4}+\dfrac{z}{16}=1;\\ x=0, y=0, z=0.\end{cases}$

6.19 $\iiint\limits_{(V)} y^2\,dxdydz$; $(V):\begin{cases}z=10(3x+y), x+y=1;\\ x=0, y=0, z=0.\end{cases}$

6.20 $\iiint\limits_{(V)} \left(5x+\dfrac{3z}{2}\right)dxdydz$; $(V):\begin{cases}y=x, y=0, x=1;\\ z=x^2+15y^2, z=0.\end{cases}$

6.21 $\iiint\limits_{(V)} \dfrac{dxdydz}{\left(1+\dfrac{x}{8}+\dfrac{y}{3}+\dfrac{z}{5}\right)^6}$; $(V):\begin{cases}\dfrac{x}{8}+\dfrac{y}{3}+\dfrac{z}{5}=1;\\ x=0, y=0, z=0.\end{cases}$

7-masala. Quyidagi chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.

7.1 $y=\dfrac{3}{x},\ y=4e^x,\ y=3,\ y=4.$

7.2 $x=\sqrt{36-y^2},\ x=6-\sqrt{36-y^2}.$

7.3 $x^2+y^2=72,\ 6y=-x^2\ \ (y\le 0).$

7.4 $x=8-y^2,\ x=-2y.$

7.5 $y=\dfrac{3}{x},\ y=8e^x,\ y=3,\ y=8.$

7.6 $y=\dfrac{\sqrt{x}}{2},\ y=\dfrac{1}{2x},\ x=16.$

7.7 $x=5-y^2,\ x=-4y.$

7.8 $y=\dfrac{2}{x},\ y=5e^x,\ y=2,\ y=5.$

7.9 $x^2+y^2=12,\ -\sqrt{6}y=x^2\ (y\le 0)$. **7.10** $x^2+y^2=36,\ 3\sqrt{2}y=x^2\ (y\ge 0)$.

7.11 $y=\frac{3}{2}\sqrt{x},\ y=\frac{3}{2x},\ x=9$. **7.12** $y=3\sqrt{x},\ y=\frac{3}{x},\ x=4$.

7.13 $y=\sin x,\ y=\cos x,\ x=0\ (x\ge 0)$. **7.14** $y=\frac{25}{4}-x^2,\ y=x-\frac{5}{2}$.

7.15 $y=20-x^2,\ y=-8x$. **7.16** $y=\frac{2}{x},\ y=7e^x,\ y=2,\ y=7$.

7.17 $y=32-x^2,\ y=-4x$. **7.18** $x=\sqrt{72-y^2},\ 6x=y^2,\ y=0\ (y\ge 0)$.

7.19 $y=\sin x,\ y=\cos x,\ x=0\ (x\le 0)$. **7.20** $y=8-x^2,\ y=-2x$.

7.21 $y=\sqrt{6-x^2},\ y=\sqrt{6}-\sqrt{6-x^2}$.

8-masala. Quyidagi chiziqlar bilan chegaralangan shaklning yuzasi topilsin.

8.1 $y^2-2y+x^2=0;\ y^2-4y+x^2=0;\ y=\frac{x}{\sqrt{3}},\ y=\sqrt{3}x$.

8.2 $x^2-2x+y^2=0;\ x^2-10x+y^2=0;\ y=0,\ y=\sqrt{3}x$.

8.3 $x^2-4x+y^2=0;\ x^2-8x+y^2=0;\ y=0,\ y=\frac{x}{\sqrt{3}}$.

8.4 $y^2-62y+x^2=0;\ y^2-10y+x^2=0;\ y=x,\ x=0$.

8.5 $y^2-6y+x^2=0;\ y^2-8y+x^2=0;\ y=\frac{x}{\sqrt{3}},\ y=\sqrt{3}x$.

8.6 $x^2-2x+y^2=0;\ x^2-4x+y^2=0;\ y=x/\sqrt{3},\ y=\sqrt{3}x$.

8.7 $x^2-2x+y^2=0;\ x^2-4x+y^2=0;\ y=0,\ y=x$.

8.8 $y^2-2y+x^2=0;\ y^2-4y+x^2=0;\ y=\sqrt{3}x,\ x=0$.

8.9 $y^2-8y+x^2=0;\ y^2-10y+x^2=0;\ y=\frac{x}{\sqrt{3}},\ y=\sqrt{3}x$.

8.10 $x^2-2x+y^2=0;\ x^2-6x+y^2=0;\ y=x/\sqrt{3},\ y=\sqrt{3}x$.

8.11 $x^2-4x+y^2=0;\ x^2-8x+y^2=0;\ y=0,\ y=x.$

8.12 $y^2-4y+x^2=0;\ y^2-6y+x^2=0;\ y=\sqrt{3}x,\ x=0.$

8.13 $y^2-4y+x^2=0;\ y^2-6y+x^2=0;\ y=x,\ x=0.$

8.14 $x^2-2x+y^2=0;\ x^2-8x+y^2=0;\ y=x/\sqrt{3},\ y=\sqrt{3}x.$

8.15 $y^2-2y+x^2=0;\ y^2-6y+x^2=0;\ y=x/\sqrt{3},\ x=0.$

8.16 $x^2-2x+y^2=0;\ x^2-6x+y^2=0;\ y=0,\ y=x/\sqrt{3}.$

8.17 $x^2-2x+y^2=0;\ x^2-4x+y^2=0;\ y=0,\ y=x/\sqrt{3}.$

8.18 $y^2-4y+x^2=0;\ y^2-10y+x^2=0;\ y=x/\sqrt{3},\ y=\sqrt{3}x.$

8.19 $y^2-2y+x^2=0;\ y^2-10y+x^2=0;\ y=x/\sqrt{3},\ y=\sqrt{3}x.$

8.20 $x^2-2x+y^2=0;\ x^2-6x+y^2=0;\ y=0,\ y=x.$

8.21 $y^2-2y+x^2=0;\ y^2-4y+x^2=0;\ y=x,\ x=0.$

9-masala. Zichligi $\rho=\rho(x,y)$ bo‘lgan va quyidagi tengsizliklar yordamida berilgan D plastinkaning massasi topilsin.

9.1 $D: x^2+\frac{y^2}{4}\le 1;\ \rho=y^2.$

9.2 $D: \frac{x^2}{4}+y^2\le 1;\ x\ge 0;\ y\ge 0,\ \rho=6x^3\cdot y^3.$

9.3 $D: 1\le\frac{x^2}{9}+\frac{y^2}{4}\le 2;\ y\ge 0;\ y\le\frac{2}{3}x;\ \rho=y/x.$

9.4 $D: 1\le\frac{x^2}{4}+y^2\le 25;\ x\ge 0;\ y\ge\frac{x}{2};\ \rho=\frac{x}{y^2}.$

9.5 $D: \frac{x^2}{4}+\frac{y^2}{25}\le 1;\ y\ge 0;\ \rho=x^2\cdot y.$

9.6 $D: \frac{x^2}{9} + \frac{y^2}{4} \le 1;\ \ \rho = x^2 y^2.$

9.7 $D: \frac{x^2}{9} + \frac{y^2}{25} \le 1;\ \ y \ge 0;\ \ \rho = \frac{7x^2}{18}.$

9.8 $D: \frac{x^2}{16} + y^2 \le 1;\ \ x \ge 0;\ \ y \ge 0;\ \ \rho = 5xy^7.$

9.9 $D: 1 \le \frac{x^2}{4} + y^2 \le 4; y \ge 0; y \ge \frac{x}{2}x. \rho = 8 y/x^3.$

9.10 $D: \frac{x^2}{4} + y^2 \le 1;\ \ x \ge 0;\ \ y \ge 0;\ \ \rho = 30x^3 y^7.$

9.11 $D: x^2/9 + y^2 \le 1;\ \ x \ge 0;\ \ \rho = 7xy^6.$

9.12 $D: 1 \le \frac{x^2}{9} + \frac{y^2}{4} \le 3; y \ge 0; y \le \frac{2}{3}x. \rho = y/x.$

9.13 $D: x^2/4 + y^2 \le 1; \rho = 4y^4.$

9.14 $D: x^2 + \frac{y^2}{25} \le 1; y \ge 0 \rho = 7x^4 y.$

9.15 $D: 1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 4; x \ge 0; y \ge 3x/2. \rho = x/y.$

9.16 $D: x^2 + \frac{y^2}{9} \le 1; y \ge 0; \rho = 35x^4 y^3.$

9.17 $D: 1 \le \frac{x^2}{16} + \frac{y^2}{4} \le 4; x \ge 0; y \ge x/2. \rho = x/y.$

9.18 $D: \frac{x^2}{4} + \frac{y^2}{9} \le 1;. \rho = x^2.$

9.19 $D: \frac{x^2}{4} + \frac{y^2}{9} \le 1; x \ge 0; y \ge 0. \rho = x^3 y.$

9.20 $D: 1 \le x^2 + \frac{y^2}{16} \le 9; y \ge 0; y \le 4x. \rho = y/x^2.$

9.21 $D: 1 \le \frac{x^2}{4} + \frac{y^2}{16} \le 25; x \ge 0; y \ge 2x; \rho = \frac{x}{y}.$

10-masala.

10.1 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ y\ge 0.$ chiziqlar bilan chegaralangan plastinkaning og'irlik markazi topilsin $(\rho=1)$.

10.2 $r^2=a^2\cos 2\varphi$ (o'ng yaproq) egri chiziq bilan chegaralangan plastinkaning og'irlik markazi topilsin. $(\rho=1)$.

10.3 $x^2+y^2=a^2, x\ge 0, y\ge 0$ tengsizliklar bilan aniqlangan plastinka uchun I_x, I_y inersiya momentlari topilsin $(\rho=1)$.

