MIRZO ULUG'BEK NOMIDAGI
O'ZBEKISTON MILLIY UNIVERSITETI

100 YIL

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IQTISODCHILAR UCHUN OLIY MATEMATIKA FANIDAN MA'RUZA VA MASHQLAR

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(O‘quv qo‘llanma)
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Tekislikda analitik geometriya

1-mavzu. Koordinatalar sistemasi va analitik geometriya bo'yicha sodda masalalar

1.1. Tekislikda dekart va qutb koordinatalari sistemasi

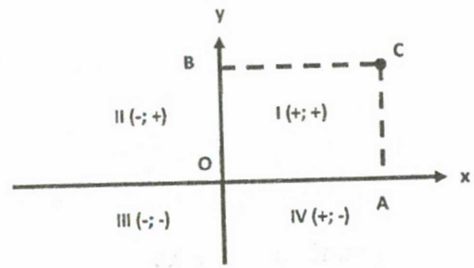
Bitta boshlang'ich nuqtasi 0 ga, bir xil masshtab birligiga ega ikki perpendikulyar Ox, Oy o'qlari tekislikda to'g'ri burchakli koordinatalar sistemasini (yoki dekart koordinatalar sistemasi) tashkil etadi.

Ox o'qi abssissa o'qi, Oy o'qi ordinata o'qi deyiladi, bu o'qlar birgalikda koordinata o'qlari deyiladi. O'qlar kesishgan O nuqta koordinata boshi deyiladi. O'qlar joylashgan tekislik koordinatalar tekisligi deyiladi va Oxy bilan belgilanadi.

Agar C tekislikdagi ixtiyoriy nuqta bo'lsa undan Ox va Oy o'qlari mos ravishda CA va CB perpendikulyarni tushuramiz. C nuqtaning dekart koordinatalari deb OA, OB yo'naltirilgan kesmalar uzinliklariga aytiladi x=OA, y=OB.

x va y koordinatalar C nuqtaning abssissasi va ordinatasi deyiladi, C(x,y) ko'rinishida yoziladi.

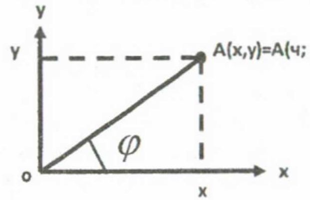
Koordinata o'qlari tekislikni to'rtta chorakka ajratadi, ular I, II, III, IV rim raqamlari bilan belgilanadi.



Endi qutb koordinatalar sistemasi deb ataluvchi koordinatalar sistemasi bilan tanishamiz.

Qutb deb ataluvchi O nuqta, undan chiqarilgan boshlang'ich nurni qaraymiz. Agar tekislikda biror A nuqta berilsa, boshlang'ich nurni soat strelkasi yo'nalishiga qarama-qarshi shunday burchakka buramiz, boshlang'ich nur A nuqtadan o'tsin. Qutb nuqta O dan A gacha masofa qutb radiusi deyiladi va r harfi bilan belgilanadi. Boshlang'ich nur A dan o'tishi uchun burilgan burchak qutb burchagi deyiladi va φ harfi bilan belgilanishi mumkin. Bunda $0 \le r < +\infty, 0 \le \varphi < 2\pi$. Agar qutb burchagi soat strelkasi yo'nalishi bo'yicha olinsa, qutb burchagi manfiy hisoblanadi.

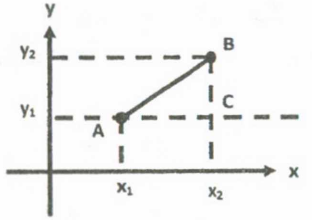
r va φ qutb koordinatalari deyiladi va $A(r;\varphi)$ tarzida yoziladi. Dekart va qutb koordinatalari orasidagi bog‘lanishni topish uchun ikkala koordinata sistemasi boshini bitta nuqtaga qo‘yamiz, boshlang‘ich nurni abssissa musbat yo‘nalishi bo‘yicha yo‘naltramiz.



Tekislikda A nuqta x,y dekart koordinatalariga va r,φ qutb koordinatalariga ega. OA gipotenuzali to‘g‘ri burchakli uchburchakdan $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ va $\begin{cases} r = \sqrt{x^2+y^2} \\ \operatorname{tg}\varphi = \frac{y}{x} \end{cases}$ formulaga ega bo‘lamiz. Bunda $0 \le \varphi < 2\pi$ ekanligidan $\operatorname{tg}\varphi = \frac{y}{x}$ ikki xil qiymat qabul qilishi mumkin. Ulardan berilgan nuqtaning koordinatalariga mosi tanlab olinadi. Masalan, A(1;1) nuqta uchun $\varphi=\frac{\pi}{4}$, A(−1;−1) nuqta uchun esa $\varphi=\frac{5\pi}{4}$ olinadi. Buni aniqlashda A(1;1) 1-chorakda, A(−1;−1) esa – 3-chorakda joylashganligini bilish kifoya.

1.2. Tekislikda ikki nuqta orasidagi masofa

Dastlab, tekislikda dekart koordinatalari bilan berilgan $A(x_1;y_1)$, $B(x_2;y_2)$ nuqtalar orasidagi d masofani aniqlaymiz. Bu nuqtalardan son o‘qlariga yordamchi parallel to‘g‘ri chiziqlar o‘tkazsak, to‘g‘ri burchakli ABC uchburchak hosil bo‘ladi.

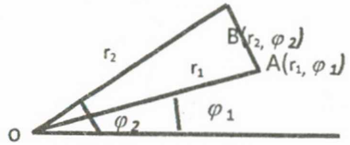


$|AC|=|x_2-x_1|$, $|BC|=|y_2-y_1|$ ekanligi Pifagor teoremasi yordamida $|AB|^2=(x_2-x_1)^2+(y_2-y_1)^2$ bo‘lishini bildiradi, ya’ni $A(x_1,y_1)$, $B(x_2,y_2)$

nuqtalar orasidagi masofa: $|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ formula yordamida topiladi.

Masalan, A(−5;2), B(3;−4) nuqtalar orasidagi masofa: $d=\sqrt{(3+5)^2+(-4-2)^2}=\sqrt{8^2+6^2}=\sqrt{100}=10$.

Endi qutb koordinatalar sistemasida berilgan $A(r_1;\ \varphi_1)$, $B(r_2,\ \varphi_2)$ nuqtalar orasidagi masofani topamiz.



OAB uchburchakda $\angle AOB=\varphi_2-\varphi_1$, cosinuslar teoremasiga ko’ra: $|AB|^2=|OA|^2+|OB|^2-2\cdot|OA|\cdot|OB|\cdot\cos\angle AOB$, ya’ni

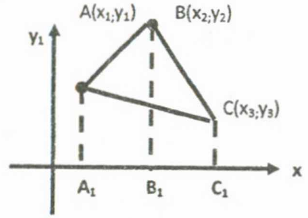
$$|AB|=\sqrt{ч_1^2+ч_2^2-2ч_1\cdot ч_2\cdot\cos(\varphi_2-\varphi_1)}.$$

Masalan, $A(5;\frac{\pi}{4})$, $B(8;\frac{\pi}{12})$ nuqtalar orasidagi masofa

$$d=\sqrt{5^2+8^2-2\cdot5\cdot8\cdot\cos(-\frac{\pi}{3})}=\sqrt{25+64-40}=\sqrt{49}=7.$$

1.3. Uchbirchak yuzi

Dekart koordinatalar sistemasida bir to‘g‘ri chiziqda yotmagan $A(x_1;y_1)$, $B(x_2;y_2)$, $C(x_3;y_3)$ nuqtalar berilgan bo‘lsin. Bu nuqtalar abssissalari (ox o‘qiga proyeksiyalari)ni Ox o‘qida A_1, B_1, C_1 deb belgilaymiz.



U holda: $S_{\Delta ABC}=S_{AA_1B_1B}+S_{BB_1C_1C}-S_{AA_1C_1A}$, ekanligi aniq. Tenglikning o‘ng tomonidagi trapetsiyalar yuzalarini topamiz:

$$S_{A1ABB1}=\frac{|A_1A|+|B_1B|}{2}\cdot|A_1B_1|=\frac{1}{2}(y_1+y_2)(x_2-x_1),$$

$$S_{B1BCC1}=\frac{|B_1B|+|C_1C|}{2}\cdot|B_1C_1|=\frac{1}{2}(y_2+y_3)(x_3-x_2),$$

$$S_{A1ACC1}=\frac{|A_1A||C_1C|}{2}\cdot|A_1C_1|=\frac{1}{2}(y_1+y_3)(x_3-x_1).$$

Demak, A, B, C nuqtalar ixtiyoriy joylashganligidan:

$$S_{\Delta ABC}=\frac{1}{2}\left|(y_1+y_2)(x_2-x_1)+(y_3+y_2)(x_3-x_2)-(y_3+y_1)(x_3-x_1)\right|,$$ yoki

$$S_{\Delta ABC}=\frac{1}{2}\left|(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)\right|$$

formulani hosil qilamiz. Masalan, uchlari A(2;−1), B(3;2), C(−2;5) nuqtalarda bo‘lgan uchburchak yuzi:

$$S_\Delta=\frac{1}{2}\cdot|(3-2)(5+1)-(-2-2)(2+1)|=\frac{1}{2}\cdot|6+12|=9\ (kv.b).$$

Qutb koordinatalar sistemasida, bir to‘g‘ri chiziqda yotmagan $A(r_1,\varphi_1), B(r_2,\varphi_2), C(r_3,\varphi_3)$ nuqtalarni qaraymiz. $S_{ABC}=S_{OAB}+S_{OBC}-S_{OAC}$ ekanligidan, ikki tomoni va ular orasidagi burchagiga ko‘ra:

$$S_{OAB}=\frac{1}{2}|OA|\cdot|OB|\cdot\sin\angle OAB=\frac{1}{2}r_1\cdot r_2\cdot\sin(\varphi_2-\varphi_1),$$

$$S_{OBC}=\frac{1}{2}|OA|\cdot|OC|\cdot\sin\angle BOC=\frac{1}{2}r_2\cdot r_3\cdot\sin(\varphi_3-\varphi_2),$$

$$S_{OAC}=\frac{1}{2}|OA|\cdot|OC|\cdot\sin\angle AOC=\frac{1}{2}r_1\cdot r_3\cdot\sin(\varphi_3-\varphi_1)$$

Berilgan nuqtalar ixtiyoriy joylashganligini hisobga olib,

$S_{\Delta ABC}=\frac{1}{2}\left|r_1r_2\sin(\varphi_2-\varphi_1)+r_2r_3\sin(\varphi_3-\varphi_2)-r_1r_3\sin(\varphi_3-\varphi_2)\right|$ formulaga ega bo‘lamiz.

Xususan, ucburchakning bir uchi qutb boshi O nuqtada bo‘lsa, $S_{OAB}=\frac{1}{2}r_1r_2\cdot|\sin(\varphi_2-\varphi_1)|$ o‘rinli bo‘ladi.

Masalan, uchlari $A(3;\frac{\pi}{8})$, $B\left(8;\frac{7\pi}{24}\right)$, $C(6;\frac{5\pi}{8})$ nuqtalarda bo‘lgan uchburchak yuzi

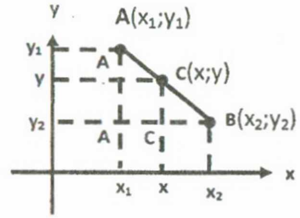
$$S_{ABC}=\frac{1}{2}\left|3\cdot8\cdot\sin\left(\frac{7\pi}{24}-\frac{\pi}{8}\right)+8\cdot6\cdot\sin\left(\frac{5\pi}{8}-\frac{7\pi}{24}\right)-3\cdot6\cdot\sin\left(\frac{5\pi}{8}-\frac{\pi}{8}\right)\right|=\frac{1}{2}\left|24\cdot\sin\frac{\pi}{6}+48\cdot\sin\frac{\pi}{3}-18\sin\frac{\pi}{2}\right|==\frac{1}{2}|12+24\sqrt{3}-18|=12\sqrt{3}-3$$

(кv.b)

1.4. Kesmani berilgan nisbatda bo‘lish

Dekart koordinatalar sistemasida uchlari $A(x_1;y_1)$, $B(x_2;y_2)$ nuqtalarda bo‘lgan kesma berilgan. Agar C(x;y) noma’lum koordinatali nuqta berilgan kesma ichida yotsa va $|AC|:|CB|=\lambda$ nisbati ma’lum bo‘lsa, C nuqta koordinatalarini topish masalasini ko‘rib chiqamiz. A,B,C nuqtalardan son o‘qlariga parallel to‘g‘ri chiziqlar o‘tkazamiz.

Uchta o‘xshash uchburchak hosil bo‘ladi: $\Delta AA_2B\sim\Delta AA_1C\sim\Delta CC_2B$



Bu uchburchaklarda mos tomonlar nisbatlari tengligidan:

$$\frac{|AA_1|}{|CC_1|}=\frac{|A_1C|}{|C_1B|}=\frac{|AC|}{|CB|}=\lambda,$$

ya’ni $\frac{y_1-y}{y-y_2}=\frac{x-x_1}{x_2-x}=\lambda$ tengliklarni olamiz.

$x-x_1=\lambda x_2-\lambda x$ *dan* $x=\frac{x_1+\lambda x_2}{1+\lambda}$, $y_1-y=\lambda y-\lambda y_2$ dan $y=\frac{y_1+\lambda y_2}{1+\lambda}$ formulaga ega bo‘lamiz. Demak, izlanayotgan nuqta $C\left(\frac{x_1+\lambda x_2}{1+\lambda};\frac{y_1+\lambda y_2}{1+\lambda}\right)$ dir.

Xususan, $|AC|=|CB|$ bo‘lsa, $x=\frac{x_1+x_2}{2}, y=\frac{y_1+y_2}{2}$ *bo‘ladi*,

Misol sifatida uchlari bir to‘g‘ri chiziqda yotmagan $A(x_1;y_1), B(x_2;y_2), C(x_3;y_3)$ nuqtalarda bo‘lgan uchburchak og‘irlik markazi (medianalari kesishgan nuqta) koordinatalarini topamiz. BC kesma o‘rtasidagi D nuqta uchun $D\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$ o‘rinlidir.

Agar O nuqta medianalari kesishish nuqtasi bo‘lsa, medianalar xossasiga ko‘ra $|AO|:|OD|=2$ *yani* $\lambda=2$ bo‘ladi. U holda O nuqta koordinatalari A va D koordinatalari yordamida quyidagicha topiladi:

$$x=\frac{x_1+\lambda x_D}{1+\lambda}=\frac{x_1+2\cdot\frac{x_2+x_3}{2}}{1+2}=\frac{x_1+x_2+x_3}{3},\quad y=\frac{y_1+\lambda y_D}{1+\lambda}=\frac{y_1+2\cdot\frac{y_2+y_3}{2}}{1+2}=\frac{y_1+y_2+y_3}{3}.$$

Demak, uchburchk og‘irlik markazi koordinatalari uning uchlari koordinatalari o‘rta arifmetigi ekan.