10.4 $y^2=4x+4$ va $y^2=-2x+4$ chiziqlar bilan chegaralangan plastinkaning og'irlik markazi topilsin $(\rho=1)$.

10.5 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ egri chiziq bilan chegaralangan plastinka uchun I_x, I_y inersiya momentlari topilsin $(\rho=1)$.

10.6 $\frac{x}{9}+\frac{y}{2}=1$, $\frac{x}{4}+\frac{y}{2}=1,\ y=0$ chiziqlar bilan chegaralangan plastinka uchun I_x, I_y lar topilsin $(\rho=1)$.

10.7 $x^2+y^2\le 16,\ x\ge 2\sqrt{3}$ tengsizliklar bilan aniqlangan plastinkaning og'irlik markazi topilsin $(\rho=1)$.

10.8 $xy=1, xy=2, y=2x, x=2y$ chiziqlar bilan chegaralangan plastinka uchun I_x, I_y inersiya momentlari topilsin $(\rho=1)$.

10.9 Agar $1\le x^2+y^2\le 4$ doiraviy halqaning har bir nuqtasidagi massa zichligi $\rho=x^2y^2$ formula bilan aniqlansa, uning massasi topilsin.

10.10 Agar $y=x^2-4x;\ y=x$ chiziqlar bilan chegaralangan plastinkaning har bir nuqtasidagi massa zichligi $\rho=x+y$ formula bilan aniqlangan bo'lsa, shu plastinka og'irlik markazi topilsin.

10.11 $xy=4,\ y=\frac{1}{2}x^2,\ y=6\ \left(y\ge\frac{1}{2}x^2\right)$ chiziqlar bilan chegaralangan plastinkaning og'irlik markazi topilsin $(\rho=5x+3)$.

10.12 $y^2=3x+4$ va $y^2+4x=11\ (y\ge 0)$ chiziqlar bilan chegaralangan plastinka massasi topilsin $(\rho=y)$.

10.13 Ikkita $\varphi=0$ va $\varphi=\pi$ nurlar hamda $r=a\varphi$ $(0\le\varphi\le\pi)$ Arximed spirali yoyi bilan chegaralangan plastinkaning og'irlik markazi topilsin $(\rho=1)$.

10.14 $y=x^3$, $x+y=2$, $x=0$ chiziqlar bilan chegaralangan plastinkaning og'irlik markazi topilsin $(\rho=1)$.

Quyidagi 10.15-10.19 misollarda plastinkaning chegarasini aniqlovshi chiziqlar berilgan. Har bir plastinkaning og'irlik markazi topilsin $(\rho=1)$.

10.15 $x=a(t-\sin t)$, $y=a(1-\cos t)$; $0\le t\le 2\pi$, $y=0$.

10.16 $x^2+y^2=a^2$; $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; $x=0$, $x\ge 0$; $y\ge 0$.

10.17 $r=a(1+\sin\varphi)$.

10.18 $r=a\sin 2\varphi$, $0\le\varphi\le\dfrac{\pi}{2}$.

10.19 $r=\sqrt{2}$, $r=2\sin\varphi$, $\dfrac{\pi}{4}\le\varphi\le\dfrac{3\pi}{4}$.

10.20 $ay=2ax-x^2$, $y=0$ chiziqlar bilan chegaralangan plastinkaning I_x, I_y inersiya momentlari topilsin $(\rho=1)$.

10.21 $r=a(1+\cos\varphi)$ kardioda bilan chegaralangan plastinkaning Ox va Ou o'qlariga nisbatan I_x, I_y inersiya momentlari topilsin $(\rho=1)$.

11-masala. Quyida ko'rsatilgan sirtlarning yuzalari topilsin.

11.1 $y^2+z^2=x^2$ sirtning $x^2-y^2=a^2$ – silindr va $y=\pm b$ – tekisliklar bilan ajratilgan qismi.

11.2 $z^2=4x$ sirtning $y^2=4x$ – silindr va $x=1$ – tekisliklar bilan ajratilgan qismi.

11.3 $(x^2+y^2)^{3/2}+z=1$ sirtning $z=0$ tekislik bilan ajratilgan qismi.

11.4 $x^2+y^2=\pm ax$ silindrlarning $x^2+y^2+z^2=a^2$ shar ichidagi qismi.

11.5 $(x+y)^2+2z^2=2a^2$ silindrik sirtning 1-oktandagi qismi $(x\ge 0,\ y\ge 0,\ z\ge 0,\ x^2+y^2+z^2\ne 0)$.

11.6 $(x+y)^2 z = x+y$ sirtning $1 \le x^2+y^2 \le 4, x>0, y>0$ sohadagi qismi.

11.7 $az = xy$ giperbolik paraboloid sirtning $\left(x^2+y^2\right)^2 = 2a^2xy$ silindr ichidagi qismi.

11.8 $z^2 = 2xy$ konus sirtning $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} < 1$, $x \ge 0$, $y \ge 0$, $z = 0$, $x^2+y^2 \ne 0$ $a>0, b>0$ sohadagi qismi.

11.9 $3z = 2\left(x\sqrt{x} + y\sqrt{y}\right)$ sirtning $x=0, y=0, x+y=1$ tekisliklar orasidagi joylashgan qismi.

11.10 $z = \sqrt{x^2+y^2}$ konus sirtining $x^2+y^2 = 2x$ silindr ichida joylashgan qismi.

11.11 $x^2+y^2 = 2az$ paraboloid sirtining $(x+y)^2 = 2a^2xy(a>0)$ silindrik sirt ichida joylashgan qismi.

11.12 $az = xy$ giperbolik paraboloid sirtning $x^2+y^2 = a^2\,(a>0)$ silindr ichida joylashgan qismi.

11.13 $\left(\frac{x}{a}+\frac{y}{b}\right)^2 + \frac{2z}{a} = 1$ sirtning $x=0, y=0, z=0$ koordinata tekisliklari orasida joylashgan qismi.

11.14 $z^2 = 2xy$ konus sirtning $x+y=1, x=0, y=0$ tekisliklar orasida joylashgan qismi.

11.15 $z = \frac{1}{2}\left(x^2-y^2\right)$ giperbolik paraboloid sirtining $\left(x^2+y^2\right)^2 = x^2-y^2$ silindr ichida joylashgan qismi.

11.16 $z = \sqrt{x^2+y^2}$ va $x+2z=a$ sirtlar bilan chegaralangan jismning to‘la sirti $(a>0)$

11.17 $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1(a,b,c>0)$ sirtning 1-oktantdagi qismi.

11.18 $x^2+y^2+z^2 = R^2$ sfera sirtining $\left(x^2+y^2\right)^2 = R^2\cdot\left(x^2-y^2\right)$ silindr ichidagi qismi.

11.19 $x^2+y^2 = 6z$ sirtning $\left(x^2+y^2\right)^2 = 9\cdot\left(x^2-y^2\right)$ silindr ichidagi qismi.

11.20 $z^2 = 4x$ sirtning $y^2 = 4x, x=1$ sirtlar bilan ajratilgan qismi.

11.21 $y^2+z^2 = x^2$ sirtning $x^2 = ay$ sirt bilan ajratilgan qismi.

12-masala. Quyidagi sirtlar bilan chegaralangan jismning hajmi hisoblansin.

12.1 $\begin{cases} x^2+y^2=2y; \\ z=5/4-x^2, \ z=0 \end{cases}$

12.2 $\begin{cases} x^2+y^2=7x, \ x^2+y^2=10x, \ y=0 \ (y\le 0); \\ z=\sqrt{x^2+y^2}, \ z=0. \end{cases}$

12.3 $\begin{cases} x^2+y^2=y, \ x^2+y^2=4y, \\ z=\sqrt{x^2+y^2}, \ z=0. \end{cases}$

12.4 $\begin{cases} x^2+y^2=8\sqrt{2}y; \\ z=x^2+y^2-64, \ z=0 \ (z\ge 0). \end{cases}$

12.5 $\begin{cases} x^2+y^2=8\sqrt{2}x; \\ z=x^2+y^2-64, \ z=0 \ (z\ge 0). \end{cases}$

12.6 $\begin{cases} x^2+y^2=2y; \\ z=\frac{13}{4}-x^2, \ z=0. \end{cases}$

12.7 $\begin{cases} x^2+y^2+4x=0; \\ z=8-y^2, \ z=0. \end{cases}$

12.8 $\begin{cases} x^2+y^2=3y, \ x^2+y^2=6y, \\ z=\sqrt{x^2+y^2}, \ z=0. \end{cases}$

12.9 $\begin{cases} x^2+y^2=6x, \ x^2+y^2=9x; \ y=0 \ (y\le 0); \\ z=\sqrt{x^2+y^2}, z=0. \end{cases}$

12.10 $\begin{cases} x^2+y^2=6\sqrt{2}x; \\ z=x^2+y^2-36, \ z=0 \ (z\ge 0). \end{cases}$

12.11 $\begin{cases} x^2+y^2=6\sqrt{2}y; \\ z=x^2+y^2-36,\ z=0\ \ (z\geq 0). \end{cases}$

12.12 $\begin{cases} x^2+y^2=2\sqrt{2}y; \\ z=x^2+y^2-4,\ \ z=0\ \ (z\geq 0). \end{cases}$

12.13 $\begin{cases} x^2+y^2=2y; \\ z=\dfrac{9}{4}-x^2,\ \ z=0. \end{cases}$ **12.14** $\begin{cases} x^2+y^2=4x, \\ z=12-y^2,\ \ z=0. \end{cases}$

12.15 $\begin{cases} x^2+y^2=2y,\ \ x^2+y^2=5y, \\ z=\sqrt{x^2+y^2},\ \ z=0. \end{cases}$

12.16 $\begin{cases} x^2+y^2=8x,\ x^2+y^2=11x,\ \ y=0\ \ (y\leq 0); \\ z=\sqrt{x^2+y^2},\ \ z=0. \end{cases}$

12.17 $\begin{cases} x^2+y^2+2\sqrt{2}y=0; \\ z=x^2+y^2-4,\ \ z=0\ \ (z\geq 0). \end{cases}$

12.18 $\begin{cases} x^2+y^2=4\sqrt{2}x, \\ z=x^2+y^2-16,\ \ z=0\ \ (z\geq 0). \end{cases}$

12.19 $\begin{cases} x^2+y^2=4x, \\ z=10-y^2,\ \ z=0. \end{cases}$ **12.20** $\begin{cases} x^2+y^2=4y, \\ z=4-x^2,\ \ z=0. \end{cases}$

12.21 $\begin{cases} x^2+y^2=4y,\ \ x^2+y^2=7y, \\ z=\sqrt{x^2+y^2},\ \ z=0. \end{cases}$

13-masala. Quyidagi sirtlar bilan chegaralangan jismning hajmi hisoblansin.