1.5. Dekart koordinatalarini almashtirish

Analitik geometriya masalalarini yechishda berilgan dekart koordinatalar sistemasidan boshqa, yangi dekart sistemasini, bu ikki sistema koordinatalari bog‘lanishini qarashga to‘g‘ri keladi. Unda koordinatalarni almashtirish formulalari hosil bo‘ladi.

Dekart koordinatalarini almashtirishning ikki turini k‘rib chiqamiz.

1. O‘qlarni parallel ko‘chirish

OXY dekart koordinatalar sistemasida koordinata boshi O(0;0) nuqta biror A(a;b) nuqtaga ko‘chiriladi, son o‘qlari yo‘nalishi eskicha qoladi.

Agar C nuqtaning eski va yangi sistemalaridagi koordinatalari C(x;y), $C(x';y')$ bo‘lsa, bu koordinatalar bog‘lanishi $\begin{cases} x' = x - a \\ y' = y - b \end{cases}$ tarzida bo‘lishi kelib chiqadi, aksincha, $\begin{cases} x = x' + a \\ y = y' + a \end{cases}$ bog‘lanishni ham yozish mumkin.

1) Parallel ko‘chirishda A(2;4) nuqtalar koordinatalari $A'(4;2)$ bo‘lsa, parallel ko‘chirish formulasini yozing.

$\begin{cases} x = x' + a \\ y = y' + b \end{cases}$ dan $\begin{cases} 2 = 4 + a \\ 4 = 2 + b \end{cases}$, ya’ni a=–2; b=2.

Demak, parallel ko‘chirish formulasi $\begin{cases} x = x' - 2 \\ y = y' + 2 \end{cases}$ ko‘rinishda bo‘ladi.

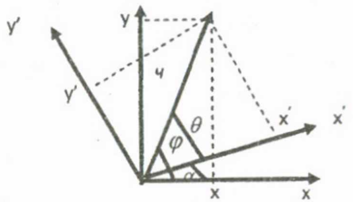
2) Parallel ko‘chirish yordamida $y=\frac{2x+1}{x-1}$ funksiyani $y=\frac{k}{x}$ ko‘rinishida yozing.

$y=\frac{2x-2+3}{x-1} = \frac{2x-2}{x-1} + \frac{3}{x-1} = 2 + \frac{3}{x-1}$. Demak, $y-2=\frac{3}{x-1}$

Agar $y' = y - 2,\ x' = x - 1$ formula yordamida parallel ko‘chirish o‘tkazilsa, funksiya $x'O'y'$ sistemada $y' = \frac{3}{x'}$ ko‘rinishda bo‘ladi.

2. Son o‘qlarini burish

Koordinata boshini o‘z joyida qoldirib, son o‘qlarini bir yo‘nalishda biror α burchakka buramiz. Unda biror A nuqtaning eski dekart (qutb) koordinatalari A(x;y)=A(r, φ) bo‘lsa, yangisida $A(x',y') = A(r,\theta)$ *bo‘ladi, chunki qutb* va A nuqta orasidagi r masofa o‘zgarmaydi.



Dekart va qutb koordinatalari bog‘lanishidan $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ va $\begin{cases} x' = r\cos\theta \\ y' = r\sin\theta \end{cases}$ tengliklarga ega bo‘lamiz.

φ=θ+α ekanligidan quyidagilar kelib chiqadi.

$$\begin{cases} x = r\cos\varphi = r\cos(\theta + \alpha) = r(\cos\theta \cdot \cos\alpha - \sin\theta \cdot \sin\alpha) = x'\cos\alpha - y'\sin\alpha \\ y = r\sin\varphi = r\sin(\theta + \alpha) = r(\sin\theta \cdot \cos\alpha + \sin\alpha \cdot \cos\theta) = y'\cos\alpha + x'\sin\alpha \end{cases}$$

Agar θ=φ-α desak, yangi koordinatalarni eskilari yordamida beriladigan $\begin{cases} x' = x\cos\alpha + y\sin\alpha \\ y' = y\cos\alpha - x\sin\alpha \end{cases}$ formulalarni ham hosil qilamiz.

Agar bir paytning o‘zida koordinata boshi biror $o'(a,b)$ nuqtaga ko‘chirilsa, son o‘qlari biror α burchakka burilsa, yuqoridagi formulalar $\begin{cases} x = x'\cos\alpha - y'\sin\alpha + a \\ y = x'\sin\alpha + y'\cos\alpha + b \end{cases}$ va $\begin{cases} x' = (x - a)\cos\alpha + (y - b)\sin\alpha \\ y' = -(x - a)\sin\alpha + (y - b)\cos\alpha \end{cases}$ ko‘rinishida bo‘ladi.

1) Son o‘qlari α=$\frac{\pi}{4}$ ga burilganda A $(\sqrt{2};\sqrt{2})$ koordinatalari topilsin.

$\begin{cases} x' = x\cos\alpha + y\sin\alpha \\ y' = y\cos\alpha - x\sin\alpha \end{cases}$ formulalardan $\begin{cases} y' = \sqrt{2}\cos\frac{\pi}{4} + \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\cdot\frac{\sqrt{2}}{2} + \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 2 \\ y' = -\sqrt{2}\sin\frac{\pi}{4} + \sqrt{2}\cos\frac{\pi}{4} = -\sqrt{2}\cdot\frac{\sqrt{2}}{2} + \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 0 \end{cases}$ kelib chiqadi, ya’ni $A'(2;0)$

2) Koordinata boshi $O'(2;-2)$ nuqtaga ko‘chirilib, son o‘qlari α=$\frac{\pi}{6}$ ga burilgandagi almashtirish formulalarini yozing.

$$\begin{cases} x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} + 2 \\ y = y'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} - 2 \end{cases} \text{yoki} \begin{cases} x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' + 2 \\ y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' - 2 \end{cases}$$

1.6. Tekislikda chiziq tenglamasi

Tekislikda biror L-chiziq berilgan bo‘lsin, uning ixtiyoriy nuqtasi C ikki koordinataga ega: C(x;y). Agar F(x, y)=0 tenglamani L dagi har bir nuqta koordinata qanoatlantirsa va L da yotmagan nuqta koordinatalari tenglamani qanoatlantirmasa, bu tenglama L-chiziqning tenglamasi deyiladi.

Masalan x-y=0 tenglamani I, III-choraklar bissektrisasini ifodalovchi to‘g‘ri chiziq nuqtalari koordinatalari qanoatlantiradi xolos. Agar L-chiziq qutb koordinatalar sistemasida berilsa, mos ravishda, tenglama F(r;φ)=0 ko‘rinishida bo‘ladi. Masalan, $r=a\cos\varphi$ (a>0) tenglama radiusi $\frac{a}{2}$ga teng bo‘lgan aylanani bildiradi, chunki A(a;0), C(r; φ), ∠OCA=90^0 ekanligidan unga yarim doira tiralganligini bildiradi.

Agar chiziq nuqta koordinatalari x va y biror t-parametrga bog‘liq bo‘lsa, u holda chiziq tenglamasi parametrik usulida berilgan deyiladi va $\begin{cases} x = \varphi(t) \\ y = \varphi(t) \end{cases}$ tarzida yoziladi.

Masalan, $x=\cos t, y=\sin t$ tenglama bilan berilgan chiziq markazi koordinatalar boshida, radiusi 1 bo‘lgan aylanadir, chunki $x^2 + y^2 = 1$.

Biror to‘g‘ri burchakli dekart koordinatalar sistemasida n-darajali algebraik tenglama bilan aniqlangan chiziq n-tartibli algebraik chiziq deyiladi.

Algebraik chiziqlarga ax+by+c=0, $Ax^2 + 2Bxy + y^2 + D + Ey + F = 0$ lar misol bo‘la oladi. Noalgebraik tenglamalarga

$y-\cos x = 0,\ y - \log_a x = 0,\ 2^x - 5^y + 1 = 0$ lar misol bo‘ladi.

n-tartibli algebraik chiziqlar parallel ko‘chirishda, o‘qlarni biror α–burchakka burishda tartibini o‘zgartirmaydi.

Mavzuga doir masalalar

1. Uchlari A(-4;2), B(0;-1), C(3;3) nuqtalarda bo‘lgan uchburchak perimetri, burchaklari, og‘irlik markazi koordinatalarini toping.
2. A(2;1) nuqtadan va ordinatalar o‘qidan 5 birlik uzoqlikdagi nuqtani toping.
3. Abssissalar o‘qida A(8;4) nuqtadan va koordinatalar boshidan barobar uzoqlikda turgan nuqtani toping.
4. A(3;-7) va B(-1;-4) nuqtalar kvadratning yonma-yon uchlari bo‘lsa, kvadrat perimetri, yuzini toping.
5. Abssissalar o‘qida shunday nuqta (lar)ni topingki, ulardan A(2;-3) nuqtagacha bo‘lgan masofa 5 ga teng bo‘lsin.
6. Ordinatalar o‘qida shunday nuqta (lar)ni ko‘rsatingki, ulardan A(-8;4) nuqtagacha bo‘lgan masofa 17 ga teng bo‘lsin.
7. A(2;2), C(5;-2) nuqtalar berilgan. Abssissalar o‘qida shunday B nuqtani topingki, ABC ucburchak to‘gri burchakli bo‘lsin.
8. Kvadratning qarama-qarshi uchlari A(3;0), C(-4;1) nuqtalarda bo‘lsa, qolgan ikki uchi koordinatalarini toping.
9. Kvadratning yonma-yon uchlari A(2;-1), B(-1;3) nuqtalarda bo‘lsa, qolgan ikki uchi koordinatalarini toping.
10. Parallelogramning uchta uchi A(3;-5), B(5;-3), C(-1;3) nuqtalarda bo‘lsa, to‘rtinchi uchi koordinatalarini toping.
11. Uchburchak uchlari A(3;6), B(-1;3), C(2;-1) nuqtalarda bo‘lsa, uchburchak yuzi va C uchidan tushirilgan balandlik uzunligini toping.
12. Qutb koordinatalar sistemasida berilgan A(6;π/2), B(5;0), C(2;π/4), D(10;-π/3), E(8;2π/3) nuqtalar dekart koordinatalarini toping.
13. Dekart koordinatalar sistemasida berilgan A(0;5), B(-3;0), D(-$\sqrt{2}$;-$\sqrt{2}$), E(1; -$\sqrt{3}$) C($\sqrt{3}$;1) nuqtalar qutb koordinatalarini toping.
14. Qutb koordinatalar sistemasida berilgan A(5;π/4), B(8;-π/12) nuqtalar orasidagi masofani hisoblang.
15. Qutb koordinatalar sistemasida kvadratning yonma-yon uchi A(12;-π/10), B(3;π/15) nuqtalarda bo‘lsa, perimetri va yuzini toping. A va B nuqtalar kvadrat qarama-qarshi uchlari bo‘lgan holat uchun masalani qayta yeching.
16. Uchburchakning bir uchi qutbda, qolgan uchlari A(5;π/4), B(4;π/12) nuqtalarda bo’lsa, uning yuzini hisoblang.

17. Uchlarti A(2;π/6), B(5;π/4), C(3;π/2) nuqtalarda bo‘lgan uchburchak yuzini hisoblang.
18. Parallel ko‘chirishda A(2;-4) nuqta A$'$(1;-1) nuqtaga o‘tsa, O(0;0) nuqta qanday nuqtaga o‘tadi?
19. Son o‘qlari $\alpha=60^0$ga burildi. Yangi sistemada A(2$\sqrt{3}$;-4), B($\sqrt{3}$;0), C(0;-2$\sqrt{3}$) bo‘lsa, bu nuqtalarning eski sistemadagi koordinatalarini toping.
20. Koordinatalarni almashtirish $x=\frac{1}{2}x'-\frac{\sqrt{3}}{2}y'$, $y=\frac{\sqrt{3}}{2}x'+\frac{1}{2}y'$ formulalar bilan berilgan. Son o‘qlari qanday burchakka burilganligini aniqlang.
21. A(5;5), B(2;-1), C(12;-6) nuqtalar berilgan. Agar koordinata boshi B nuqtaga ko‘chirilib, son o‘qlari α=arctg3/4 burchakka burilsa, yuqoridagi nuqtalar koordinatalari qanday bo‘ladi?
22. Parallel ko‘chirish yordamida $y'=kx'^2$ ko‘rinishiga keltiring:
a) $y=2x^2-8x+14$, b) $y=x^2-4x+7$, b) $y=6-4x-2x^2$
23. Parallel ko‘chirish yordamida $y'=\frac{k}{x'}$ ko‘rinishiga keltiring:
a) $y=\frac{4x-3}{4x+3}$, b) $y=\frac{2x+1}{x-1}$, c) $y=\frac{1-x}{4x-3}$
24. xy-1=0 berilgan. Son o‘qlari $\alpha=45^0$ga burilsa, tenglama qanday ko‘rinishga ega bo‘ladi?
25. Qutb koordinatalar sistemasida quyidagi chiziqlarni yasang.
1) $r=a\varphi$ (a>0), 2) $r=a(1+\cos\varphi)$, 3) $r=a\sin3\varphi$.

2-mavzu. To‘g‘ri chiziq tenglamalari

Tekislikda birinchi tartibli chiziqlar – to‘g‘ri chiziqlardir. Bu bobda to‘g‘ri chiziqning tenglamalari, ular haqidagi asosiy masalalar o‘rganiladi.

2.1. To‘g‘ri chiziqning umumiy tenglamasi

Tekislikda dekart koordinatalar sistemasi berilgan bo‘lsin. $A(x_1;y_1)$, $B(x_2;y_2)$ nuqtalarni aniqlaymiz. Bu nuqtalardan bir xil masofada yotuvchi C(x;y) nuqtalar to‘plami to‘g‘ri chiziq hosil qilib, AB o‘rta perpendikulyari hisoblanadi. $|AC| = |CB|$ tenglikdan

$\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}$ ga ega bo‘lamiz. Tomonlarini kvadratga oshirib, qavslarni ochamiz:

$x^2-2xx_1+x_1^2+y^2-2yy_1+y_1^2 = x^2-2xx_2+x_2^2+y^2-2yy_2+y_2^2$ o‘xshash hadlarni ixchamlab, $2(x_2-x_1)x+2(y_2-y_1)y+x_1^2+y_1^2-x_2^2-y_2^2=0$ tenglamaga ega bo‘lamiz.

Agar $A=2(x_2-x_1)$, $B=2(y_2-y_1)$, $C=x_1^2+y_1^2-x_2^2-y_2^2$ belgilashlar kiritsak, tenglama:

$$Ax+By+C=0 \qquad (1)$$

ko‘rinish oladi.