13.1 $\begin{cases} z=2-12\left(x^2+y^2\right), \\ z=24x+2. \end{cases}$ **13.2** $\begin{cases} z=24\left(x^2+y^2\right)+1, \\ z=48x+1. \end{cases}$

13.3 $\begin{cases} z=10\left[(x-1)^2+y^2\right]+1, \\ z=21-20x. \end{cases}$

13.4 $\begin{cases} z=2-18\left[(x-1)^2+y^2\right], \\ z=-36x-34. \end{cases}$

13.5 $\begin{cases} z=8\left(x^2+y^2\right)+3, \\ z=16x+3. \end{cases}$

13.6 $\begin{cases} z=-16\left(x^2+y^2\right)-1, \\ z=-32x-1: \end{cases}$

13.7 $\begin{cases} z=2-20\left[(x+1)^2+y^2\right], \\ z=-40x-38. \end{cases}$

13.8 $\begin{cases} z=30\left[(x+1)^2+y^2\right]+1, \\ z=-60x-61. \end{cases}$

13.9 $\begin{cases} z=4-14\left(x^2+y^2\right), \\ z=-34-28x. \end{cases}$

13.10 $\begin{cases} z=26\left(x^2+y^2\right)-2, \\ z=-52x-2. \end{cases}$

13.11 $\begin{cases} z=28\cdot\left[(x+1)^2+y^2\right]+3, \\ z=56x+59. \end{cases}$

13.12 $\begin{cases} z=-2\left[(x-1)^2+y^2\right]-1, \\ z=4x-5. \end{cases}$

13.13 $\begin{cases} z=32\left(x^2+y^2\right)+3, \\ z=3-64x. \end{cases}$

13.14 $\begin{cases} z=-2\left(x^2+y^2\right)-1, \\ z=4y-1. \end{cases}$

13.15 $\begin{cases} z=4-6\left[(x-1)^2+y^2\right], \\ z=12x-8. \end{cases}$

13.16 $\begin{cases} z=26\left[(x-1)^2+y^2\right]-2, \\ z=50-52x \end{cases}$

13.17 $\begin{cases} z=2-4\left(x^2+y^2\right)+3, \\ z=8x+2. \end{cases}$

13.18 $\begin{cases} z=30\left(x^2+y^2\right)+1, \\ z=60y+1 \end{cases}$

13.19 $\begin{cases} z=22\cdot\left[(x-1)^2+y^2\right]+3, \\ z=47-44x. \end{cases}$

13.20 $\begin{cases} z=-16\left[(x+1)^2+y^2\right]-1, \\ z=-32x-33. \end{cases}$

13.21 $\begin{cases} z=2-18\left(x^2+y^2\right), \\ z=2-36y. \end{cases}$

14-masala. Quyidagi sirtlar bilan chegaralangan jismning hajmini uch karrali integral yordamida hisoblang.

14.1 $(x^2+y^2)^3+z^6=a^3xyz.$ **14.2** $(x^2+y^2+z^2)^3=a^6\cdot(x^2+y^2)^{-1}.$

14.3 $(x^2+y^2)^2+z^4=a^3\cdot(x-y).$ **14.4** $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2+\frac{z^4}{c^4}=\frac{z}{h}.$

14.5 $\begin{cases} x^2+y^2+z^2=1, x^2+y^2+z^2=16, \\ z^2=x^2+y^2 (x\ge 0, y\ge 0, z\ge 0) \end{cases}.$ **14.6** $z=10(x^2+y^2)^2+1, z=1-20y.$

14.7 $\frac{x^2}{4}+\frac{y^2}{9}=z^2, 2z=\frac{x^2}{4}+\frac{y^2}{9}.$ **14.8** $z=24(x^2+y^2)^2+1, z=48x+1.$

14.9 $x^2+y^2=3z, x+y=6.$ **14.10** $z=2-20(x^2+y^2)^2, z=2-40y.$

14.11 $z=6\cdot\sqrt{x^2+y^2}, z=16-x^2+y^2.$ **14.12** $\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{3}}+\sqrt{\frac{z}{5}}=1, 1, x=0, y=0, z=0.$

14.13 $z=\sqrt{64-x^2-y^2}, z=1, x^2+y^2=60$ ***(silindr tashqarisida).***

14.14 $2z=2x^2+\frac{y^2}{3}, 4x^2+\frac{y^2}{9}=1, z=0.$

14.15 $z=\frac{15}{2}\sqrt{x^2+y^2}, z=\frac{17}{2}-x^2-y^2.$

14.16 $y^2+z^2=a^2, y^2+z^2=x^2, x=b(0<a<b).$

14.17 $z=\sqrt{\frac{16}{9}-x^2-y^2}, 2z=x^2+y^2.$

14.18 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\cdot\frac{z}{c}.$

14.19 $z=4-14(x^2+y^2)^2, z=4-28x.$

14.20 $x+y+z=a, x+y+z=2a, x+y=z, x+y=2z, y=x, y=3x.$

14.21 $x^2+y^2+z^2=2az, x^2+y^2=z^2, x^2+y^2=\frac{1}{3}z^2.$

15-masala. Egri chiziqli integrallar hisoblansin.

15.1 $\int\limits_{\gamma}\sin y dx+\sin x dy$, $\gamma: A(0,\pi)$ va $B(\pi,0)$ nuqtalarni tutashtiruvchi kesma.

15.2 $\oint\limits_{ABCDA}\frac{dx+dy}{|x|+|y|}$, $A(1;\ 0)$, $B(0;\ 1)$, $C(-1;\ 0)$, $D(0;\ 1)$.

15.3 $\oint\limits_{\gamma}\frac{(x+y)dx-(x-y)dy}{x^2+y^2}$; $\gamma: x^2+y^2=a^2$.

15.4 $\int\limits_{\gamma}(2a-y)dx+xdy$; $\gamma: x=a(t-\sin t)$, $y=a(1-\cos t)$, $0\le t\le 2\pi$.

15.5 $\oint\limits_{\gamma}(x+y)dx+(x-y)dy$, $\gamma:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

15.6 $\int\limits_{\gamma}(x^2+y^2)dx+(x^2-y^2)dy$, $\gamma: y=1-|1-x|$, $0\le x\le 2$.

15.7 $\int\limits_{\gamma}(x^2-2xy)dx+(y^2-2xy)dy$, $\gamma: y=x^2$, $-1\le x\le 1$.

15.8 $\int\limits_{\gamma}\frac{xdx+ydy}{\sqrt{1+x^2+y^2}}$, $\gamma:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x\ge 0$, $y\ge 0$.

15.9 $\int\limits_{\gamma}ydx-xdy$, $\gamma: x=a\cos^3 t$, $y=a\sin^3 t$ $\left(0\le t\le\frac{\pi}{2}\right)$.

15.10 $\int\limits_{\gamma}\frac{ydx-xdy}{x^2+y^2}$, $\gamma: x=a\cos^3 t$, $y=a\sin^3 t$ $\left(0\le t\le\frac{\pi}{2}\right)$.

15.11 $\oint\limits_{\gamma}y\cos x dx+\sin x dy$, γ: uchlari $(1;\ 0)$, $(0;\ 2)$, va $(2;\ 0)$ nuqtalarda bo'lgan uchburchak konturi.

15.12 $\oint\limits_{\gamma}2xdx-(x+2y)dy$, γ: uchlari $(-1;\ 0)$, $(0;\ 2)$, va $(2;\ 0)$ nuqtalarda bo'lgan uchburchak konturi.

15.13 $\int\limits_{\gamma}(x^2+y^2)dx+xydy,\ \gamma: y=e^x$ chiziqning $(0;\ 1)$ va $(1;\ e)$ nuqtalari orasidagi yoyi.

15.14 $\int\limits_{\gamma}xdy,\ \gamma: \frac{x}{a}+\frac{y}{b}=1$ to'g'ri chiziqning $(a;\ 0)$ va $(0;\ b)$ nuqtalar orasidagi kesmasi.

15.15 $\int\limits_{\gamma}|y|ds,\ \gamma:\ (x^2+y^2)^2=a^2(x^2+y^2)$ -lemniskata yoyi.

15.16 $\int\limits_{\gamma}\left(x^{4/3}+y^{4/3}\right)ds,\ \gamma: x^{2/3}+y^{2/3}=a^{2/3}$ – astroida.

15.17 $\int\limits_{\gamma}(x+y)ds,\ \gamma: \rho^2=a^2\cdot\cos 2\varphi$ - lemniskata.

15.18 $\int\limits_{\gamma}x^2ds,\ \gamma: x^2+y^2=a^2,\ x\geq 0.$

15.19 $\int\limits_{\gamma}yds,\ \gamma:\ y^2=2x$, γ: parabolaning $(0;\ 0)$ va $(1;\sqrt{2})$ nuqtalari orasidagi yoy.