Bu tenglama **to‘g‘ri chiziq umumiy tenglamasi** deyiladi.

Masalan, P(4;1), Q(-1;2) nuqtalardan bir xil masofoda yotuvchi to‘g‘ri chiziq tenglamasini topamiz. $\sqrt{(x-4)^2+(y-1)^2} = \sqrt{(x+1)^2+(y-2)^2}$, $x^2-8x+16+y^2-2y+1 = x^2+2x+1+y^2-4y+4$ o‘xshash hadlarni ixchamlab, 10x-2y-12=0 yoki 5x-y-6=0 tengamaga egamiz.

To‘g‘ri chiziq umumiy tenglamasidagi A, B, C sonlari tenglama koeffitsiyentlari deyilib, quyidagicha xususiy holatlar bo‘lishi mumkin:

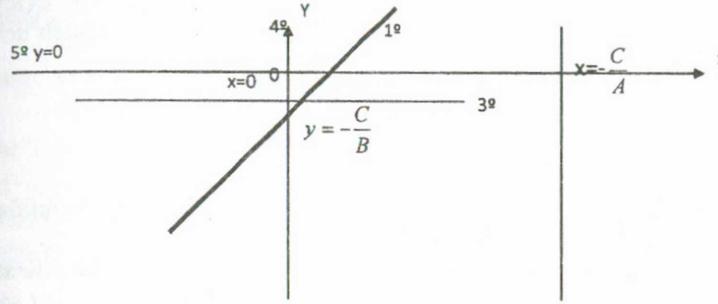
1°. $A\neq 0$, $B\neq 0$, $C=0$ bu holda tenglama $Ax+By=0$ ko‘rinish olib, koordinata boshidan o‘tuvchi to‘g‘ri chiziq bo‘ladi, chunki, O(0;0) nuqta tenglamani qanoatlantiradi.

2°. $A\neq 0$, $B=0$, $C\neq 0$. Bu holda tenglama $Ax+C=0$ bo‘lib, uni $x=-\frac{C}{A}$ ko‘rinishda yozish mumkin. Demak, abssissa biror o‘zgarmas songa teng, ordinata ixtiyoriy qiymat qabul qiladi. Bu to‘g‘ri chiziqning Oy o‘qiga parallelligini bildiradi.

3°. $A=0$, $B\neq 0$, $C\neq 0$. Bu holda $By+C=0$ hosil bo‘lib, $y=-\frac{C}{B}$ tarzida yoziladi. To‘g‘ri chiziq Ox o‘qiga parallel.

4°. $A\neq 0$, $B=C=0$. Tenglama $Ax=0$ ko‘rinishida bo‘lib, $x=0$ tenglama kelib chiqadi va Oy o‘qini ifodalaydi.

5°. $B\neq 0$, $A=C=0$. Bu holda y=0 kelib chiqadi va bu tenglama Ox o‘qini bildiradi.



2.2. To‘g‘ri chiziqning burchak koeffitsiyentli tenglamasi

Dekart koordinatalar sistemasida ordinatalar o‘qidan O(0;0) dan hisobla’nganda uzinligi b ga teng kesma ajratadigan, abssissa o‘qi bilan α burchak hosil qiluvchi to‘g‘ri chiziqni aniqlaymiz. To‘g‘ri chiziqning ixtiyoriy C(x;y) nuqtasini olamiz.

Hosil bo‘lgan to‘g‘ri burchakli uchburchakdan $\frac{y-b}{x}=\operatorname{tg}\alpha$ ekanligini topamiz. Bu tenlamadagi $\operatorname{tg}\alpha$ to‘g‘ri chiziqning burchak koeffitsiyenti deyiladi va k bilan belgilanadi: $k=\operatorname{tg}\alpha$.

To‘g‘ri chiziq tenglamasi $\frac{y-b}{x}=k$ ko‘rinish oladi. Undan to‘g‘ri chiziqning burchak koeffitsiyentli tenglamasi deb ataluvchi

$$y=kx+b \qquad (2)$$

tenglamani olamiz.

To‘g‘ri chiziq holati k va b koeffitsiyentlari bilan to‘la aniqlanadi. To‘g‘ri chiziq umumiy Ax+By+C=0 tenglamasidan burchak koeffitsiyentlisiga o‘tish uchun bu tenglamani y ga nisbatan yechish kifoya.

$$y=-\frac{A}{B}x-\frac{C}{B}$$

Bunda $k=-\frac{A}{B}$, $b=-\frac{C}{B}$ belgilashlar kiritilsa, tenglama y=kx+b ko‘rinishda bo‘ladi.

Ma‘lumki, y=kx+b funksiya chiziqli deyilar edi. Demak, chiziqli funksiya grafigi to‘g‘ri chiziq bo‘lar ekan. b=0 bo‘lsa y=kx hosil bo‘lib, x va y o‘zaro proporsional, k-esa proporsionallik koeffitsiyenti deyiladi.

2.3. To‘g‘ri chiziqning kesmalar bo‘yicha tenglamasi

Tekislikda abssissa o‘qidan a= OA, ordinata o‘qidan b=OB kesmalar ajratadigan to‘g‘ri chiziqni aniqlaymiz. To‘g‘ri chiziq ixtiyoriy C(x;y) nuqta abssissasini A_1, ordinatasini B_1 bilan belgilasak, uchta o‘xshash uchburchak hosil bo‘ladi: $\Delta AOB \sim \Delta AA_1C \sim \Delta CB_1B$, ya’ni $\frac{OA}{OB}=\frac{A_1A}{A_1C}=\frac{CB_1}{B_1B}$.

Demak, $\frac{a}{b}=\frac{a-x}{y}=\frac{x}{y-b}$.

Bu tengliklarning birortasini soddalashtirsak,

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (3)$$

tenglama hosil bo‘ladi. Bu tenglama **to‘g‘ri chiziqning kesmalar bo‘yicha tenglamasi** deyiladi.

To‘g‘ri chiziqning kesmalar bo‘yicha tenglamasini koordinata boshidan o‘tuvchi to‘g‘ri chiziqlar uchun yozib bo‘lmaydi, chunki ular son o‘qlaridan kesmalar ajratmaydi.

Ax+By+C= 0 ($C\neq 0$) tenglamadan kesmalar bo‘yicha tenglamaga o‘tish uchun Ax+By= $-C$ tarzida yozib, tomonlarni (-C)ga bo‘lib yuboriladi: $\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1$, a= $-\frac{C}{A}$, b= $-\frac{C}{B}$ belgilash kiritsak, tenglama $\frac{x}{a}+\frac{y}{b}=1$ ko‘rinishga keladi.

Masalan, 3x-4y-12=0 to‘g‘ri chiziq kesmalar bo‘yicha tenglamasi $\frac{x}{4}+\frac{y}{-3}=1$ ko‘rinishida bo‘lib, abssissa o‘qidan musbat yo‘nalishda 4 ga teng kesma, ordinatalar o‘qida manfiy yo‘nalish bo‘yicha 3 ga teng kesmalar ajratar ekan.

2.4. To‘g‘ri chiziqning normal tenglamasi

Tekislikda biror to‘g‘ri chiziqni qaraylik. Koordinata boshidan bu to‘g‘ri chiziqqa tushirilgan normal deb ataluvchi, perpendikulyar uzunligi p, normal bilan abssissa musbat yo‘nalishi orasidagi burchak α($\alpha\neq 0$, $\alpha\neq\frac{\pi}{2}$) bo‘lsin. To‘g‘ri chiziqda biror C(x;y) nuqta olib, uning abssissasidagi proyeksiyasini C_1 deb belgilaymiz. $\angle AON=\angle ACC_1=\angle ABO=\alpha$ ekanligi $\Delta AON, \Delta ACC_1, \Delta ABC$ uchburchaklar to‘g‘ri burchakli o‘xshash ekanligini bildiradi: OB= $\frac{p}{\sin\alpha}$, OA= $\frac{p}{\cos\alpha}$ tengliklarni hisobga olib: $\frac{y}{\frac{p}{\sin\alpha}}=\frac{\frac{p}{\cos\alpha}-x}{\frac{p}{\cos\alpha}}$ munosabatga ega bo‘lamiz. Uni soddalashtirib, normal tenglama deb ataluvchi quyidagi

$$x\cos\alpha+y\sin\alpha-p=0 \qquad (4)$$

tenglamani keltirib chiqaramiz.

Bu tenglamani a= $\frac{p}{\cos\alpha}$, b= $\frac{p}{\sin\alpha}$ kesmalar bo‘yicha ham keltirib chiqarish mumkin. Normal tenglamadagi p soni to‘g‘ri chiziqning markazdan qancha masofada o‘tganligini bildiradi.

To‘g‘ri chiziqning umumiy Ax+By+C= 0 tenglamasini normal tenglamaga keltirish masalasini ko‘raylik.

$\mu\neq 0$ normallovchi ko‘paytiruvchi bo‘lsa, $\mu Ax+\mu By+\mu C=0$ normal tenglama bo‘ladi, ya’ni $\mu A=\cos\alpha$, $\mu B=\sin\alpha$, $\mu C=-p$.

$(\mu A)^2+(\mu B)^2=\cos^2\alpha+\sin^2\alpha$ munosabatdan $\mu^2=\frac{1}{A^2+B^2}$ yoki $\mu=\pm\frac{1}{\sqrt{A^2+B^2}}$ bo‘lishi kelib chiqadi. Demak, Ax+By+C=0 tenglama $\pm\frac{A}{\sqrt{A^2+B^2}}x\pm\frac{B}{\sqrt{A^2+B^2}}y\pm\frac{C}{\sqrt{A^2+B^2}}=0$ normal ko‘rinishga keladi. p soni oldida manfiy ishora hosil qilishi uchun μ ning ishorasi C ning ishorasiga qarama-qarshi olinadi.

Masalan, 6x-8y+5=0 tenglama normallovchi ko‘paytiruvchisi $\mu=\pm\frac{1}{6^2+8^2}=\pm\frac{1}{10}$, C=5 ekanligidan $\mu=-\frac{1}{10}$ oilinishi, normal tenglama esa $-\frac{3}{5}x+\frac{4}{5}y-\frac{1}{2}=0$ bo‘lishi kelib chiqadi.

2.5. To‘g‘ri chiziqning qutb koordinatalardagi tenglamasi

Qutb koordinatalar sistemasida biror to‘g‘ri chiziq, qutbdan unga tushirilgan, normal deb ataluvchi, uzunligi p ga teng perpendikulyar va unga mos qutb burchagi α berilgan bo‘lsin. Normal va to‘g‘ri chiziq kesishgan nuqtani A deb belgilaymiz.

To‘g‘ri chiziq ixtiyoriy C(r;φ) nuqtasini qaraymiz. To‘g‘ri burchakli OAC uchburchakdagi $\frac{p}{r}=\cos(\alpha-\varphi)$ munosabatdan, to‘g‘ri chiziq tenglamasi: r= $\frac{p}{\cos(\alpha-\varphi)}$ yoki r = $\frac{p}{\cos(\varphi-\alpha)}$ ko‘rinishida bo‘lishi kelib chiqadi.

Bu tenglamani normal tenglamadan $x=r\cos\varphi$, $y=r\sin\varphi$ almashtirishlar yordamida ham topish mumkin. Unda $x\cos\alpha+y\sin\alpha-P=0$ tenglama $r\cos\varphi\cos\alpha+r\sin\varphi\sin\alpha-p=0$ ko‘rinish oladi va $r=\frac{p}{\cos(\varphi-\alpha)}$ tenglamaga ega bo‘lamiz.

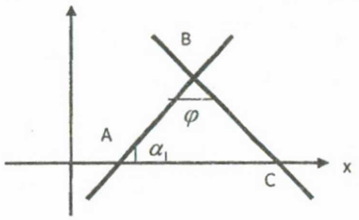
2.6. To‘g‘ri chiziqning parametrik tenglamasi

Ba’zi hollarda to‘g‘ri chiziq ixtiyoriy nuqtasi koordinatalari biror λ parametrga bog‘liq bo‘lib qoladi: $\begin{cases} x=m\lambda+x_0 \\ y=n\lambda+y_0 \end{cases}$

Bunday tenglamadan avvalgi tenglamalarni hosil qilish uchun, dastlab, λ lar topiladi, $\lambda=\frac{x-x_0}{m}$, $\lambda=\frac{y-y_0}{n}$ so‘ngra ular tenglashtirilib, λ parametr yo‘qotiladi: $\frac{x-x_0}{m}=\frac{y-y_0}{n}$.

2.7. Ikki to‘g‘ri chiziq orasidagi burchak

Tekislikda ikki $y=k_1x+b_1$, $y=k_2x+b_2$ to‘g‘ri chiziqlar orasidagi φ burchakni topish masalasini ko‘ramiz, bunda $k_1=\mathrm{tg}\,\alpha_1$; $k_2=\mathrm{tg}\,\alpha_2$.



Uchburchak tashqi burchagi xossasidan: $\alpha_2=\varphi+\alpha_1$.

Izlanayotgan burchak $\varphi=\alpha_2-\alpha_1$ burchakni esa burchak koeffitsiyentlari orqali topish qulay: $\mathrm{tg}\,\varphi=\mathrm{tg}(\alpha_2-\alpha_1)=\frac{\mathrm{tg}\,\alpha_2-\mathrm{tg}\,\alpha_1}{1+\mathrm{tg}\,\alpha_1\cdot\mathrm{tg}\,\alpha_2}=\frac{k_2-k_1}{1+k_1\cdot k_2}$

Berilgan to‘g‘ri chiziqlar orasidagi o‘tkir burchakni topish uchun $\mathrm{tg}\,\varphi=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right|$ ko‘rinishida yozish kifoya.

Masalan, y=-2x va y=3x-4 to‘g‘ri chiziqlar uchun $\mathrm{tg}\,\varphi=\frac{3-(-2)}{1+(-2)\cdot 3}=-1$, demak, ular orasidagi o‘tmas burchak $\frac{3\pi}{4}$ ga, o‘tkir burchak esa $\frac{\pi}{4}$ga teng.

1. Agar to‘g‘ri chiziqlar parallel bo‘lsa, $\varphi=0$ yoki $\varphi=\pi$ bo‘lib $k_2-k_1=0$ kelib chiqadi. Demak, to‘g‘ri chiziqlar parallellik sharti $k_2=k_1$ dir.

2. To‘g‘ri chiziqlar o‘zaro perpendikulyar bo‘lsa, $\pi=\frac{\pi}{2}$, $\mathrm{tg}\frac{\pi}{2}=\infty$, $1+k_1k_2=0$ shart kelib chiqadi. Demak, to‘g‘ri chiziqlar perpendikulyarlik sharti $k_2=-\frac{1}{k_1}$ dir.

Agar to‘g‘ri chiziqlar $A_1x+B_1y+C_1=0$, $A_2x+B_2y+C_2=0$ formulalar bilan berilsa, ularni y ga nisbatan ychib $k_1=-\frac{A_1}{B_1}$, $k_2=-\frac{A_2}{B_2}$ bo‘lishini topamiz.

Demak, to‘g‘ri chiziqlar umumiy tenglamasi bilan berilsa,

$$\mathrm{tg}\,\varphi=\frac{-\frac{A_2}{B_2}-\left(-\frac{A_1}{B_1}\right)}{1+\left(-\frac{A_1}{B_1}\right)\cdot\left(-\frac{A_2}{B_2}\right)}=\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}$$

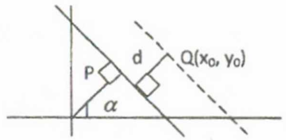
formulaga ega bo‘lamiz. Unda to‘g‘ri chiziqlar parallel bo‘lishi uchun $A_1B_2-A_2B_1=0$, ya’ni $\frac{A_1}{A_2}=\frac{B_1}{B_2}$ bo‘lishi, perpendikulyar bo‘lishi uchun esa $A_1A_2+B_1B_2=0$ bo‘lishi kerak.

1) y=2x-5, y=2x+1, y=$-\frac{1}{2}$x+5 to‘g‘ri chiziqlarning dastlabki ikkitasi parallel, uchinchisi ularga perpendikulyardir.

2) 2x-3y+5=0, 4x-6y+1=0, 3x+2y+5=0, to‘g‘ri chiziqlarning dastlabki ikkitasi parallel, uchinchisi ularga perpendikulyardir.

2.8. Nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa

Normal tenglamasi bilan berilgan $x\cos\alpha+y\sin\alpha-P=0$ to‘g‘ri chiziq va unda yotmagan biror $Q(x_0;y_0)$ nuqta berilgan bo‘lsin. $Q(x_0;y_0)$ nuqtadan berilgan to‘g‘ri chiziqqacha bo‘lgan d masofani topish masalasini ko‘rib chiqamiz. $Q(x_0;y_0)$dan o‘tib, $x\cos\alpha+y\sin\alpha-p=0$ga parallel to‘g‘ri chiziq $x\cos\alpha+y\sin\alpha-q=0$ tenglama bilan beriladi, bunda q=p+d, lekin $q=x_0\cos\alpha+y_0\sin\alpha$ ekanligidan



$$d=q-p=x_0\cos\alpha+y_0\sin\alpha-p$$

kelib chiqadi. Agar $q<p$ bo’lsa $d=p-q$ bo‘lishini hisobga olsak,

$$d=|x_0\cos\alpha+y_0\sin\alpha-p|$$

formulaga ega bo‘lamiz.

Agar to‘g‘ri chiziq Ax+By+C=0 umumiy tenglamasi bilan berilsa, masofa formulasi $d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$ ko‘rinishida bo‘ladi. Masalan, A(4;2), B(1;1) dan 3x-4y-4=0 gacha masofani hisoblaymiz. $d_A=\frac{|3\cdot 4-4\cdot 2-4|}{\sqrt{3^2+(-4)^2}}=0$, $d_B=\frac{|3\cdot 1-4\cdot 1-4|}{\sqrt{3^2+(-4)^2}}=\frac{|-5|}{5}=1$, demak, A to‘g‘ri chiziqqa tegishli, B nuqta esa to‘g‘ri chiziqdan 1 birlik uzoqlikda joylashgan.

Normal tenglamasi bilan berilgan $x\cos\alpha+y\sin\alpha-p=0$ va $x\cos\alpha+y\sin\alpha-q=0$ to‘g‘ri chiziqlar orasidagi masofa $d=|p-q|$ bo‘lishi tushunarli. Agar to‘g‘ri chiziqlar $x\cos\alpha+y\sin\alpha-p=0$, $\lambda x\cos\alpha+\lambda y\sin\alpha-q=0$ tenglamalar bilan berilsa, masofa $d=\left|p-\frac{q}{\lambda}\right|$ bo‘ladi. Demak, ikki parallel $A_1x+B_1y+C_1=0$, $\lambda A_1x+\lambda B_1y+C_2=0$ to‘g‘ri chiziqlar orasidagi masofa $d=\left|\frac{C_1-\frac{C_2}{\lambda}}{\sqrt{A^2+B^2}}\right|$ formula yordamida topiladi.

Masalan, 6x+8y+7=0, 3x-4y-7=0 to‘g‘ri chiziqlar o‘zaro parallel, ularni $3x+4y+\frac{7}{2}=0$, $3x-4y-7=0$ tarzida yozsak, $d=\frac{\left|\frac{7}{2}+7\right|}{\sqrt{3^2+4^2}}=\frac{21}{10}=2{,}1$ ekanligi kelib chiqadi.

Ba’zi hollarda birinchi to‘g‘ri chiziqdan biror nuqta tanlab olinib, ikkinchisigacha masofani hisoblasa ham bo‘ladi, masalan, $C(0;\frac{7}{8})$ nuqta birinchi to‘g‘ri chiziqqa tegishli, undan ikkinchi to‘g‘ri chiziqqacha masofa esa:

$$d=\frac{\left|3\cdot 0-4\frac{7}{8}-7\right|}{\sqrt{3^2+(-4)^2}}=\frac{\left|-\frac{7}{2}-7\right|}{5}=\frac{21}{10}=2.1 .$$

Masofa formulasi yordamida ikki kesishuvchi $A_1x+B_1yC_1=0$ va $A_2x+B_2y+C_2=0$ to‘g‘ri chiziq bissektrisalari tenglamasini keltirib chiqaramiz.

Bissektrisadagi ixtiyoriy C(x;y) nuqtadan berilgan to‘g‘ri chiziqqacha masofalar tengligidan $\frac{|A_1x+B_1y+C_1|}{\sqrt{A^2+B^2}}=\frac{|A_2x+B_2y+C_2|}{\sqrt{A_2^2+B_2^2}}$ yoki

$\frac{A_1x+B_1y+C_1}{\sqrt{A_1^2+B_1^2}}=\pm\frac{A_2x+B_2y+C_2}{\sqrt{A_2^2+B_2^2}}$ kelib chiqadi.

2.9. Bitta va ikkita nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamalari

To‘g‘ri chiziq y=kx+b tenglama bilan berilib, dastlab, uning bitta $A(x_0;y_0)$ nuqtasi ma’lum bo‘lsin, demak, $y_0=kx_0+b$. Berilgan tenglamadan topilgan sonli tenglikni ayirsak, $y-y_0=k(x-x_0)$ tenglama hosil bo‘ladi. U $A(x_0;y_0)$ nuqtadan o‘tuvchi barcha to‘g‘ri chiziqlar tenglamasidir. Bu to‘g‘ri chiziqlar $A(x_0;y_0)$dan o‘tuvchi to‘g‘ri chiziqlar dastasi deyiladi. Agar dastadagi biror to‘g‘ri chiziq $B(x_1;y_1)$ nuqtadan ham o‘tsa $y_1-y_0=k(x_1-x_0)$ tenglik bajariladi. Undan k=$\frac{y_1-y_0}{x_1-x_0}$ topiladi. Demak, A va B nuqtalardan o‘tuvchi chiziq tenglamasi $y-y_0=\frac{y_1-y_0}{x_1-x_0}(x-x_0)$ yoki $\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}$ ko‘rinishida bo‘ladi.

Masalan, A(2;-1), va B(1;2) nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi $\frac{x-2}{1-2}=\frac{y+1}{2-(-1)}$ yoki $y=-3x+5$ ko‘rinishida bo‘ladi.

Endi biror $C(x_0;y_0)$ nuqtadan o‘tib, berilgan $y=k_1x+b_1$ to‘g‘ri chiziqqa parallel (perpendikulyar) to‘g‘ri chiziq tenglamasi formulasini keltirib chiqaramiz.

Izlanayotgan to‘g‘ri chiziq $C(x_0;y_0)$dan o‘tadi, demak, tenglamasi $y-y_0=k(x-x_0)$ ko‘rinishda bo‘ladi. Bundan tashqari, agar u $y=k_1x+b_1$ga parallel (perpendikulyar) bo‘lsa, $k=k_1$ $(k=-\frac{1}{k_1})$ bo‘lib tenglamasi

$y-y_0=k_1(x-x_0)\left[y-y_0=-\frac{1}{k_1}(x-x_0)\right]$ ko‘rinishida bo‘ladi.

Masalan, C(2;-1)dan o‘tib, y=4x+3 ga parallel (perpendikulyar) bo‘lgan to‘g‘ri chiziq tenglamasi y+1=4(x-2) $\left[y+1=-\frac{1}{4}(x-2)\right]$ ko‘rinishda bo‘ladi.

Mavzuga doir masalalar

1. To‘g‘ri chiziq $(a+2)x+(a^2-9)y+3a^2-8a+5=0$ tenglama bilan berilgan. a ning to‘g‘ri chiziq qanday qiymatida berilgan?

a) abssissalar o‘qiga parallel;
b) ordinatalar o‘qiga parallel;
c) koordinatalar boshidan o‘tuvchi bo‘ladi?

2. To‘g‘ri chiziq (m+2n-3)x+(2m-n+1)y+6m+9=0 tenglama bilan berilgan m va n ning qanday qimatida bu to‘g‘ri chiziq abssissalar o‘qiga parallel va ordinatalar o‘qida koordinatalar boshidan hisoblaganda -3 ga teng kesma ajratadi? Ushbu to‘g‘ri chiziq tenglamasini yozing.

3. To‘g‘ri chiziq (2m-n+5)x+(m+3n-2)y+2m+7n+19=0 tenglama bilan berilgan. m va n ning qanday qimatida bu to‘g‘ri chiziq ordinatalar o‘qiga parallel va abssissalar o‘qida +5 ga teng kesma ajraladi? Bu to‘g‘ri chiziq tenglamasini yozing.

4. A(0;1) va B(1;2) nuqtalardan bir xil masofada yotuvchi to’g’ri chiziq tenglamasini yozing.

5. Ordinata o‘qidan b=3 kesma ajratib, abssissa o‘qi bilan a) 45^0; b)135^0 burchak tashkil etuvchi to‘g‘ri chiziq tenglamalarini yozing.

6. Koordinatalar boshidan o‘tib, abssissa o‘qi bilan a) 60^0; b) 120^0 burchak tashkil etuvchi to‘g‘ri chiziqlar tenglamasini yozing.

7. 2x-3y-6=0 va 12x+5y-60=0 to‘g‘ri chiziqlar kesmalar bo‘yicha tenglamalarini yozing.

8. A(4;3) nuqtadan o‘tib, koordinatalar burchagidan yuzi 30 kv birlikka teng uchburchak ajratuvchi to‘g‘ri chiziq tenglamasini yozing.

9. 3x-4y-20=0, y=kx+b, $\frac{x}{a}+\frac{y}{b}=1$ to‘g‘ri chiziqlar normal tenglamalarini yozing.

10. Koordinatalar boshidan 12x-5y+52=0 to‘g‘ri chiziqqacha bo‘lgan masofa topilsin.

11. Koeffitsiyentlari noldan farqli Ax+By+C=0 to‘g‘ri chiziq va son o‘qlari bilan chegaralangan uchburchak yuzi S=$\frac{1}{2}\frac{C^2}{|AB|}$ formula bilan topilishini isbotlang.

12. Quyidagi to‘g‘ri chiziqlar orasidagi burchakni toping:
1) 5x-y+7=0 va 3x+2y=0, 2) x-2y+4=0 va 2x-4y+3=0, 3) 3x-2y+7=0 va 2x+3y-3=0, 4) 3x+2y-1=0 va 5x-2y+3=0.

13. Qutb koordinatalar sistemasida berilgan $r_1=\frac{P_1}{\cos(\varphi-\alpha_1)}$ va $r_2=\frac{P_2}{\cos(\varphi-\alpha_2)}$ to‘g‘ri chiziqlar orasidagi burchakni topish formulasini yozing.

14. Parametrik usulda berilgan $\{x=m\lambda+x_0,\ y=n\lambda+y_0\}$ to'g'ri chiziq va abssissa o'qi orasidagi burchak $tg\varphi=n/m$ formula bilan hisoblanishini isbotdang.
15. Parametrik usulda berilgan $\{x=m_1\lambda+x_1,\ y=n_1\lambda+y_1\}$ va $\{x=m_2\lambda+x_2,\ y=n_2\lambda+y_2\}$ to'g'ri chiziqlar orasidagi burchak $\cos\varphi=\frac{|m_1m_2+n_1n_2|}{\sqrt{m_1^2+n_1^2}\sqrt{m_2^2+n_2^2}}$ formula bilan topilishini isbotlang. Parallellik va perpendikulyarlik shartlarini yozing.
16. Uchburchak tomonlari $x+3y=0$, $x=3$, $x-2y+3=0$ tenglamalar bilan berilgan. Uning uchlari koordinatalari, ichki burchaklarini toping.
17. $y=kx+5$ to'g'ri chiziq koordinatalar boshidan $d=\sqrt{5}$ masofa uzoqlikda bo'lsa, k qanday qiymatlar qabul qiladi?
18. Berilgan nuqtadan to'g'ri chiziqqacha bo'lgan masofani toping:
1) A(2;-1), $4x+3y+10=0$; 2) B(0;-3), $5x-12y-23=0$;
3) C(-2;3), $3x-4y-2=0$; 4) D(1;-2), $x-2y-5=0$.
19. Quyidagi parallel to'g'ri chiziqlar orasidagi masofani toping:
1) $3x-4y-10=0$, $6x-8y+5=0$; 2) $5x-12y+26=0$, $5x-12y-13=0$;
3) $4x-3y+15=0$, $8x-6y+25=0$; 4) $24x-10y+39=0$, $12x-5y-26=0$.
20. Kvadratning ikki tomoni tenglamalari $5x-12y-65=0$ va $5x-12y+26=0$ bo'lsa, uning perimetri va yuzini toping.
21. $3x-y-4=0$ va $2x+6y+3=0$ to'g'ri chiziqlar hosil qilgan burchak bissektrisalaridan koordinata boshidan o'tuvchisi tenglamasini toping.
22. $3x+4y-5=0$ va $5x-12y+3=0$ hosil qilgan o'tkir burchak bissektrisasi tenglamasini yozing.

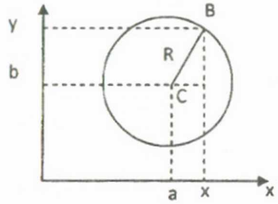
3-mavzu. Ikkinchi tartibli chiziqlar

Bu bobda Ikkinchi tartibli chiziqlar (ITCh), aylana, ellips, giperbola, parabola, ularning xossalari, ITCh lar umumiy tenglamasi va uni kanonik ko'rinishiga keltirish o'rganiladi.

3.1. Aylana tenglamasi

Markaz deb ataluvchi C(a;b) nuqtadan bir xil R masofada yotuvchi nuqtalar to'plami **aylana** deyiladi.