15.20 $\int\limits_{\gamma}\frac{1}{\sqrt{x^2+y^2+4}}ds,\ \gamma:(0;\ 0)$ va $(1;\ 2)$ nuqtalarni tutashtiruvchi to'g'ri chiziq kesmasi.

15.21 $\int\limits_{\gamma}xyds,\ \gamma: 3|x|+4|y|=12,\ y\geq 0.$

16-masala. Hisoblang.

16.1 $\int\limits_{(-1;2)}^{(2;3)} xdy+ydx.$ **16.2** $\int\limits_{(0;1)}^{(2;3)}(x+y)dx+(x-y)dy.$

16.3 $\int\limits_{(2;1)}^{(1;2)}\frac{ydx-xdy}{x^2}.$

16.4 $\int\limits_{(2;1)}^{(1;2)}\frac{ydx-xdy}{x^2}.$ -Oy o'qini kesmaydigan chiziqlar bo'ylab.

16.5 $\int\limits_{(0;2)}^{(1;3)} \left(4xy-15x^2y\right)dx+\left(2x^2-5x^3+7\right)dy.$

16.6 $\int\limits_{(1;0)}^{(6;8)} \frac{xdx+ydy}{\sqrt{x^2+y^2}}$ – koordinata boshini kesib o'tmaydigan chiziqlar bo'ylab.

16.7 $\int\limits_{(1;-1)}^{(1;1)} (x-y)\cdot(dx-dy).$

16.8 $\int\limits_{(-2;-1)}^{(0;0)} \left(x^4+4xy^3\right)dx+\left(6x^2y^2-5y^4\right)dy.$

16.9 $\int\limits_{(0;1)}^{(3;-4)} xdx+ydy.$

16.10 $\int\limits_{(0;-1)}^{(1;0)} \frac{xdy-ydx}{(x-y)^2}$ $-\ y=x$ *to'g'ri chiziqni kesmaydigan chiziqlar bo'ylab.*

16.11 $\int\limits_{(0;0)}^{(a;b)} f(x+y)\cdot(dx+dy), f(u)-$ uzluksiz funksiya.

16.12 $\int\limits_{(0;0)}^{(a;b)} e^x\left(\cos ydx-\sin ydy\right).$

16.13-16.21 misollardagi ifodalarning biror $F(x,y)$ funksiyaning to'liq differensiali bo'lishi yoki bo'lmasligini aniqlang. Agar u to'liq differensial bo'lsa, $F(x,y)$ funksiyani toping.

16.13 $\left(x^2+2x-y^2\right)dx+\left(x^2-2xy-y^2\right)dy.$

16.14 $\left(12x^2y+\frac{1}{y^2}\right)dx+\left(4x^3-\frac{2x}{y^2}\right)dy.$

16.15 $\dfrac{x}{y\cdot\sqrt{x^2+y^2}}dx-\dfrac{x^2+\sqrt{x^2+y^2}}{y^2\cdot\sqrt{x^2+y^2}}dy.$

16.16 $\dfrac{\left(x^2+2xy+5y^2\right)dx+\left(x^2-2xy+y^2\right)dy}{(x+y)^3}.$

16.17 $\dfrac{ydx-xdy}{3x^2-2xy+3y^2}.$

16.18 $e^x\cdot\left[e^y(x-y+2)+y\right]dx+e^x\cdot\left[e^y\cdot(x-y)+1\right]dy.$

16.19 $\dfrac{xdy-ydx}{x^2+4y^2}.$

16.20 $\dfrac{2x\left(1-e^y\right)}{\left(1+x^2\right)^2}dx+\dfrac{e^y}{1+x^2}dy.$

16.21 $\left(\dfrac{x}{\sqrt{x^2+y^2}}+y\right)dx+\left(\dfrac{y}{\sqrt{x^2+y^2}}+x\right)dx.$

17-masala. Quyidagi I-tur sirt integrallari hisoblansin.

17.1 $\iint\limits_{(S)}\sqrt{x^2+y^2}ds,\ (S)-\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}-\dfrac{z^2}{b^2}=0,\ 0\le z\le b$ konusning yon sirti

17.2 $\iint\limits_{(S)}\left(x^2+y^2\right)ds,\ (S)-$ushbu $\sqrt{x^2+y^2}\le z\le 1$ jismni chegaralovchi sirt.

17.3 $\iint\limits_{(S)}(xy+xz+yz)ds,\ (S)-\ z=\sqrt{x^2+y^2}$ konus sirtining $x^2+y^2=ax$ sirt bilan ajratilgan qismi.

17.4 $\iint\limits_{(S)}\left(x^2+y^2\right)zds,\ (S)-x^2+y^2+z^2=a^2,\ z>0.$

17.5 $\iint\limits_{(S)}zds,\ (S)-$birinchi oktantdagi $x+y+z=1$ tekislik bilan ajratilgan tetraedrning to'liq sirti.

17.6 $\iint\limits_{(S)}(x^2+y^2)ds,\ (S)-x^2+y^2+z^2=a^2$

17.7 $\iint\limits_{(S)}(x+y+z)ds,\ (S)-$ ushbu $0\le x\le a, 0\le y\le a, 0\le z\le a$ kubning to'liq sirti.

17.8 $\iint\limits_{(S)}(6x+4y+3z)ds,\ (S)-x+2y+3z=6$ tekislikning I-oktantdagi qismi.

17.9 $\iint\limits_{(S)}zds,\ (S)-z=\sqrt{16-x^2-y^2}$ sirtning $x\ge 0, y\ge 0, x+y\le 4$ sohadagi qismi.

17.10 $\iint\limits_{(S)}(x^2+y^2+z^2)ds,\ (S)-x^2+y^2+4x=0,\ 2\le z\le 4$ silindrning to'liq sirti

17.11 $\iint\limits_{(S)}zds,\ (S)-z=xy$ sirtning $x^2+y^2=4$ silindr ichidagi qismi.

17.12 $\iint\limits_{(S)}yds,\ (S)-x=2y^2+1(y>0)$ sirtning $x=y^2+z^2, x=2, x=3$ sirtlar orasidagi qismi.

17.13 $\iint\limits_{(S)}\sqrt{y^2-x^2}ds,\ (S)-x^2+y^2=z^2$ konus sirtining $x^2+y^2=a^2$ silindr bilan ajratilgan qismi.

17.14 $\oiint\limits_{(S)}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}ds,\ (S)\ -\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ (a>b>c>0).$

17.15 $\oiint\limits_{(S)}\frac{ds}{(x+y+1)^2},\ (S)\ -\ x+y+z\le 1,\ x\ge 0,\ y\ge 0,\ z\ge 0$ tetraedrning chegarasi.

17.16 $\oiint\limits_{(S)}(xy+yz+zx)ds,\ (S)-z=\sqrt{x^2+y^2},\ x^2+y^2<2ax.$

17.17 $\oint\!\!\!\int_{(S)} \left(x^2+y^2\right)ds$, (S) – $\sqrt{x^2+y^2}\le z\le 1$ jism chegarasi.

17.18 $\oint\!\!\!\int_{(S)} \left(x^2+y^2+z-\frac{1}{2}\right)ds$, (S) – $2z=2-x^2-y^2$, $z\ge 0$ paraboloid qismi

17.19 $\oint\!\!\!\int_{(S)} \left(3x^2+5y^2+3z^2-2\right)ds$, (S)–$y=\sqrt{x^2+z^2}$ konusning $y=0$ va $y=b$ tekisliklar orasidagi qismi.

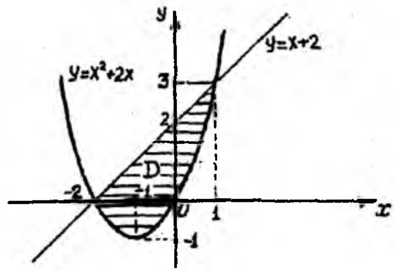
17.20 $\iint_{(S)} xyzds$, (S) – $z^2=2xy$, $z\ge 0$ konusning $x^2+y^2=a^2$ silindr ichidagi qismi.

17.21 $\iint_{(S)} |xyz|ds$, (S)–$z=x^2+y^2$, sirtning $z=1$ tekislik bilan ajratilgan qismi.

-D-
Namunaviy variant yechimi

1.21-masala. Ushbu $y \ge x^2+2x, y=x+2$ **chiziqlar bilan chegaralangan** D **soha uchun** $\iint\limits_D f(x,y)dxdy$ **ikki karrali integral takroriy integralga keltirilsin va integrallash chegaralari ikki xil tartibda qo'yilsin.**

◁ Birinchi navbatda $y \ge x^2+2x=(x+1)^2-1$ va $y=x+2$ chiziqlarning kesishish nuqtalari $M_1(-2;0)$, $M_2(1;3)$ larni topamiz va sohani chizmada tasvirlaymiz (14-chizma).



14-chizma.

14-chizmadan ko'rinadiki, D sohani tengsizliklar yordamida quyidagicha ifodalash mumkin:

$$D=\{(x,y):-2\le x\le 1,\ x^2+2x\le y\le x+2\}=\{(x,y):-1-\sqrt{y+1}\le x\le -1+\sqrt{y+1};-1\le y\le 0\}\cup$$

$$\cup\{(x,y):y-2\le x\le -1+\sqrt{y+1};0\le y\le 3\}.$$

Bu yerdan 2⁰-punktdagi 3 va 4-teoremalarga ko'ra quyidagi tengiklarni hosil qilamiz:

$$\iint\limits_D f(x,y)dxdx=\int_{-2}^{1}\left[\int_{x^2+2x}^{x+2} f(x,y)dy\right]dx=\int_{-1}^{0}\left[\int_{-1-\sqrt{y+1}}^{-1+\sqrt{y+1}} f(x,y)dx\right]dy+\int_{0}^{3}\left[\int_{y-2}^{-1+\sqrt{y+1}} f(x,y)dx\right]dy \ ▷.$$

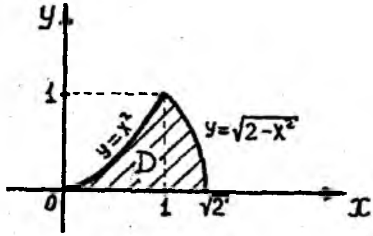
2.21-masala. Integrallash tartibini o'zgartiring.

$$\int_0^1 dx\int_0^{x^2} fdy+\int_1^{\sqrt{2}} dx\int_0^{\sqrt{2-x^2}} fdy.$$

◁Masala shartiga ko'ra

$$D=\{(x,y):0\le x\le 1,\ 0\le y\le x^2\}\cup\{(x,y):1\le x\le\sqrt{2},\ 0\le y\le\sqrt{2-x^2}\}$$

D soha 15-chizmada tasvirlangan.



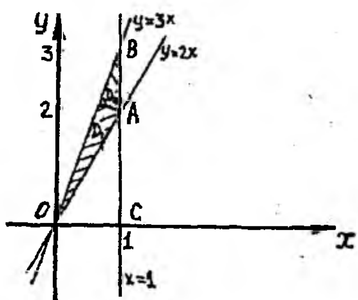
15-chizma.

15-chizmada ko‘rinadiki, D sohani quyidagicha ham ifodalash mumkin:

$$D=\left\{(x,y):\sqrt{y}\le x\le\sqrt{2-y^2},\ 0\le y\le 1\right\}\Rightarrow\int_0^1 dx\int_0^{x^2} fdy+\int_1^{\sqrt{2}} dx\int_0^{\sqrt{2-x^2}} fdy=\int_0^1 dy\int_{\sqrt{y}}^{\sqrt{2-y^2}} fdx.\triangleright$$

3.21-masala. $D=\{(x,y):\ 0\le x\le 1,\ 2x\le y\le 3x\}$ **soha uchun** $\iint\limits_D f(x,y)\,dxdy$ **integralda qutb, koordinatalariga** $x=r\cos\varphi, y=r\sin\varphi$ **o‘tib, integrallash chegaralari ikki xil tartibda qo‘yilsin.**

$\triangleleft$ Integrallash chegarasini qo‘yish uchun avval D sohani chizmada tasvirlab olamiz (16-chizma).



16-chizma.

Ikki karrali integralda o‘zgaruvchilarni almashtirish uchun birinchi navbatda akslantirish yakobianini hisoblaymiz.

$$I=\frac{D(x,y)}{D(r,\varphi)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\varphi}\\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

Undan so'ng D sohani qutb koordinatalar sistemasida ifodalaymiz, ya'ni D sohaning akslantirish natijasidagi obrazi Δ ni topamiz va (10)-formuladan foydalanamiz.

$$< AOC = arctg2,\ \ < BOC = arctg3,\ \ OA = \sqrt{1^2+2^2} = \sqrt{5},\ \ OB = \sqrt{1^2+3^2} = \sqrt{10};\ \ x = 1 \Rightarrow$$

$$\Rightarrow r\cos\varphi = 1 \Rightarrow r = \frac{1}{\cos\varphi} \Rightarrow D = \left\{(r,\varphi): arctg2 \le \varphi \le arctg3,\ \ 0 \le r \le \frac{1}{\cos\varphi}\right\} =$$

$$= D_1 \cup D_2 = \left\{(r,\varphi): arctg2 \le \varphi \le arctg3, 0 \le r \le \sqrt{5}\right\} \cup$$

$$\cup \left\{(r,\varphi): arctg\frac{1}{r} \le \varphi \le arctg3,\ \ \sqrt{5} \le r \le \sqrt{10}\right\} \Rightarrow$$

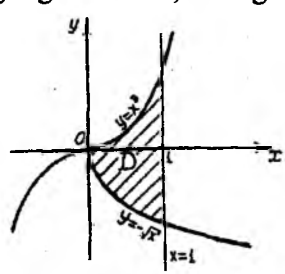
$$\iint_D f(x,y)\,dxdy = \int_{arctg2}^{arctg3} d\varphi \int_0^{\sqrt{5}} rf(r\cos\varphi, r\sin\varphi)dr = \int_0^{\sqrt{5}} dr \int_{arctg2}^{arctg3} rf(r\cos\varphi, r\sin\varphi)\,d\varphi +$$

$$+ \int_{\sqrt{5}}^{\sqrt{10}} dr \int_{arctg\frac{1}{r}}^{arctg3} rf(r\cos\varphi, r\sin\varphi)\,d\varphi \rhd$$

4.21-masala. Hisoblang.

$$I = \iint_D \left(xy - 4x^3y^3\right)dxdy;\quad D: x = 1,\ \ y = x^3,\ \ y = -\sqrt{x}.$$

$\lhd$ D sohaning shaklini chizib olamiz (17-chizma) va karrali integralni takroriy integralga keltirib, uning qiymatini hisoblaymiz:



17-chizma.

$$I = \iint_D \left(xy - 4x^3y^3\right)dxdy = \left(\left(D: \left\{(x,y): 0 \le x \le 1,\ \ -\sqrt{x} \le y \le x^3\right\}\right)\right) = \int_0^1 dx \int_{-\sqrt{x}}^{x^3} \left(xy - 4x^3y^3\right)dy =$$

$$= \int_0^1 \left(x \cdot \frac{y^2}{2} - x^3y^4\right)\Bigg|_{-\sqrt{x}}^{x^3} dx = \int_0^1 \left(\frac{x^7}{2} - x^{15} - \frac{x^2}{2} + x^5\right)dx = \left(\frac{x^8}{16} - \frac{x^{16}}{16} - \frac{x^3}{6} + \frac{x^6}{6}\right)\Bigg|_0^1 = 0 \rhd.$$

5.21-masala. Hisoblang.

$$I=\iint\limits_D y\cos xy\,dxdy;\quad D: y=\pi,\ \ y=3\pi,\ \ x=\frac{1}{2},\ \ x=1.$$

◁ **Masala shartiga ko'ra** $D=\left\{(x,y):\ \frac{1}{2}\le x\le 1,\ \ \pi\le y\le 3\pi\right\}$ **soha to'g'ri to'rtburchakdan iborat. Unda**

$$I=\iint\limits_D y\cos xy\,dxdy=\int_{\frac{1}{2}}^{1}dx\int_{\pi}^{3\pi}y\cos xy\,dy=\int_{\pi}^{3\pi}dy\int_{\frac{1}{2}}^{1}y\cos xy\,dx=\int_{\pi}^{3\pi}\left(y\cdot\frac{1}{y}\sin xy\right)\Bigg|_{\frac{1}{2}}^{1}dy=$$

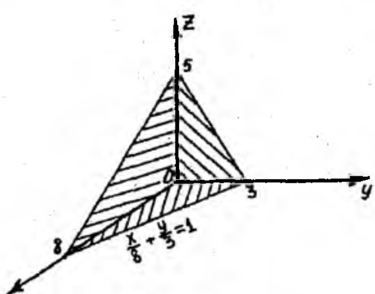
$$=\int_{\pi}^{3\pi}\left(\sin y-\sin\frac{y}{2}\right)dy=\left(-\cos y+2\cos\frac{y}{2}\right)\Bigg|_{\pi}^{3\pi}=0.\ ▷$$

6.21-masala. Hisoblang.

$$I=\iiint\limits_{(V)}\frac{dxdydz}{\left(1+\frac{x}{8}+\frac{y}{3}+\frac{z}{5}\right)^6};\qquad (V):\begin{cases}\frac{x}{8}+\frac{y}{3}+\frac{z}{5}=1,\\ x=0,y=0,z=0.\end{cases}$$

◁ **Masala shartidan ko'rinadiki (V) jism uchburchakli piramida bo'ladi va uning shakli 18-chizmada tasvirlangan. (V) jismni tengsizliklar yordamida quyidagicha yozish mumkin:**

$$(V):\ \left\{(x,y,z):\ 0\le x\le 8,\ \ 0\le y\le 3\left(1-\frac{x}{8}\right),\ \ 0\le z\le 5\left(1-\frac{x}{8}-\frac{y}{3}\right)\right\}.$$



18-chizma.

Berilgan uch karrali integralni takroriy integralga keltirish yo'li bilan hisoblaymiz:

$$I=\int_0^8 dx\int_0^{3\left(1-\frac{x}{8}\right)} dy\int_0^{5\left(1-\frac{x}{8}-\frac{y}{3}\right)}\frac{dz}{\left(1+\frac{x}{8}+\frac{y}{3}+\frac{z}{5}\right)^6}=\int_0^8 dx\int_0^{3\left(1-\frac{x}{8}\right)}\left[-\frac{1}{\left(1+\frac{x}{8}+\frac{y}{3}+\frac{z}{5}\right)^5}\Bigg|_0^{5\left(1-\frac{x}{8}-\frac{y}{3}\right)}\right]dy=$$

$$=\int_0^8 dx\int_0^{3\left(1-\frac{x}{8}\right)}\left[\frac{1}{\left(1+\frac{x}{8}+\frac{y}{3}\right)^5}-\frac{1}{32}\right]dy=\int_0^8\left[-\frac{3}{4\left(1+\frac{x}{8}+\frac{y}{3}\right)^4}-\frac{y}{32}\right]_0^{3\left(1-\frac{x}{8}\right)}dx=$$

$$=\int_0^8\left[-\frac{3}{64}-\frac{3}{32}\left(1-\frac{x}{8}\right)+\frac{3}{4\left(1+\frac{x}{8}\right)^4}\right]dx=\left[-\frac{3}{64}x-\frac{3}{32}\left(x-\frac{x^2}{16}\right)-\frac{2}{\left(1+\frac{x}{8}\right)^3}\right]_0^8=$$

$$=-\frac{3}{8}-\frac{3}{32}(8-4)-\frac{2}{8}+2=-\frac{3}{8}-\frac{3}{8}-\frac{2}{8}+2=1.\triangleright$$

Izoh. Agar 6.21-misolda o‘zgaruvchilarni almashtirsak, ya’ni $x=8u,\ y=3v,\ z=5w$ almashtirish bajarsak, integralni hisoblash ancha yengillashadi.