Aylana tenglamasini olish uchun uning ixtiyoriy B(x;y) nuqtasini olamiz.



Pifagor teoremasiga ko'ra $(x-a)^2+(y-b)^2=R^2$ *aylana* umumiy tenglamasini hosil qilamiz.

Agar aylana markazi koordinata boshi O(0;0) bo'lsa, tenglamasi $x^2+y^2=R^2$ ko'rinishida bo'ladi.

Masalan, $x^2-4x+y^2+8x-5=0$ aylana markazi koordinatalari va radiusini topish uchun tenglama $(x-2)^2-4+(y+4)^2-16-5=0$ yoki $(x-2)^2+(y+4)^2=5^2$ ko'rinishda yoziladi.

Demak, aylana markazi C(2;-4) nuqtada va radiusi R=5 dir.

3.2 Ellipsning kanonik tenglamasi, uning xossalari

Har bir nuqtasidan berilgan fokus deb ataluvchi ikki F_1 *va* F_2 nuqtalargacha masofalarning yig'indisi $|F_1F_2|$dan katta o'zgarmas 2a soniga teng nuqtalar to'plami **ellips** deyiladi.

Ellips tenglamasini keltirib chiqarish uchun fokus deb ataluvchi nuqtalarni abssissa o'qida koordinata boshiga nisbatan simmetrik joylashtiramiz: $F_2(c;0)$, $F_1(-c;0)$, ya'ni $|F_1F_2|=2c$

Agar M(x;y) ellips ixtiyoriy nuqtasi bo'lsa, $r_1=|MF_1|$ va $r_2=|MF_2|$ uzinliklar ellipsning fokal radiuslari deyiladi. Shartga ko'ra $r_1+r_2=2a$

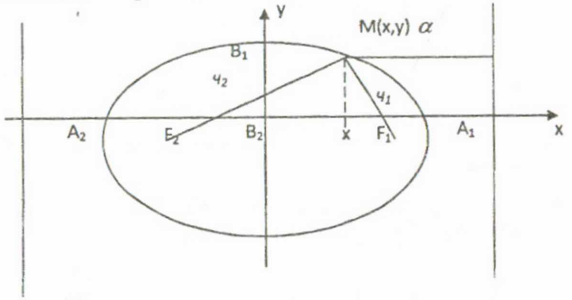
$r_1=\sqrt{(x+c)^2+y^2}$, $r_2=\sqrt{(x-c)^2+y^2}$ ekanligidan ellips tenglamasi $\sqrt{(x+c)^2+y^2}$ $+\sqrt{(x-c)^2+y^2}=2a$ ekanligini topamiz. Ellipsning bu tenglamasi ishlatish uchun noqulay, ikkinchi radikalni o'ng tomonga o'tkazib, tomonlarni kvadratga oshiramiz:

$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$ yoki $a\sqrt{(x-c)^2+y^2}=a^2-cx$.

Bu tenglamani ham kvadratiga oshirib, $a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$ yoki $(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$ tenglamaga ega bo'lamiz. a>c bo'lganligi uchun $b=\sqrt{a^2-c^2}$ belgilash kiritish mumkin.

Ellips tenglamasi $b^2x^2+a^2y^2=a^2b^2$ ko'rinish oladi, bundan **ellipsning kanonik tenglamasi** deb ataluvchi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamani olamizl.

Bunda $y^2=b^2(1-\frac{x^2}{a^2})$ ekanligini hisobga olsak, fokal radiuslari uchun $r_1=\sqrt{(x+c)^2+b^2(1-\frac{x^2}{a^2})}$ $=\sqrt{(a-\frac{cx}{a})^2}$ $=a-\frac{cx}{a}$, $r_2=\sqrt{(x-c)^2+b^2(1-\frac{x^2}{a^2})}$ $=a-\frac{cx}{a}$ formulalarni topamiz.



Koordinata tekisligining 1-choragida y≥0 bo'lib, ellips $y=\frac{a}{b}\sqrt{a^2-x^2}$ tenglama bilan beriladi. Unda quyidagilar kelib chiqadi:

1) $x=0$ bo'lsa, $y=b$. Agar x odan a gacha o'sadi, y kamayadi.

2) $x=a$ bo'lsa, $y=0$

3) $x>a$ bo'lsa, y aniqlanmagan.

Boshqa choraklarda ham simmetriklikdan ellipsni to'la chizishimiz mumkin, chunki koordinata boshi simmetriya markazidir. a va b kattaliklar ellipsning katta va kichik yarim o'qlari deyiladi. $\varepsilon=\frac{c}{a}$ soni ellips **ekssentrisiteti** deyiladi. Ellipsda $0\le\varepsilon<1$ bo'lib, ε =0 da aylana hosil bo'ladi.

$x=\pm\frac{a}{\varepsilon}$ tenglamalar bilan berilgan to'g'ri chiziqlar **direktrisalar** deyiladi.

Agar ellips M(x;y) nuqtasidan biror direkrisagacha masofa d unga mos fokal radius r bo'lsa, $\varepsilon=\frac{r}{d}$ bo'ladi, haqiqatdan $d=\frac{a}{\varepsilon}-x$, $r=a-\varepsilon x$ ekanligi buni tasdiqlaydi.

Ma'lumki planetalar va ba'zi kometalar bir fokusida Quyosh joylashgan elliptik trayektoriyalar bo'ylab harakatlanadi. Unda planetalar ekssentrisiteti nolga yaqin, kometalar ekssentriteti esa birga yaqin ellips bo'ylab harakatlanadi.

Yerning 1 yilda 1 marta, Galley kometasining esa 72 yilda bir marta Quyosh atrofida aylanishini eslash kifoya.

3.3. Giperbolaning kanonik tenglamasi xossalari

Har bir nuqtasidan berilgan fokuslar deb ataluvchi ikki F_1 va F_2 nuqtalargacha masofalari ayirmasi absolyut qiymati o'zgarmas $2a$ songa teng nuqtalarning to'plami **giperbola** deyiladi.

Giperbola tenglamasini hosil qilish uchun F_1 va F_2 fokuslarni abssissa o'qiga, koordinata boshiga simmetrik qilib joylaymiz: $F_1(-c;0)$, $F_2(c;0)$

Agar M(x;y) giperbola ixtiyoriy nuqtasi bo'lsa, ta'rifga ko'ra: $|MF_1|-|MF_2|=2a$, yoki $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$ tenglamani $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\pm 2a$ ko'rinishda yozib, tomonlarini kvadratga oshiramiz.

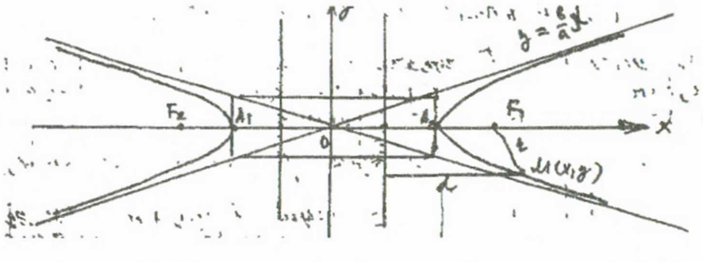
Soddalashtirib: $\pm a\sqrt{(x-c)^2+y^2}=cx^2-a^2$ Bu tenglamani ham kvadratga oshirib, guruhlab; $(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$ tenglikni olamiz. $c>a$ bo'lganligidan $b=\sqrt{c^2-a^2}$ belgilash kiritamiz, natijada, giperbola tenglamasi $b^2x^2-a^2y^2=a^2\cdot b^2$ ko'rinishga keladi. Tomonlarni $a^2\cdot b^2$ ga bo'lib, **giperbola kanonik tenglamasi** deb ataluvchi $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tenglamani hosil qilamiz .

Bu tenglamada x, y lar kvadrat darajada bo'lishi, grafikning son o'qlari va koordinata boshiga nisbatan simmetrikligini bildiradi. Demak, giperbola grafigini I choragida chizish kifoya. Tenglama I chorakda $y=\frac{b}{a}\sqrt{x^2-a^2}$ ko'rinishda bo'ladi. Undan quyidagilar kelib chiqadi:

1) $0\le x<a$ da funksiya aniqlanmagan

2) $x=a$ da $y=0$

3) $x>a$ da $y>0$. Agar x→+∞ bo'lsa, y→+∞.



$y=\pm\frac{b}{a}x$ to‘g‘ri chiziqlari asimptotalari deyiladi. I chorakda giperbola va asimptota farqini baholaymiz:

$$\frac{b}{a}x-\frac{b}{a}\sqrt{x^2-a^2}=\frac{b}{a}(x-\sqrt{x^2-a^2})\cdot\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}=\frac{ab}{x+\sqrt{x^2-a^2}}$$

Oxirgi kasr $x\to+\infty$ da nolga intiladi, demak, giperbola $y=\frac{b}{a}x$ to‘g‘ri chiziqqa yaqinlashib boradi.

Agar giperbola tenglamasi $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ko‘rinishida bo‘lsa, giperbola fokuslari OY o‘qida bo‘lib, giperbola shohlari oy o‘qi bo‘lib yo‘naladi. Bu giperbola oldingisiga nisbatan qo‘shma deyiladi.

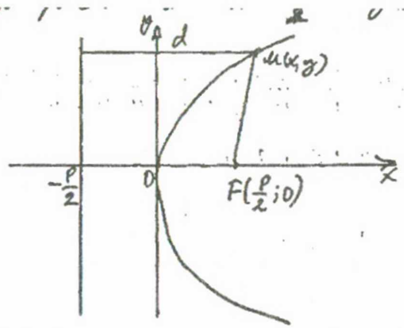
Agar $a=b$ bo‘lsa, giperbola teng tomonli deyilib, tenglamasi $x^2-y^2=a^2$ ko‘rinishida bo‘ladi.

$\varepsilon=\frac{c}{a}$ soni giperbola ekssentrisiteti deyilib, $\varepsilon>1$. $x=\pm\frac{a}{\varepsilon}$ to‘g‘ri chiziqlar giperbola direktrisalari deyiladi. Giperbolada ham $\frac{r}{d}=\varepsilon$ o‘rinlidir.

3.4. Parabola kanonik tenglamasi, uning xossalari

Fokus deb ataluvchi F nuqtadan va direktrisa deb ataluvchi to‘g‘ri chiziqdan bir xil uzoqlikda joylashgan nuqtalar to‘plami **parabola** deyiladi.

Fokusdan direktrisagacha bo‘lgan masofa p bilan belgilanib, y parabola parametri deyiladi. Parabola tenglamasini olish uchun F nuqtani Ox o‘qi bo‘ylab koordinata boshidan $\frac{p}{2}$ masofada joylashtramiz. Direktrisa esa $x=-\frac{p}{2}$ bo‘ladi. Parabola ixtiyoriy M(x;y) nuqtasi uchun $|MF|=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}$ va direktrisagacha bo‘lgan masofa $x+\frac{p}{2}$ ekanligidan $\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}=x+\frac{p}{2}$ kelib chiqadi. Bu tenglikni kvadratga oshirib: $x^2-px+\frac{p^2}{4}+y^2=x^2+px+\frac{p^2}{4}$ soddalashtirsak, parabolaning kanonik tenglamasi kelib chiqadi: $y^2=2px$.



Parabola grafigi Ox o‘qiga nisbatan simmetrik bo‘lib, koordinata boshidan o‘tadi. Parabolada r=d ekanligidan $\varepsilon=1$ bo‘ladi. $y^2=2px$ uchun abssissa o‘qi simmetriya o‘qi bo‘lib, parabola o‘qi deyiladi. p soni fokusdan direktrisagacha masofani bildiradi.

Agar parabola $y^2=-2px$ ko‘rinishda bo‘lsa, uning grafigi $(-\infty;0]$ da aniqlanadi.

3.5. ITCh lar qutbiy tenglamalari

ITCh lardan birini olamiz. Uning fokusi joylashgan nuqtaga qutbni joylab, boshlang‘ich nurni abssissa musbat yo‘nalishi bo‘yicha yo‘naltiramiz. ITCh direktrisasi L bo‘lsin, ixtiyoriy nuqtasini M($r;\varphi$) bilan belgilaymiz. Fokusadan ITCh gacha oy o‘qidagi kesma p-fakal parametr bo‘lsin $\frac{p}{|DF|}=\varepsilon=\frac{r}{d}$ ekanligidan $|DF|=\frac{p}{\varepsilon}$, ya’ni $d=|DF|+|FN|=\frac{p}{\varepsilon}+r\cos\varphi$ Demak, $\frac{r}{\varepsilon}=\frac{p}{\varepsilon}+r\cos\varphi$, yoki r = p+r $\varepsilon\cos\varphi$ bundan ixtiyoriy ITCh qutb tenglamasi $r=\frac{p}{1-\varepsilon\cos\varphi}$ ko‘rinishida bo‘lishi kelib chiqadi, unda p-fokal parametr ε qaralayotgan chiziq ekssentrisiteti.

Bu tenglama $\varepsilon=0$ da aylana, $0<\varepsilon<1$ da ellips, $\varepsilon=1$ da parabola, $\varepsilon>1$ da esa giperbola tenglamasidir.

3.6. ITCh larni kanonik ko‘rinishga keltiring

Ikkinchi tartibli chiziqlar umumiy $Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0$ (1) tenglama bilan beriladi. Agar dekart koordinatalarda son o‘qlarini parallel ko‘chirsak, yangi ($o'x'y'$) sistema, qo‘shimchasiga o‘qlarni biror α burchakka bursak, $o''x''y''$ sistema hosil bo‘ladi. So‘nggi sistemani OXY tarzida belgilaymiz.

Teorema. ITCh umumiy tenglamasi koordinata o‘qlarini parallel ko‘chirish va biror burchakka burish yordamida quyidagi hollardan biriga keltiriladi:

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellips, $a=b$ *da aylana*)
2. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (giperbola)
3. $y^2=2px$ (parabola)
4. $y^2-k^2x^2=0$ (ikki kesishuvchi to‘g‘ri chiziq)
5. . $y^2-k^2=0$ yoki $x^2-k^2=0$ (ikki parallel to‘g‘ri chiziq)
6. $y^2=0$ yoki $x^2=0$ (ustma-ust tushgan to‘g‘ri chiziqlar)
7. $y^2+k^2x^2=0$ (bitta nuqta)
8. $y^2+k^2x^2=-1$ (bo‘sh to‘plam)

Isbot. Dastlab, O(0;0) koordinatalar boshini biror $P(x_0;y_0)$ nuqtaga parallel ko‘chiramiz. Unda $\begin{cases} x=x'+x_0 \\ y=y'+y_0 \end{cases}$ almashtrish o‘tkaziladi. Umumiy tenglama:

$A(x'^2+2x'x_0+x_0^2)+2B(x'y'+x'y_0+x_0y'+x_0y_0)+C(y'^2+2y'y_0+y_0^2)+2D(x'+x_0)+2E(y'+y_0)+F=0$ yoki

$Ax'^2+2Bx'y'+Cy'^2+(2Ax_0+2By_0+2D)x'+(2Bx_0+2Cy_0+2E)y'+(Ax_0^2+2Bx_0y_0+Cy_0^2+2x_0D+2y_0E+F)=0$

ko‘rinishiga keladi.