Bunda yakobian $\dfrac{D(x,y,v)}{D(x,y,w)}=\begin{vmatrix}8&0&0\\0&3&0\\0&0&5\end{vmatrix}=120$ bo‘lib,

(V) jism ushbu $(\Delta)=\{(u,\ v,\ w):\ 0\le u\le 1,\ 0\le v\le 1-u,\ 0\le w\le 1-u-v\}$ jismga akslanadi va o‘zgaruvchilarni almashtirish formulasiga ko‘ra quyidagilarga ega bo‘lamiz.

$$I=120\int_0^1 du\int_0^{1-u} dv\int_0^{1-u-v}\frac{dw}{(1+u+v+w)^6}.$$

Bu integralni hisoblab, $I=1$ ekanligini tekshirish qiyin emas.

7.21-masala. Quyidagi

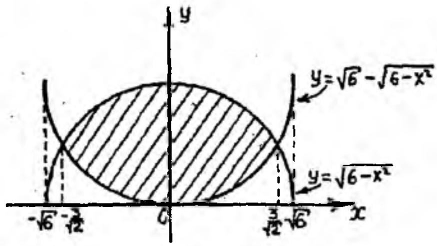
$$y=\sqrt{6-x^2},\ y=\sqrt{6}-\sqrt{6-x^2};$$

chiziqlar bilan chegaralangan shaklning yuzasi topilsin.

◁ $\begin{cases} y=\sqrt{6-x^2} \\ y=\sqrt{6}-\sqrt{6-x^2} \end{cases}$ **sistemani echamiz va bu chiziqlarni kesishish nuqtalarini topamiz.**

$$\sqrt{6-x^2}=\sqrt{6}-\sqrt{6-x^2} \Rightarrow x_1=-\frac{3}{\sqrt{2}};\ x_2=\frac{3}{\sqrt{2}}.$$

Berilgan chiziqlar bilan chegaralangan D sohaning shaklini chizamiz (19-chizma) va bu sohaning yuzasini (15)-formula yordamida hisoblaymiz.



19-chizma.

$$S=\iint_D dxdy=\int_{-\frac{3}{\sqrt{2}}}^{3/\sqrt{2}}dx\int_{\sqrt{6}-\sqrt{6-x^2}}^{\sqrt{6-x^2}}dy=2\cdot\int_0^{\frac{3}{\sqrt{2}}}dx\int_{\sqrt{6}-\sqrt{6-x^2}}^{\sqrt{6-x^2}}dy=2\int_0^{3/\sqrt{3}}\left(2\sqrt{6-x^2}-\sqrt{6}\right)dx=$$

$$\left(\left(\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+c,\ \text{formuladan foydalanamiz}\right)\right)=$$

$$=2\left(x\sqrt{6-x^2}+6\arcsin\frac{x}{\sqrt{6}}-\sqrt{6}x\right)\Bigg|_0^{\frac{3}{\sqrt{2}}}=2\cdot\left(\frac{3}{\sqrt{2}}\cdot\sqrt{\frac{3}{2}}+6\arcsin\frac{\sqrt{3}}{2}-3\sqrt{3}\right)=$$

$$=2\cdot\left(2\pi-\frac{3\sqrt{3}}{2}\right)=4\pi-3\sqrt{3}\quad \text{kv.birl.} \triangleright$$

8.21-masala. Quyidagi

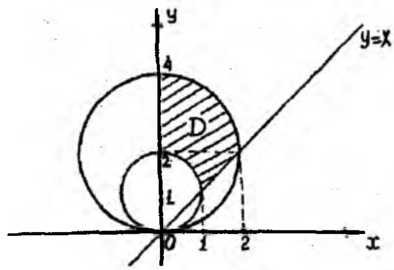
$$y^2-2y+x^2=0,\ y^2-4y+x^2=0,\ y=x,\ x=0.$$

chiziqlar bilan chegaralangan shaklning yuzasi topilsin.

$$y^2-2y+x^2=0 \;\Rightarrow\; x^2+(y-1)^2=1;$$

◁ $$y^2-4y+x^2=0 \;\Rightarrow\; x^2+(y-2)^2=4.$$

Bu tengliklardan foydalanib, berilgan chiziqlar bilan chegaralangan D sohaning chizmasini osongina chizamiz (20-chizma).



20-chizma.

D sohani tengsizliklar yordamida yozib olamiz:

$$D=\left\{(x,y):\ \sqrt{2y-y^2}\le x\le y;\ 1\le y\le 2\right\}\cup\left\{(x,y):\ 0\le x\le\sqrt{4y-y^2};\ 2\le y\le 4\right\}.$$

Bu munosabatdan foydalanib, D sohaning yuzasini hisoblaymiz:

$$S=\iint_D dxdy=\int_1^2 dy\int_{\sqrt{2y-y^2}}^{y}dx+\int_2^4 dy\int_0^{\sqrt{4y-y^2}}dx=\int_1^2\left(y-\sqrt{2y-y^2}\right)dy+\int_2^4\sqrt{4y-y^2}\,dy=$$

$$\int_1^2\left(y-\sqrt{1-(y-1)^2}\right)dy+\int_2^4\sqrt{4-(y-2)^2}\,dy=\left(\frac{y^2}{2}-\frac{y-1}{2}\sqrt{1-(y-1)^2}-\frac{1}{2}\arcsin y-1\right)\Bigg|_1^2+$$

$$+\left(\frac{y-2}{2}\sqrt{4-(y-2)^2}+2\arcsin\frac{y-2}{2}\right)\Bigg|_2^4=\left(2-\frac{1}{2}\arcsin 1\right)-\frac{1}{2}+2\arcsin 1=\frac{3}{2}+\frac{3}{2}\arcsin 1=$$

$$=\frac{3}{2}\left(1+\frac{\pi}{2}\right)=\frac{3(2+\pi)}{4}\ \text{kv. birlik.}$$ ▷

9.21-masala. Zichligi $\rho=\dfrac{x}{y}$ bo'lgan

$$D=\left\{(x,y):\ 1\le\frac{x^2}{4}+\frac{y^2}{16}\le 5;\ x\ge 0;\ y\ge 2x\right\};$$

plastinkaning massasi topilsin.

◁ Plastinkaning massasini (17)-formula, ya'ni $M=\iint_D \rho(x,y)dxdy$ formuladan foydalanib, topamiz. Bu integralni hisoblashni yengillashtirish uchun umumlashgan qutb koordinatalar sistemasiga o'tamiz:

$$\begin{matrix} x=2r\cos\varphi \\ y=4r\sin\varphi \end{matrix} \Rightarrow |I|=\frac{D(x,y)}{D(r,\varphi)}=\begin{vmatrix} 2\cos\varphi & -2r\sin\varphi \\ 4\sin\varphi & 4r\cos\varphi \end{vmatrix}=8r;$$

$$\left.\begin{matrix} y=2x \\ x=0 \end{matrix}\right\} \Rightarrow \begin{cases} 4r\sin\varphi=4r\cos\varphi \\ 2r\cos\varphi \end{cases} \Rightarrow \begin{cases} \varphi=\frac{\pi}{4} \\ \varphi=\pi/2 \end{cases} \Rightarrow$$ Bajarilgan almashtirishdan so'ng berilgan D plastinkaga ushbu $\Delta=\left\{(r,\varphi): \frac{\pi}{4}\le\varphi\le\pi/2;\ 1\le r\le 5\right\}$ plastinka akslanadi $\Rightarrow M=\iint_D \rho(x,y)dxdy=$

$$=\iint_\Delta 8r\rho(2r\cos\varphi,\ 4r\sin\varphi)drd\varphi=8\int_{\frac{\pi}{4}}^{\pi/2} d\varphi\int_1^5 r\cdot\frac{2r\cos\varphi}{4r\sin\varphi}dr=4\int_{\pi/4}^{\pi/2}\frac{\cos\varphi}{\sin\varphi}\cdot\left.\frac{r^2}{2}\right|_1^5 d\varphi=$$

$$=48\ln|\sin\varphi|\Big|_{\frac{\pi}{4}}^{\pi/2}=48\left(\ln\sin\frac{\pi}{2}-\ln\sin\frac{\pi}{4}\right)=48\left(-\ln\frac{1}{\sqrt{2}}\right)=48\cdot\ln\sqrt{2}=24\ln 2. \triangleright$$

10.21-masala. $r=a(1+\cos\varphi)$ **kardioda bilan chegaralangan plastinkaning 0x va 0y o'qlariga nisbatan** I_x **va** I_y **inersiya momentlari topilsin.** $(\rho=1)$

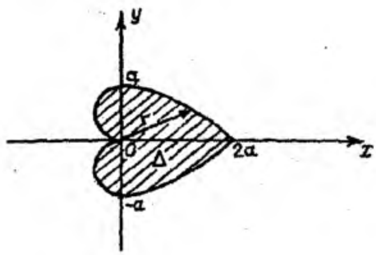
◁ Berilgan plastinkaning 0x va 0y o'qlariga nisbatan inersiya momentlari (20)-formulalarga ko'ra

$$I_x=\iint_D \rho\cdot y^2dxdy=\iint_D y^2dxdy \quad \text{va} \quad I_y=\iint_D \rho x^2dxdy=\iint_D x^2dxdy$$ tengliklar yordamida topiladi.

Bu formulalar qutb koordinatalar sistemasida quyidagi ko'rinishga keladi:

$$I_x=\iint_\Delta r^3\sin^2\varphi drd\varphi \quad \text{va} \quad I_y=\iint_\Delta r^3\cos^2\varphi drd\varphi,$$

bu yerda $\Delta=\left\{(r,\varphi):\ -\frac{\pi}{2}\le\varphi\le\pi;\ 0\le r\le a(1+\cos\varphi)\right\}$ (21-chizma).



21-chizma.

$$I_x = \iint_\Delta r^3 \sin^2 \varphi dr d\varphi = \int_{-\pi}^{\pi} \sin^2 \varphi d\varphi \int_0^{a(1+\cos\varphi)} r^3 dr = 32a^4 \int_0^{\pi} \sin^2 \frac{\varphi}{2} \cdot \cos^{10} \frac{\varphi}{2} d\varphi =$$

$$= \left(\begin{pmatrix} \sin\frac{\varphi}{2} = z \Rightarrow \varphi = 2\arcsin z \Rightarrow d\varphi = \frac{2dz}{\sqrt{1-z^2}} \\ \varphi = 0 \Rightarrow z = 0 \\ \varphi = \pi \Rightarrow z = 1 \end{pmatrix} \right) = 64a^4 \int_0^1 z^2 \cdot (1-z^2)^{9/2} dz =$$

$$= \left(\left(z^2 = t \Rightarrow dz = \frac{1}{2} t^{-\frac{1}{2}} dt \right) \right) = 32a^4 \int_0^1 t^{\frac{1}{2}} \cdot (1-t)^{\frac{9}{2}} dt = 32a^4 \cdot \int_0^1 t^{\frac{3}{2}-1} \cdot (1-t)^{\frac{11}{2}-1} dt =$$

$$= 32a^4 \cdot B\left(\frac{3}{2}; \frac{11}{2}\right) = 32a^4 \cdot \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{11}{2}\right)}{\Gamma(7)} = \frac{21\pi a^4}{32};$$

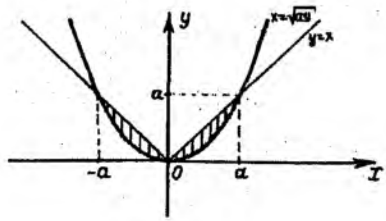
$$I_y = \iint_\Delta r^3 \cos^2 \varphi dr d\varphi = \iint_\Delta r^3 dr d\varphi - I_x = \int_{-\pi}^{\pi} d\varphi \int_a^{a(1+\cos\varphi)} r^3 dr - I_x =$$

$$= 8a^2 \int_0^{\pi} \cos^8 \frac{\varphi}{2} - d\varphi - I_x = 8a^4 \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{9}{2}\right)}{\Gamma(5)} - I_x = \frac{49\pi a^4}{32}.$$

Demak, $I_x = \frac{21}{32}\pi a^4$, $I_y = \frac{49}{32}\pi a^4$.▷

11.21-masala. Quyida ko'rsatilgan sirtning yuzasi topilsin. (S): $y^2 + z^2 = x^2$ **sirtning** $x^2 = ay$ **sirt bilan ajratilgan qismi.**

◁ Yuzasini topishimiz kerak bo'lgan sirtning 0xy tekisligidagi proyeksiyasi 22-chizmada tasvirlangan.



22-chizma.

Sirtning yuzasini (16)-formuladan foydalanib, hisoblaymiz. Agar $D=\{(x,y):\ 0\le y\le a,\ y\le x\le\sqrt{ay}\}$ desak, unda $S=4\cdot\iint\limits_D\sqrt{1+(z_x')^2+(z_y')^2}\,dxdy$ bo'ladi, chunki, $y^2+z^2=x^2$ konus sirtning Oxy tekislikka nisbatan simmetrik joylashgan $z\ge 0$ va $z\le 0$ tengsizliklar bilan tasvirlanadigan qismlari bor, ham D soha (S) sirtning 0xy tekislikdagi proyeksiyasining yarim bo'lagi.

$$z=\sqrt{x^2-y^2}\Rightarrow z_x'=\frac{x}{\sqrt{x^2-y^2}},\ z_y'=-\frac{y}{\sqrt{x^2-y^2}}\Rightarrow\sqrt{1+(z_x')^2+(z_y')^2}=\frac{\sqrt{2}x}{\sqrt{x^2-y^2}}\Rightarrow$$

$$S=4\sqrt{2}\int_0^a dy\int_y^{\sqrt{ay}}\frac{xdx}{\sqrt{x^2-y^2}}=4\sqrt{2}\int_0^a\sqrt{x^2-y^2}\Big|_y^{\sqrt{ay}}dy=4\sqrt{2}\int_0^a\sqrt{ay-y^2}\,dy=$$

$$4\sqrt{2}\cdot\int_0^a\sqrt{\frac{a^2}{4}-\left(y-\frac{a}{2}\right)^2}\,dy=4\sqrt{2}\cdot\left[\frac{y-\frac{a}{2}}{2}\sqrt{ay-y^2}+\frac{a^2}{8}\arcsin\frac{y-\frac{a}{2}}{\frac{a}{2}}\right]_0^a=\frac{\pi a^2}{\sqrt{2}}.$$

Shunday qilib, $S=\dfrac{\pi a^2}{\sqrt{2}}$ kv. birl. ▷

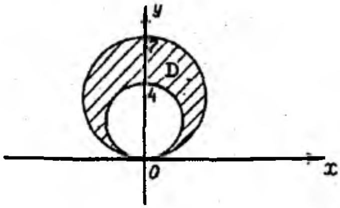
12.21-masala. Quyidagi

$$x^2+y^2=4y,\ x^2+y^2=7y,\ z=\sqrt{x^2+y^2},\ z=0;$$

sirtlar bilan chegaralangan jismning hajmi hisoblansin.

◁ Jismning hajmini ikki karrali integral yordamida (14)-formuladan foydalanib, hisoblaymiz:

$$V=\iint\limits_D f(x,y)\,dxdy=\iint\limits_D\sqrt{x^2+y^2}\,dxdy,$$



23-chizma.

bu yerda D soha $x^2+y^2=4y$ va $x^2+y^2=7y$ aylanalar bilan chegaralangan (23-chizma).

Hisoblashni yengillashtirish uchun qutb koordinatalar sistemasiga o'tamiz: $x=r\cos\varphi,\ y=r\sin\varphi$.

Unda D soha ushbu $\Delta=\{(r,\varphi):\ 0\le\varphi\le\pi,\ 4\sin\varphi\le r\le 7\sin\varphi\}$ sohaga akslanadi va hajm osongina hisoblanadi.

$$V=\iint_{\Delta} r^2 dr d\varphi=\int_0^{\pi} d\varphi\int_{4\sin\varphi}^{7\sin\varphi} r^2 dr=2\cdot\int_0^{\pi/2} d\varphi\int_{4\sin\varphi}^{7\sin\varphi} r^2 dr=2\int_0^{\pi/2}\frac{r^3}{3}\Bigg|_{4\sin\varphi}^{7\sin\varphi} d\varphi=186\int_0^{\pi/2}\sin^3\varphi d\varphi=$$

$$=186\int_0^{\frac{\pi}{2}}(\cos^2\varphi-1)d(\cos\varphi)=186\cdot\left(\frac{\cos^3\varphi}{3}-\cos\varphi\right)\Bigg|_0^{\pi/2}=186\cdot\frac{2}{3}=124.$$

Demak, $V=124$ kub. birlik. ▷

13.21-masala. Quyidagi $z=2-18(x^2+y^2)$ va $z=2-36y$ sirtlar bilan chegaralangan jismning hajmi hisoblansin.

◁ Bu jismning hajmini ham (14)-formuladan foydalanib, hisoblaymiz. Avval jismning 0xy tekisligidagi proyeksiyasi D ni topamiz:

$$\begin{cases} z=2-18(x^2+y^2) \\ z=2-36y \end{cases}\Rightarrow 2-18(x^2+y^2)=2-36y\Rightarrow x^2+y^2-2y=0\Rightarrow x^2+(y-1)^2=1\Rightarrow$$

$$D=\{(x,y):x^2+(y-1)^2=1\}.$$

Demak,

$$V=\iint_D\left[2-18(x^2+y^2)-(2-36y)\right]dxdy=-18\iint_D(x^2+y^2-2y)dxdy=-18\iint_D\left[x^2+(y-1)^2-1\right]dxdy=$$

$$=\left(\begin{cases} x=r\cos\varphi\Rightarrow|y|=r,\ \Delta=\{(r,\varphi):0\le\varphi\le 2\pi,\ 0\le r\le 1\} \\ y-1=r\sin\varphi \end{cases}\right)=-18\int_0^{2\pi}d\varphi\int_0^1 r(r^2-1)dr=$$

$$=-18\int_0^{2\pi}\left(\frac{r^4}{4}-\frac{r^2}{2}\right)\Bigg|_0^1 d\varphi=\frac{18}{4}\cdot 2\pi=9\pi \text{ kub. birlik.}$$ ▷

14.21-masala. Quyidagi

$$x^2+y^2+z^2=2az,\ x^2+y^2=z^2,\ x^2+y^2=\frac{1}{3}z^2$$

sirtlar bilan chegaralangan jismning hajmini uch karrali integral yordamida hisoblang.