Demak, $(x_0;y_0)$ $\begin{cases} Ax_0+By_0+D=0 \\ Bx_0+Cy_0+E=0 \end{cases}$ sistema yechimi bo‘lsa, umumiy tenglama $Ax'^2+2Bx'y'+Cy'^2+F'=0$ (2) ko‘rinishiga kelar ekan.

(2)-tenglamadagi $x'y'$ ko‘paytmani yo‘qotish uchun o‘qlarni biror α burchakka buramiz, ya’ni $\begin{cases} x'=x\cos\alpha-y\sin\alpha \\ y'=x\sin\alpha+y\cos\alpha \end{cases}$ almashtrish o‘tkazamiz. Yangi sistemada $x\cdot y$ ko‘paytma koeffitsiyenti

$-A\sin 2\alpha+2B\cos 2\alpha+C\sin 2\alpha=0$ bo‘lsa, teorema isbotlanadi. Buning uchun $ctg2\alpha=\frac{A-C}{2B}$ shart o‘rinli bo‘ladigan α burchak tanlash yetarli. (2)-tenglama $A'x^2+C'y^2+F'=0$ tenglamaga keladi.

Bu tenglama berilgan ITCh ning kanonik ko‘rinishi deyiladi.

Kanonik tenglama olinguncha A,B,C-sonlari o‘zgarmaydi. $AC-B^2$ ifoda ITCh tenglamasi invarianti deyiladi. Bu ifoda uchun, $A'\cdot C'-B'^2=AC-B^2$ bo‘ladi.

ITCh $AC-B^2$ ifoda ishorasiga ko‘ra quyidagi turlarga bo‘linadi:

1. $AC-B^2>0$ bo‘lsa, ITCh elliptik tipda,
2. $AC-B^2=0$ bo‘lsa, ITCh parabolik tipda,
3. $AC-B^2<0$ bo‘lsa, ITCh giperbolik tipda bo‘ladi.

1) $3x^2+10xy+3y^2-2x-14y-13=0$ tenglamani kanonik ko‘rinishga keltiring.

$\begin{cases} 3x_0+5y_0-1=0 \\ 5x_0+3y_0-7=0 \end{cases}$ sistema yechimi $x_0=2,\ y_0=-1$ bo‘lganligi uchun $\begin{cases} x=x'+2 \\ y=y'-1 \end{cases}$ almashtirish o‘tkazamiz:

$3(x'^2+4x'+4)+10(x'y'-x'+2y'-2)+3(y'^2-2y'+1)-2(x'+2)-14(y'-1)-13=0$ yoki

$3x'^2+10x'y'+3y'^2-8=0$. Endi $ctg2\alpha=\frac{3-3}{10}=0$ dan $\alpha=45^\circ$ ekanligini topib, $\begin{cases} x'=\frac{\sqrt{2}}{2}(x-y) \\ y'=\frac{\sqrt{2}}{2}(x+y) \end{cases}$ almashtirish o‘tkazamiz:

$3\cdot\frac{1}{2}(x^2-2xy+y^2+10\cdot\frac{1}{2}(x^2-y^2)+3\cdot\frac{1}{2}(x^2+2xy+y^2)-8=0$, yoki

$$8x^2-2y^2=8$$

Tekshirilgan ITCh $x^2-\frac{y^2}{2^2}=1$ tenglamaga ega giperbola bo‘ladi. $AC-B^2=3\cdot 3-5^2<0$ ekanligi ham buni tasdiqlaydi.

2). $8x^2+4xy+5y^2+16x+4y-28=0$ kanonik ko‘rinishga keltirilsin.

$\begin{cases} 8x_0+2y_0+8=0 \\ 2x_0+5y_0+2=0 \end{cases}$ yechimi (-1;0) bo‘lganligi uchun $\begin{cases} x=x'-1 \\ y=y' \end{cases}$ almashtirish o‘tkazamiz.

$8(x'^2-2x'+1)+4(x'\cdot y'-y')+5y'^2+16(x'-1)+4y'-28=0$ yoki

$8x'^2+4x'y'+5y'^2-36=0$.

So‘ngra $ctg2\alpha=\frac{8-5}{4}=\frac{3}{4}$ ekanligini topamiz. Bu holda α burchakni aniqlab bo‘lmaydi, shuning uchun $\sin\alpha, \cos\alpha$ larni topishga harakat qilamiz: $\frac{1}{tg2\alpha}=\frac{3}{4}$ yoki $\frac{1-tg^2\alpha}{2\,tg\,\alpha}=\frac{3}{4}$, $2\,tg^2\alpha+3\,tg\,\alpha-2=0$ tenglamaga egamiz. Bundan $tg\,\alpha=\frac{1}{2}$ yechimni olishimiz mumkin ($0<\alpha<\frac{\pi}{2}$ bo‘lishiga harakat qildik xolos). $1+tg^2\alpha=\frac{1}{\cos^2\alpha}$ ayniyatdan $\cos\alpha=\frac{1}{\pm\sqrt{1+tg^2\alpha}}=\pm\frac{2}{\sqrt{5}}$, undan $\sin\alpha=\pm\frac{1}{\sqrt{5}}$ biz $\cos\alpha=\frac{2}{\sqrt{5}}$, $\sin\alpha=\frac{1}{\sqrt{5}}$ deb almashtirish o‘tkazamiz: $x'=\frac{1}{\sqrt{5}}(2x-y)$, $y'=\frac{1}{\sqrt{5}}(x+2y)$ ekanligidan, $\frac{8}{5}(4x^2-4xy+y^2)+\frac{4}{5}(2x^2+3xy-2y^2)+\frac{5}{5}(x^2+4xy+4y^2)-36=0$.

Soddalashtirib, $9x^2-4y^2-36=0$ *yoki* $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$ ellips kanonik tenglamasini hosil qilamiz.

Mavzuga doir masalalar

1. Quyidagi aylanalar markazi koordinatalari va radiusini toping:
a) $x^2+y^2-4x+6y-3=0$, b) $x^2+y^2-8x=0$, c) $x^2+y^2+4y=0$
2. A(-4;6) nuqta berilgan. Diametri OA kesmadan iborat aylana tenglamasini yozing:
3. A(1;2) nuqtadan o‘tuvchi va koordinata o‘qlariga urinuvchi aylana tenglamasini yozing:

4. A(-1;3), B(0;2), C(1;-1) nuqtalardan o'tuvchi aylana tenglamasini yozing.

5. $y = -\sqrt{-x^2 - 4x}$ chiziq shaklini chizing.

6. Berilgan nuqtadan berilgan aylanagacha bo'lgan eng qisqa (eng uzun) masofani toping.
a) A(6;-8), $x^2+y^2=9$. b) B(3;9), $x^2+y^2-26x+30y+313=0$
7. Aylanalar orasidagi eng qisqa va eng katta masofani toping.
a) $x^2+y^2+4x-4y+7=0$ va $x^2+y^2-8x-8y+23=0$. b) $x^2+y^2+4x-4y+7=0$ va $x^2+y^2=25$
8. Qutb koordinatalarida berilgan aylanalar markazi va radiusini aniqlang.
a) $r=3\cos\varphi$, b) $r=-4\cos\varphi$, c) $r=\cos\varphi-\sin\varphi$.

9. Fokuslari abssissalar o'qida koordinata boshiga nisbatan simmetrik joylashgan ellips tenglamasini quyidagi shartlarda yozing:
a) yarim o'qlari 5 va 2; b) katta o'qi 10, 2c=8; c) kichik o'qi 24, 2c=10;
d) 2c=6, ε=0,6; e) direktrisalar orasidagi masofa 32 va ε=0,5.

10. $9x^2+25y^2=225$ ellips berilgan. Quyidagilarni toping:
a) yarim o'qlari; b) fokuslari; c) ekssentrisiteti d) direktrisa tenglamasi
11. Quyidagi ellipslar fokuslari koordinatalari, yarim o'qlari, ekssentrisiteti va direktrisa tenglamalarini toping:
a) $5x^2+9y^2-30x+18y+9=0$; b) $16x^2+25y^2+32x-100y-284=0$; c) $4x^2+3y^2-8x+12y-32=0$.
12. Quyidagi chiziqlar shaklini chizing:
a) $y = -7 + \frac{2}{5}\sqrt{16 + 6x - x^2}$, b) $x = -2\sqrt{-5 - 6y - y^2}$.

13. Fokuslari abssissa o'qida koordinata boshiga nisbatan simmetrik joylashgan giperbola tenglamasini quyidagi shartlarda tuzing:
a) 2a=10, 2b=8; b) 2c =2, 2b=8; c) 2c=6, ε=1,5; d) 2a=16, ε=1,25;

e) 2c=20 va asimptotalari $y = \pm\frac{4}{3}x$.

14. $16x^2-9y^2=144$ giperbolada a,b, fokuslar koordinatalari, ekssentrisiteti, asimptota va direktrisa tenglamalarini toping.
15. Quyidagi chiziqlar shaklini chizing:
a) $y = \frac{2}{3}\sqrt{x^2 - 9}$, b) $x = -\frac{4}{3}\sqrt{y^2 + 9}$.

16. Quyidagi giperbolalar fokuslari koordinatalari, yarim o'qlari, ekssentrisiteti, asimptota va direktrisa tenglamalarini toping:
a) $16x^2-9y^2-64x-54y-161=0$, b) $9x^2-16y^2+90x+32y-367=0$.
17. Uchi koordinata boshida joylashgan va quyidagi shartga bo'ysunuvchi parabola tenglamasini tuzing:
a) Abssissaga nisbatan simmetrik va A(9;6) nuqtadan o'tuvchi;

b) Ordinatalar o'qiga nisbatan simmetrik va C(1;1) dan o'tuvchi.
18. Quyidagi chiziqlar shaklini chizing:
a) $y = 2\sqrt{x}$, b) $y = -3\sqrt{-2x}$, c) $x = -\sqrt{3y}$, d) $x = 4\sqrt{-y}$

19. $r = \frac{12}{3-\sqrt{2}\cos\varphi}$ ellipsda r=6 bo'ladigan nuqtani aniqlang.

20. $r = \frac{15}{3-4\cos\varphi}$ giperbolada r=3 bo'ladigan nuqtani aniqlang.

21. $r = \frac{p}{1-\cos\varphi}$ parabolada eng kichik radiusli nuqtani aniqlang.

22. Kanonik ko'rinishiga keltiring:
1. $4x^2 + 9y^2 - 40x + 36y + 100 = 0$
2) $x^2 - 2xy + y^2 - 12x + 12y - 14 = 0$
3) $2x^2 + 6\sqrt{3}xy - 4y^2 - 9 = 0$
4) $x^2 - 3\sqrt{3}xy - 2y^2 - 10 = 0$
5) $9x^2 - 24xy + 16y^2 - 20x + 110y - 50 = 0$.

Tekislikda analitik geometriyaga doir, joriy nazorat uchun uy vazifalari

(N-talabaning guruh ro'yhatidagi nomeri)

I. Tekislikda A(-1;-1), B(1;N), C(N;1) berilgan. Quyidagilarni toping:
1) ABC uchburchak perimetri;
2) ABC uchburchak og'irlik markazi koordinatalari;
3) ABC uchburchak yuzi;
4) C nuqtadan o'tuvchi to'g'ri chiziqlar dastasi tenglamasi;
5) A va B nuqtalardan o'tuvchi to'g'ri chiziq barcha tenglamalari;
6) C nuqtadan o'tib, AB ga parallel (perpendikulyar) to'g'ri chiziq;
7) C nuqtadan AB gacha masofa;
8) ABC uchburchak ichki burchaklari;
9) C uchidan tushirilgan mediana, bissektrisa, balandlik tenglamalari;
10) C nuqtaning AB dagi proyeksiyasi koordinatalari;
11) Shunday E(x;o), F(o;y) nuqtalarni topingki, ulardan A gacha masofa (N+5) bo'lsin;
12) Shunday D(x; y) nuqta topingki, ABCD parallelogramm bo'lsin.

II. Tekislikda shunday nuqtalar tenglamasini tuzingki, ular quyidagi shartlarni qanoatlantirsin:
1) A(N-10;2) va B(3;20-N) nuqtalardan bir xil uzoqlikda;
2) A(-N;0) va B(N;0) gacha masofalar 1:N nisbatda;

3) A(-N;0) va B(N;0) gacha masofalar yig'indisi 4N;
4) A(-N;0) va B(N;0) gacha masofalar ayirmasi N/2;
5) x+N=0 va B(N;0) gacha masofalar teng.

III. Qutb koordinatalar sistemasida $A(N;-\pi/6)$, $B(N/2;\pi/4)$, $C(N;\pi/3)$ berilgan. Quyidagilarni toping:
1. ABC uchburchak perimetri; 2. ABC uchburchak yuzi.
IV. Qutb koordinatalar sistemasida $r=\frac{1}{1-\frac{N}{15}\cos\varphi}$ chiziq berilgan.
1. $\varphi=0,\ \pi/12,\ \pi/6,\ 2\pi$ qiymatlarda hisoblang, chizing;
2. Dekart koordinatalariga o'tkazing va kanonik tenglamasini yozing.
3. ITCh ga mos parametrlari (fokuslar koordinatalari, ekssentrisiteti, fokal radiuslari) topilsin.
V. Kanonik ko'rinishiga keltiring.
1. $Nx^2+(-1)^N Ny^2+4(-1)^{N+1}x+8(-1)^N Ny=0$; 2. $x^2+4xy+y^2-Nx+Ny-N=0$;
2. $3N\sqrt{3}xy+(-1)^N 3Ny^2-100=0$.

Oliy algebra elementlari

4-mavzu. Kompleks sonlar

$x^2+1=0$ kabi tenglamalarda, kvadrati-1 ga teng haqiqiy sonning mavjud emasligi, haqiqiy sonlar to'plamini kengaytirish zarurligini taqozo etadi.

Kvadrati -1 ga teng bo'ladigan son mavhum birlik deyiladi va i harfi bilan belgilanadi, ya'ni $i=\sqrt{-1}$.

Tarkibida mavhum birlik i qatnashgan son kompleks son (mavhum son) deyiladi.

4.1. Kompleks sonning algebraik formasi

$x,\ y\in R$ bo'lganda $z=x+iy$ son kompleks son, yozuv esa kompleks son algebraik formasi deyiladi.

x soni kompleks son haqiqiy qismi deyiladi va Rez ko'rinishida, y soni esa mavhum qismi deyilib, Imz tarzida belgilanadi.