◁ Izlangan hajmni topish uchun avval (13)-formulalardan foydalanib,

$$\begin{cases} x=\rho\cos\varphi\cdot\sin\psi, \\ y=\rho\sin\varphi\cdot\sin\psi, \\ z=\rho\cos\psi. \end{cases}$$

akslantirish yordamida sferik koordinatalar sistemasiga o'tamiz. Bunda yakobian

$$\frac{D(x,y,z)}{D(\rho,\varphi,\psi)}=\rho^2\cdot\sin\psi.$$

bo'lib, (V) jism (Δ) jismga akslanadi. Izlangan hajmni hisoblash uchun

$$V=\iiint\limits_{(V)}dxdydz=\iiint\limits_{(\Delta)}\rho^2\cdot\sin\psi d\rho d\varphi d\psi.$$

formuladan foydalanamiz.

Berilgan sirtlarning tenglamalarini sferik koordinatalarida yozamiz.

$$\{x^2+y^2+z^2=2az\}\to\{\rho=2a\cos\psi\},$$

$$\{x^2+y^2=z^2\}\to\{tg^2\psi=1\}\Rightarrow\left\{\psi=\frac{\pi}{4}\right\},$$

$$\left\{x^2+y^2=\frac{1}{3}z^2\right\}\to\left\{tg^2\psi=\frac{1}{3}\right\}\Rightarrow\left\{\psi=\frac{\pi}{6}\right\},$$

Demak, (V) jism quyidagicha bo'ladi.

$$(\Delta)=\left\{(\rho,\varphi,\psi):\ 0\le\varphi\le2\pi,\ \frac{\pi}{6}\le\psi\le\frac{\pi}{4},\ 0\le\rho\le2a\cos\psi\right\}$$

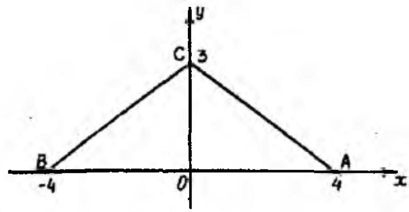
Shunday qilib, izlangan hajmni osongina hisoblaymiz:

$$V=\int\limits_0^{2\pi}d\varphi\int\limits_{\pi/6}^{\pi/4}\sin\psi d\psi\int\limits_0^{2a\cos\psi}\rho^2d\rho=\frac{16\pi a^3}{3}\int\limits_{\pi/6}^{\pi/4}\cos^3\psi\cdot\sin\psi d\psi=\frac{5}{12}\pi a^3\quad\text{(kub.birl)}\ \triangleright.$$

15.21-masala. Egri chiziqli integral hisoblansin.

$$\int\limits_{\gamma} xyds, \quad \gamma : 3|x| + 4|y| = 12, \quad y \geq 0.$$

◁ γ yoy 0xy tekislikda ABS siniq chiziqni beradi. (24-chizma).



24-chizma.

$$AC: 3x + 4y = 12, 0 \leq x \leq 4; \qquad AC: y = -\frac{3}{4}x + 3, 0 \leq x \leq 4.$$

$$\Rightarrow$$

$$BC: -3x + 4y = 12, -4 \leq x \leq 0; \qquad BC: y = \frac{3}{4}x + 3, -4 \leq x \leq 0.$$

Birinchi tur egri chiziqli integralning qiymatini (24)-formuladan foydalanib, hisoblaymiz:

$$\int\limits_{AB} xyds = \int\limits_{AC} xyds + \int\limits_{BC} xyds = \int\limits_0^4 x \cdot \left(-\frac{3}{4}x + 3\right) \cdot \sqrt{1 + \left(-\frac{3}{4}\right)^3} dx + \int\limits_{-4}^{0} x\left(\frac{3}{4}x + 3\right) \cdot \sqrt{1 + \left(\frac{3}{4}\right)^2} dx =$$

$$= \int\limits_0^4 x\left(-\frac{3}{4}x + 3\right) \cdot \frac{5}{4} dx + \int\limits_{-4}^{0} x\left(\frac{3}{4}x + 3\right)\frac{5}{4} dx = \frac{5}{4} \cdot \left(\frac{3x^2}{2} - \frac{x^3}{4}\right)\Bigg|_0^4 + \frac{5}{4} \cdot \left(\frac{x^3}{4} + \frac{3x^2}{2}\right)\Bigg|_{-4}^{0} =$$

$$= \frac{5}{4}(24 - 16) + \frac{5}{4}(16 - 24) = 0 \ \triangleright$$

16.21-masala. Ushbu

$$\left(\frac{x}{\sqrt{x^2 + y^2}} + y\right)dx + \left(\frac{y}{\sqrt{x^2 + y^2}} + x\right)dy.$$

ifodaning biror $F(x, y)$ funksiyaning to'liq differensiali bo'lishi yoki bo'lmasligini aniqlang. Agar u to'liq differensiali bo'lsa, $F(x, y)$ funksiyani toping.

◁ $P(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}} + y$ va $Q(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}} + x,$

deb belgilasak, $Pdx+Qdy$ ifodaning to'liq differensiali bo'lishi uchun (38)-tenglik, ya'ni

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x};$$

munosabat bajarilishi kerak. Shuni tekshiramiz:

$$\frac{\partial P}{\partial y}=\left(\frac{x}{\sqrt{x^2+y^2}}+y\right)'_y=-\frac{xy}{\left(\sqrt{x^2+y^2}\right)^3}+1=\frac{\partial Q}{\partial x}\Rightarrow$$ Berilgan ifoda

biror $F(x,y)$ funksiyaning to'liq differensiali. $F(x,y)$ funksiyani topish uchun (39)-formuladan foydalanamiz. Soddalik uchun $x_0=0$, $y_0=1$ deb olamiz.

$$F(x,y)=\int_0^x P(x,y)dx+\int_1^y Q(0,y)dy=\int_0^x\left(\frac{x}{\sqrt{x^2+y^2}}+y\right)dx+\int_1^y dy+c=\sqrt{x^2+y^2}+xy+c.$$

Demak, $F(x,y)=\sqrt{x^2+y^2}+xy+c.$ ▷

17.21-masala. Quyidagi I-tur sirt integrali hisoblansin.

$\iint\limits_{(S)}|xyz|ds$, $(S)-z=x^2+y^2$, (S): $z=x^2+y^2$ sirtning $z=1$ **tekislik bilan ajratilgan qismi.**

◁ $z=x^2+y^2$ paraboloid aylanma sirtdir, unda $z\geq 0$.

Demak, integral ostidagi funksiya

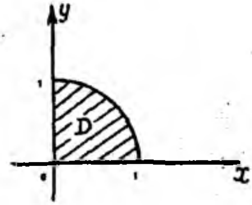
$$f(x,y,z)=|xyz|=z\cdot|xy|$$

ko'rinishda yozilishi mumkin. To'rtta oktantda olingan $M_1(x,y,z)$, $M_2(-x,y,z)$, $M_3(-x,-y,z)$, $M_4(x,-y,z)$ nuqtalarda bu funksiyaning qiymati o'zaro teng.

Shuning uchun integrallashni I-oktantda (unda $f(x,y,z)=xyz$) olib boramiz va natijani 4 ga ko'paytiramiz.

$$I=4\cdot I_1=4\cdot\iint\limits_{(S_1)}xyzds=4\cdot\iint\limits_{D}xyz\cdot\sqrt{1+\left(z'_x\right)^2+\left(z'_y\right)^2}dxdy,$$

bu yerda (S_1) sirt (S) sirtning I-oktantdagi qismi, D esa (S_1) ning Oxu tekisligidagi proyeksiyasi (25-chizma).



25-chizma.

$$z_x' = 2x,\quad z_y' = 2y \;\Rightarrow I = 4\iint\limits_D xy\left(x^2+y^2\right)\sqrt{1+4x^2+4y^2}\,dxdy = \left(\left(\begin{array}{ll} x = r\cos\varphi & \Rightarrow\; 0 \le r \le 1 \\ y = r\sin\varphi & \Rightarrow\; 0 \le \varphi \le \dfrac{\pi}{2} \end{array}\right)\right) =$$

$$= 4\int_0^1 r^5\cdot\sqrt{1+4r^2}\,dr\int_0^{\pi/2}\sin\varphi\cdot\cos\varphi\, d\varphi = 2\cdot\int_0^1 r^5\cdot\sqrt{1+4r^2}\cdot\sin^2\varphi\Big|_0^{\frac{\pi}{2}}dr = 2\int_0^1 r^5\cdot\sqrt{1+4r^2}\,dr =$$

$$= \left(\left(\begin{array}{l} \sqrt{1+4r^2} = t,\ \text{desak},\quad r=0 \Rightarrow t=1,\ r^2 = \dfrac{1}{4}\left(t^2-1\right) \\ \qquad\qquad\qquad\qquad r=1 \Rightarrow t=\sqrt{5},\ rdr = \dfrac{1}{4}tdt \end{array}\right)\right) = 2\int_1^{\sqrt{5}}\left(\frac{1}{4}\left(t^2-1\right)\right)^2\cdot t\cdot\frac{1}{4}tdt =$$

$$= \frac{1}{32}\int_1^{\sqrt{5}} t^2\cdot\left(t^2-1\right)^2dt = \frac{1}{420}\left(125\sqrt{5}-1\right).\triangleright$$

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Bahodir Allaberdiyevich SHOIMQULOV
Tohir Tursunbayevich TUYCHIYEV
Davlatboy Halillayevich DJUMABOYEV

MATEMATIK ANALIZDAN MUSTAQIL ISHLAR

Oliy texnika o'quv yurtlarining bakalavriat ta'lim yo'nalishi talabalari uchun o'quv qo'llanma

«O'zbekiston faylasuflari milliy jamiyati» nashriyoti.
100029, Toshkent shahri, Matbuotchilar ko'chasi, 32-uy.
Tel: 236-55-79; faks: 239-88-61.

Nashr uchun mas'ul *M. Tursunova*
Muharrir *I. Karimov*
Texnik muharrir *A. Berdiyeva*
Musahhih *H. Zokirova*
Sahifalovchi *Z. Boltayev*

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