$x+iy$ va $x-iy$ sonlar o'zaro qo'shma kompleks sonlar deyiladi. Ulardan biri z bo'lsa, ikkinchisi $\bar{z}$ ko'rinishida belgilanadi. O'zaro qo'shma $z,\bar{z}$ sonlar yig'indisi, ko'paytmasi haqiyqiy bo'lishi ravshan. Bundan tashqari, $\overline{z_1 z_2}=\overline{z_1}\cdot\overline{z_2}$, $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$, $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}}$ tengliklarni keyinchalik isbotlash mumkin.

$z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ sonlari ustida amallar quyidagicha kiritiladi:

1). $z_1\pm z_2=(x_1+iy_1)\pm(x_2+iy_2)=(x_1\pm x_2)+i(y_1\pm y_2)$,

2). $z_1\cdot z_2=(x_1+iy_1)\cdot(x_2+iy_2)=x_1x_2+ix_1y_2+iy_1x_2-y_1y_2=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$,

3). $\frac{z_1}{z_2}=\frac{x_1+iy_1}{x_2+iy_2}\cdot\frac{x_2-iy_2}{x_2-iy_2}=\frac{x_1x_2-ix_1y_2+iy_1x_2+y_1y_2}{x_2^2+y_2^2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}$.

Masalan, $z_1=1+2i$, $z_2=1-i$ bo'lsa,

$z_1+z_2=1+2i+1-i=2+i$, $\quad z_1-z_2=1+2i-1+i=3i$,

$z_1\cdot z_2=(1+2i)\cdot(1-i)=1-i+2i+2=3+i$,

$$\frac{z_1}{z_2}=\frac{1+2i}{1-i}\cdot\frac{1+i}{1+i}=\frac{1+i+2i-2}{1+1}=-\frac{1}{2}+\frac{3}{2}i$$

$\overline{z_1z_2}=\overline{z_1}\cdot\overline{z_2}$ tenglikni tekshiramiz:

$\overline{z_1z_2}=\overline{(x_1+iy_1)\cdot(x_2+iy_2)}=\overline{(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)}=(x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1)$

$\overline{z_1}\cdot\overline{z_2}=(x_1-iy_1)\cdot(x_2-iy_2)=x_1x_2-ix_1y_2-iy_1x_2-y_1y_2=(x_1x_2-y_1y_2)-i(x_1y_2+y_1x_2)$.

4.2. Kompleks son trigonometrik formasi. Muavr formulalari

Tekislikda dekart koordinatalari sistemasini kiritib, kompleks son haqiqiy qismini abssissalar o‘qiga, mavhum qismini ordinatalar o‘qiga joylashtiramiz. Tekislikdagi M(x;y) nuqta z=x+iy kompleks sonning tekislikdagi geometrik tasviri deyiladi. Turli xil kompleks sonlarga tekislikning turli nuqtalari mos keladi, bu moslik o‘zaro bir qiymatlidir.

Agar qutb koordinatalari ham kiritilsa, $M(r\cos\varphi; r\sin\varphi)$ bo‘ladi. Koordinatalar boshidan M(x;y)gacha masofa berilgan kompleks son moduli deyiladi, $|z|$ tarzida belgilanadi.

Ravshanki, $r = |z| = \sqrt{x^2+y^2}$.

Qutb burchagi φ esa z kompleks son argumenti deyiladi, $\arg z$ tarzida belgilanadi: $\varphi = \arg z, 0 \le \arg z < 2\pi$

Argumentni $\operatorname{tg}\varphi = \frac{y}{x}$ munosabatdan topish qulay.

$x = r\cos\varphi, y = r\sin\varphi$ bog‘lanishdan foydalanib:

$z = x + iy = r\cos\varphi + ir\sin\varphi = r[\cos\varphi + i\sin\varphi]$

Kompleks sonning $z = r[\cos\varphi + i\sin\varphi]$ ko‘rinishi **trigonometrik formasi** deyiladi.

$z = 1 - \sqrt{3}i$ bo‘lsa, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$, $\operatorname{tg}\varphi = \frac{-\sqrt{3}}{1}$, $x > 0, y < 0$ bo‘lganligidan $\varphi \in IV$ va $\varphi = \frac{5\pi}{3}$.

Demak, $z = 1 - \sqrt{3}i = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$ trigonometrik formada berilgan $z_1 = r_1[\cos\varphi_1 + i\sin\varphi_1]$ va $z_2 = r_2[\cos\varphi_2 + i\sin\varphi_2]$ kompleks sonlar ustida ammallar quyidagicha kiritiladi: 1) $z_1 \pm z_2 = (r_1\cos\varphi_1 \pm r_2\cos\varphi_2) + i(r_1\sin\varphi_1 \pm r_2\sin\varphi_2)$.

2) $z_1 \cdot z_2 = r_1 \cdot r_2[\cos\varphi_1\cos\varphi_2 + i\cos\varphi_1\sin\varphi_2 + i\sin\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2] =$
$= r_1 \cdot r_2 \cdot [\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)]$,

3) $\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} \cdot \frac{\cos\varphi_2 - i\sin\varphi_2}{\cos\varphi_2 - i\sin\varphi_2} = = \frac{r_1}{r_2} \cdot \frac{\cos\varphi_1\cos\varphi_2 - i\cos\varphi_1\sin\varphi_2 + i\sin\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2}{\cos^2\varphi + \sin^2\varphi} =$
$= \frac{r_1}{r_2} \cdot [\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)]$

Agar $z_1 \cdot z_2$ amalida $z = z_1 = z_2 = r(\cos\varphi + i\sin\varphi)$ bo‘lsa, $z^2 = r^2[\cos2\varphi + i\sin2\varphi]$, $z^3 = r^3[\cos3\varphi + i\sin3\varphi], \ldots,$
$z^n = [r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos\varphi + i\sin\varphi)^n = r^n[\cos n\varphi + i\sin n\varphi]$

Oxirgi tenglik Muavrning **darajaga oshirish formulasi** deyiladi.

Misol sifatida $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$ ni hisoblaymiz.

Dastlab, kompleks son algebraik formulasini topamiz:

$\frac{1+i\sqrt{3}}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i+i\sqrt{3}-\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2}$

$r = \sqrt{\left(\frac{1-\sqrt{3}}{2}\right) + \left(\frac{1+\sqrt{3}}{2}\right)} = \sqrt{2}$, $\operatorname{tg}\varphi = \frac{1+\sqrt{3}}{1-\sqrt{3}} = -(2+\sqrt{3})$ va y>0, x<0 ekanligidan y є II, $\varphi = 105° = \frac{7\pi}{12}$ kelib chiqadi.

Demak, Muavr darajaga oshirish formulasiga ko‘ra:

$\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20} = \left[\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)\right]^{20} = 2^{10} \cdot \left[\cos20 \cdot \frac{7\pi}{12} + i\sin20 \cdot \frac{7\pi}{12}\right] = 2^{10}$
$\cdot \left[\cos\frac{35\pi}{3} + i\sin\frac{35\pi}{3}\right] = 2^{10} \cdot \left[\cos\left(12\pi - \frac{\pi}{3}\right) + i\sin\left(12\pi - \frac{\pi}{3}\right)\right] = 2^{10} \cdot \left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2^{10}$
$\cdot \left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = 2^9 \cdot (1 - i\sqrt{3})$.

Kompleks sondan ildiz chiqarish masalasini qaraymiz .

$\sqrt[n]{z} = \sqrt[n]{x+iy} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)}$

Albatta, bu ildizdan qandaydir $p(\cos\theta + i\sin\theta)$ kompleks son chiqqan deb faraz qilish mumkin, u holda

$[p(\cos\theta + i\sin\theta)]^n = p^n[\cos n\theta + i\sin n\theta)] = r(\cos\varphi + i\sin\varphi)$ tenglik bajarilishi kerak.

$\sin\varphi$, $\cos\varphi$ larning davri 2π ekanligini hisobga olib,

$p^n = r$, $n\theta = u + 2\pi k$ lardan $p = \sqrt[n]{r}$, $\theta = \frac{\varphi + 2k}{n}$, к є Z kelib chiqadi natijada, Muavrning ildiz chiqarish formulasi.

$\sqrt[n]{r(\cos\varphi + \sin\varphi)} = \sqrt[n]{r}\left[\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right]$, к є Z hosil bo‘ladi. Bunda $\kappa = \overline{0, n-1}$, bo‘lganda n ta ildiz topiladi. к = n, n+1,…qiymatlarda esa davr hisobiga. avvalgi ildizlar bilan ustma-ust tushadigan ildizlar kelib chiqadi.

Misol $\sqrt[4]{-\sqrt{3} - i}$

$r = \sqrt{-(\sqrt{3})^2 + (-1)^2}$, $\operatorname{tg}y = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$, x<0, y<0 bo‘lgani uchun $\varphi \in III$, $\varphi = \frac{7\pi}{6}$

Muavr ildiz chiqarish formulasidan

$\sqrt[4]{-\sqrt{3} - i} = \sqrt[4]{2\left(\cos\frac{7\pi}{6} i\sin\frac{7\pi}{6}\right)} = \sqrt[4]{2} \cdot \left[\cos\frac{\frac{7\pi}{6} + 2k\pi}{4} + i\sin\frac{\frac{7\pi}{6} + 2k\pi}{4}\right]$,
$\sqrt[4]{-\sqrt{3} - i} = \sqrt[4]{2\left(\cos\frac{7\pi}{6} i\sin\frac{7\pi}{6}\right)} = \sqrt[4]{2} \cdot \left[\cos\frac{\frac{7\pi}{6} + 2k\pi}{4} + i\sin\frac{\frac{7\pi}{6} + 2k\pi}{4}\right]$,

$\kappa = 0$ da $z_0 = \sqrt[4]{2} \cdot \left[\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right]$, $\kappa = 1$ da $z_1 = \sqrt[4]{2} \cdot \left[\cos\frac{19\pi}{24} + i\sin\frac{19\pi}{24}\right]$,
$\kappa = 2$ da $z_2 = \sqrt[4]{2} \cdot \left[\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right]$, $\kappa = 3$ da $z_3 = \sqrt[4]{2} \cdot \left[\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right]$ kelib chiqadi.

$\sqrt[5]{-1}$ ning ildizlarini topamiz, $r = \sqrt{(-1)^2 + 0^2} = 1$ va $\operatorname{tg}\varphi = \frac{0}{-1} = 0$, x<0 ekanligidan $\varphi = \pi$, u holda

$\sqrt[5]{-1} = \sqrt[5]{\cos\pi + i\sin\pi} = \cos\frac{\pi+2k\pi}{5} + \sin\frac{\pi+2k\pi}{5}$

$z_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, $z_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$,
$z_2 = \cos\pi + i\sin\pi = -1$, $z_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$,
$z_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$ Demak, ildizlarning bittasi haqiqiy, qolganlari ikki juft o‘zaro qo‘shma kompleks sonlar ekan.

4.3. Kompleks sonning ko‘rsatkichli formasi

Dastlab, Eyler ayniyati deb ataluvchi $e^{i\varphi}=\cos\varphi+\sin\varphi$ formulani hozircha isbotsiz qabul qilamiz. U holda z= $x+iy=r[\cos\varphi+\sin\varphi]=r\cdot e^{i\varphi}$ kelib chiqadi. z= $r\cdot e^{i\varphi}$ yozuv kompleks son ko‘rsatkichli formasi deyiladi, kompleks sonning bu formasi ixcham yozilishi bilan ajralib turadi, masalan $z_1=e^{i\varphi 1}, z_2=e^{i\varphi 2}$ kompleks sonlar ko‘paytmasi $z_1\cdot z_2=r_1r_2e^{i(\varphi_1+\varphi_1)}$ bo‘linmasi esa $\frac{z_1}{z_2}=\frac{r_1}{r_2}e^{i(\varphi_1-\varphi_2)}$ tarzida yoziladi. Muavr darajaga oshirish formulasi $z^n=r^n\cdot e^{i\varphi}$ ko‘rinishida yoziladi.

Kompleks sonlar to‘plami C harfi bilan belgilanadi.

4.4. Radikallarda yechiladigan tenglamalar

Talabaga ax = b chiziqli tenglama, a ≠ 0 da $x=\frac{-b}{a}$ yechim bo‘lishi,

$ax^2+bx+c=0$ kvadrat tenglama ildizlari $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ formula bilan,

$x^2+px+q=0$ keltirilgan kvadrat tenglama yechimlari esa $x_{1,2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$ formula bilan topilishi, ular uchun $x_1+x_2=-p$, $x_1\cdot x_2=q$ tengliklarni ifadalovchi **Viyet teoremasi** tanish deb o‘ylaymiz. Bundan tashqari, $ax^2+2\kappa x+c=0$ uchun $x_{1,2}=\frac{-\kappa\pm\sqrt{\kappa^2-ac}}{a}$, $x^2+2\kappa x+c=0$ uchun $-\kappa\pm\sqrt{\kappa^2-c}$ ildiz topish formulalari o‘rinli ekanligini eslatib o‘tamiz.

$x^3+a_1x^2+a_2x+a_3=0$ kubik tenglamani olaylik. Tomonlarini a_0 ga bo‘lib, $x^3+ax^2+bx+c=0$ tenglamaga ega bo‘lamiz.

Shunday $x=y+\alpha$ almashtirish o‘tkazamizki, oxirgi tenglama soddalashsin.

$y^3+(3\alpha+a)y^2+(3\alpha^2+2a\alpha+b)y+(\alpha^3+a\alpha^2+bd+c)=0$

Demak, $x=y-\frac{a}{3}$ almashtrish o‘tkazilsa, kubik tenglama $y^3+py+q=0$ ko‘rinishga ega bo‘ladi.

Oxirgi tenglama yechimlarini y=u+v ko‘rinishida qidiriladi, bunda $u\cdot v=-\frac{p}{3}$ sharti shunday yechim mavjudligini ta’minlaydi, ya’ni ular $t^2-yt-\frac{p}{3}=0$ tenglama yechimlaridir.

$(u+v)^3+p(u+v)+q=0,\ldots,(u^3+v^3+q)+(3uv+p)(u+v)=0$

3uv+p=0 ekanligidan, $\begin{cases}u^3+v^3=-q\\ u^3\cdot v^3=-\frac{p^3}{27}\end{cases}$ sistemaga ega bo‘lamiz, Viyet teoremasidan u^3, v^3лap $z^2-qz-\frac{p3}{27}=0$ tenglama ildizlari ekanligi kelib chiqadi $z_1=u^3=-\frac{q}{2}+\sqrt{\frac{q2}{4}+\frac{p3}{27}}$, $z_2=v^3=-\frac{q}{2}-\sqrt{\frac{q2}{4}+\frac{p3}{27}}$, Demak, $y^3+py+q=0$ tenglama yechimi:

$$y=u+v=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q2}{4}+\frac{p3}{27}}}+\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q2}{4}+\frac{p3}{27}}}$$

formuladan topilib, u **Kardano formulasi** deyiladi.

Har bir kub ildizdan uchta qiymatga, u+v uchun esa 9 ta qiymatga ega bo‘lamiz. Bu qiymatlardan $v\cdot u=-\frac{p}{3}$ shartga bo‘ysunuvchi uchtasigina tenglama yechimi bo‘ladi.

$\Delta=\frac{q^2}{4}+\frac{p3}{27}$ soni $y^3+py+q=0$ kubik tenglama diskriminanti deyiladi. Uning yordamida ildizlar quydaigicha topiladi:

1) $\Delta>0$ bo‘lsa, bitta haqiqiy, ikkita o‘zaro qo‘shma kompleks ildizlar mavjud bo‘ladi:

$y_1=u_1+v_1$; $y_{2,3}=-\frac{v1+v1}{2}\pm\frac{v1-v1}{2}\sqrt{3i}$, bunda u_1,v_1 lar u, v ning haqiqiy qiymatlari.

2) $\Delta=0$, bo‘lsa uchta haqiqiy (ikkitasi o‘zaro teng) ildiz mavjud: $y_1=\frac{3q}{p}, y_2=y_3=\frac{-3q}{2p}=-\frac{y_1}{2}$

3) $\Delta=0$, bo‘lsa uchta turli haqiqiy ildizlar mavjud: $y_1=2\sqrt{-\frac{p}{3}}$, $\cos\frac{\varphi}{3}$,

$y_{2,3}=2\sqrt{-\frac{p}{3}}\cos\left(\frac{\varphi}{3}\pm120°\right)$, bunda $\cos\varphi=\frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}}$

1) $x^3+6x-7=0$ uchun, Kardano formulasidan,

$=\sqrt[3]{\frac{7}{2}+\sqrt{\frac{49}{4}+\left(\frac{6}{3}\right)^3}}+\sqrt[3]{\frac{7}{2}-\sqrt{\frac{49}{4}+\left(\frac{6}{3}\right)^3}}=\sqrt[3]{8}+\sqrt[3]{-1}$ kelib chiqadi. Ularning ildizlari 2, $-1\pm\sqrt{3i}$ va -1; $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ bo‘lganligidan, $\Delta>0$ ekanligini hisobga olsak, $x_1=1$, $x_{2,3}=-\frac{1}{2}\pm\frac{3\sqrt{3}}{2}i$ kelib chiqadi.

2) $x^3-12x+16=0$, $\Delta=\frac{16\cdot16}{4}+\frac{(-12)3}{27}=0$ ekanligidan $x_1=\frac{3\cdot16}{-12}=-4, x_{2,3}=\frac{-4}{2}=2$.

To‘rtinchi darajali $x^4+ax^3+bx^2+cx+d=0$ tenglama ildizlarini topish uchun uni $(x^2+\frac{a}{2}x+y)^2-[(2y+\frac{a^2}{4}-b)x2+(ay-c)x+(y^2-d)=0]$ ko‘rinishida yozib olamiz, bunda y yangi yordamchi kattalik.

Ayriluvchi uchhad biror $(\alpha x+\beta)$ning to‘la kvadrati bo‘lishi uchun diskriminanti nol bo‘lishi, ya’ni $(ay-c)^2-4(2y+\frac{a^2}{4}-b)(y^2-d)=0$ bo‘lishi zarur va yetarlidir. Hosil bo‘lgan kubik tenglamaning kamida bitta haqiqiy ildizi bor. Agar bu ildiz y_0 bo‘lib, uni topa olsak, to‘rtinchi darajali tenglama:

$$(x^2+\frac{a}{2}x+y_0)^2-(\alpha x+\beta)^2=0$$

$$\text{yoki}\left[x^2+(\frac{a}{2}+\alpha)x+(y_0+\beta)\right]\cdot\left[x^2+(\frac{a}{2}+\alpha)x+(y_0-\beta)\right]=0$$

ko'rinish oladi. Hosil bo'lgan ikki kvadrat tenglama yechilib, izlanayotgan to'rtta ildiz topiladi. Bu usul Kardano shogirdi A.Ferrari tomonidan ko'rsatilgan.

$x^4+2x^3-13x^2-38x-24=0$ tenglamani yechib ko'ramiz.

$(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)$ formuladan foydalanib,

$(x^2+x+y)^2-x^2-y^2-2x^2y-2xy-13x^2-38x-24=0$ ko'rinishga keltiramiz, chunki $(x^2+x+y)^2=x^4+x^2+y^2+2x^3+2x^2y+2xy$.

Demak, $(x^2+x+y)^2-[(2y+14)x^2+2(y+19)x+(y^2+24)]=0$.

Ayriluvchining diskriminanti $(y+19)^2-(2y+14)(y^2+24)=0$ bo'lishi kerak. Soddalashtirib, $2y^3+13y^2+10y-25=0$.

Bu tenglamani $(y-1)(2y^2+15y+25)=0$ ko'rinishda yozsak, $y_0=1$ deyish mumkinligi ko'rinadi. U holda $(x^2+x+1)^2-[16x^2+40x+25]=0$, bundan $(x^2+x+1)^2-(4x-5)^2=0$ kelib chiqadi.

$[x^2+5x+6][x^2-3x-4]=0$ kvadrat tenglamalarni yechib: $x_1=-1$, $x_2=4$, $x_3=-3$, $x_4=-2$ ekanligini topamiz.

Agar tenglama darajasi besh yoki undan katta bo'lsa, bunday tenglama umumiy hollarda radikallarda yechilmaydi (Abel teoremasi).

Mavzuga doir masalalar

1. $n\in N$ bo'lganda i^n ni hisoblang.

2. Amallarni bajaring:

1) $(2+3i)-(1-i)$, 2) $(1-i)(4+3i)$, 3) $\frac{1+i}{1-i}$, 4) $\frac{2i}{1+i}$, 5) $(1+i)^3$, 6) $(\frac{1-i}{1+i})^2$,

7) $(\frac{-1+i\sqrt{3}}{2})^2$ 8) $(\frac{-1+i\sqrt{3}}{2})^3$.

3. Tenglamalarni yeching:

1) $x^2-(2+i)x+(-1+7i)=0$ 2) $x^2-(3-2i)x+(5-5i)=0$ 3) $(2+i)x^2-(5-i)x+(2-2i)=0$

4. Muavr formulalaridan foydalanib hisoblang:

1) $(1-i)^{25}$ 2) $(\frac{1+\sqrt{3}+2}{1-i})^{20}$ 3) $(1-\frac{\sqrt{3}-i}{2})^{20}$ 4) $(\sqrt{3}-i)^{20}$

5) $\sqrt[2]{2-2i}$ 6) $\sqrt[4]{-4}$ 7) $\sqrt[6]{\frac{1-i}{1+\sqrt{3}+i}}$ 8) $\sqrt[3]{\frac{1-i}{1+\sqrt{3}i}}$

5. $(\frac{-1+i\sqrt{3}}{2})^n+(\frac{-1-\sqrt{3}}{2})^n=\begin{cases} 2, agar\ n=3k \\ -1, agar\ n=3k+1 \\ -1, agar\ n=3k+2 \end{cases}$ ekanligini isbotlang.

6. Agar $x+\frac{1}{x}=2\cos\varphi$ bo'lsa, $x^n+\frac{1}{x^n}=2\cos n\varphi$ ekanligini isbotlang.

7. $(1+i)^n=2^{\frac{n}{2}}(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4})$, $(\sqrt{3}-i)^n=2^n(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6})$ tengliklarni isbotlang .

8. Kardano formulasi bo'yicha yeching.

1) $x^3+6x+9=0$ 2) $x^3+12x+63=0$ 3) $x^3+9x^2+18x+28=0$ 4) $x^3+6x^2+30+25=0$ 5) $x^3-6x+4=0$ 6) $x^3+6x+2=0$ 7) $x^3+18x+15=0$ 8) $x^3+24x-56=0$ 9) $x^3+45x-98=0$

9. Ferrari usuli bilan yeching.

1) $x^4+4x^3-2x^2-12x+9=0$ 2) $x^4-2x^3-8x^2+13x-24=0$ 3) $x^4-2x^3+2x^2+4x-8=0$ 4) $x^4-6x^3+6x^2+27x-56=0$ 5) $x^4-4x^3+5x^2-2x-6=0$ 6) $x^4+2x^3+2x^2+10x+25=0$

10. $\sqrt[5]{1}$ ildizlari yordamida $\sin 18^0$, $\cos 18^0$larni hisoblang.

5-mavzu. Ko'phadlar

5.1. Tenglama ratsional ildizlarini topish. Viyet teoremasi

$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ tenglamani olamiz.

Agar $\frac{p}{q}$ ratsional son ($p\epsilon Z$, $q\epsilon N$) berilgan tenglama ildizlari bo'lsa, $a_n(\frac{p}{q})^n+a_{n-1}(\frac{p}{q})^{n-1}+\ldots+a_1\frac{p}{q}+a_0=0$ ayniyat bo'ladi.

Undan $a_np^n+a_{n-1}p^{n-1}\cdot q+\ldots+a_1p.q^{n-1}+a_0q^n=0$

Bu ayniyatdan quyidagi tenglamalarni yozish mumkin: $a_np^n=-q[a_{n-1}p^{n-1}+\ldots+a_1pq^{n-2}+a_0q^{n-1}]$, $a_0p^n=-p[a_np^{n-1}+a_{n-1}p^{n-2}q+\ldots+a_1q^{n-1}]$.

p va q qisqarmas ekanligidan, a_n ning q ga a_0 ning p ga bo'linishi kelib chiqadi.

Demak, quyidagi alomat o'rinli ekan: Agar $\frac{p}{q}$ ratsional son tenglama ildizi bo'lsa, q soni a_n ning p soni a_0 ning bo'luvchisi bo'ladi.

Xususan, $a_n=1$ bo'lsa, ratsional ildizlar a_0 ning bo'luvchilari bo'lishi mumkin, xolos.

1) $x^4+2x^3-13x^2-38x-24=0$ tenglamada $a_n=1$, $a_0=-24$ bo'lganligi uchun $\{\frac{p}{q}\}=\{\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24\}$

O'rniga qo'yib tekshirishlar ildizlar $x_1=-1$, $x_2=-2$, $x_3=-3$, $x_4=4$ ekanligini ko'rsatadi.

2) $24x^5+10x^4-x^3-19x^2-5x+6=0$ tenglamada $a_n=24$, $a_0=6$ ekanligidan $p=\{\pm1,\pm2,\pm3,\pm6,\}$, $q=\{1,2,3,4,6,8,12,24\}$. va $\frac{p}{q}$ ko'rinishdagi kasrlardan $\frac{1}{2},-\frac{2}{3},\frac{3}{4}$ lar tenglama ildizlari bo'lishi kelib chiqadi.

Demak, qolgan ikki ildiz yoki irratsional sonlar, yoki qo'shma kompleks sonlardir.

Agar $x^2+px+q=0$ tenglama ildizlari x_1 va x_2 bo'lsa, $(x-x_1)(x-x_2)=x^2-(x_1+x_2)x+x_1x_2$ tenglikdan $\begin{cases} x_1+x_2=-p \\ x_1\cdot x_2=q \end{cases}$

Viyet teoremasi keltirib chiqarilgan edi.

Bu teorema yuqori darajali tenglamalar uchun qanday ko'rinishda bo'lishini tekshirib ko'ramiz.

$x^3+a_0x^2+a_1x+a_2=0$ tenglama ildizlari x_1, x_2, x_3 bo'lsa, $(x-x_1)(x-x_2)(x-x_3)=x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_1x_3+x_2x_3)x-x_1x_2x_3$ tenglikdan Viyet teoremasi $\begin{cases} x_1+x_2+x_3=-a_0 \\ x_1x_2+x_1x_3+x_2x_3=a_1 \\ x_1x_2x_3=-a_2 \end{cases}$ ko'rinishida bo'lishi kelib chiqadi.

$x^4+a_0x^3+a_1x^2+a_2x+a_3=0$ uchun

$$\begin{cases} x_1+x_2+x_3+x_4=-a_0 \\ x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=a_1 \\ x_1x_2x_3+x_2x_2x_4+x_2x_3x_4+x_1x_3x_4=-a_2 \\ x_1x_2x_3x_4=a_3 \end{cases}$$

bo'lishi ravshan.

Umuman, $x^n+a_0x^{n-1}+a_1x^{n-2}+\ldots+a_{n-2}x+a_{n-1}=0$ tenglama uchun Viyet tengliklari $\begin{cases} a_0=-(x_1+x_2+\cdots x_n) \\ a_1=x_1x_2+x_1x_3+\cdots x_1x_n+x_2x_3+\cdots x_{n-1}x_n \\ a_2=-(x_1x_2x_3+x_1x_2x_4+\cdots x_{n-2}x_{n-1}x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1}=(-1)^n x_1x_2\ldots x_n \end{cases}$ ko'rinishida bo'ladi.

Masala. Tomonlari $x^3-ax^2+bx-c=0$ tenglama yechimlari bo'lgan uchburchakka tashqi chizilgan doira yuzini toping.

Yechish. Tenglama yechimlari x_1; x_2; x_3 bo'lsin. U holda Viyet teoremasiga ko'ra $x_1+x_2+x_3=a$, $x_1x_2+x_1x_3+x_2x_3=b$; $x_1x_2x_3=c$ bo'ladi. Tashqi chizilgan aylana radiusi uchun $R=\frac{x_1+x_2+x_3}{4S_\Delta}$ formuladan foydalanamiz. $p=\frac{x_1+x_2+x_3}{2}=\frac{a}{2}$ ekanligidan, **Geron formulasiga** ko'ra:

$$S_\Delta=\sqrt{p(p-x_1)(p-x_2)(p-x_3)}=\sqrt{\frac{a}{2}\left(\frac{a}{2}-x_1\right)\left(\frac{a}{2}-x_2\right)\left(\frac{a}{2}-x_3\right)}=$$

$$=\sqrt{\frac{a}{2}\left[\frac{a3}{8}-(x_1+x_2+x_3)\frac{a2}{4}+(x_1x_2+x_1x_3+x_2x_3)\frac{a}{2}-x_1x_2x_3\right]}=$$

$$=\sqrt{\frac{a}{2}\left[\frac{a^3}{8}-\frac{a^3}{4}+\frac{ab}{2}-c\right]}=\frac{1}{4}\sqrt{a(4ab-a^3-8c)}.$$

U holda $R=\frac{c}{\sqrt{a(4ab-a^3-8c)}}$. Demak, tashqi chizilgan doira yuzi $S=\frac{\pi c^2}{\sqrt{a(4ab-a^3-8c)}}$.

5.2. Ko'phadlar. Algebraning asosiy teoremasi, natijalari

Natural darajani $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ funksiya n-darajali ko'phad deyiladi, bunda $a_n\neq0$, $a_{n-1},\ldots,a_1,a_0\in R$

Ikki $P_n(x)$ va $Q_n(x)=b_nx^n+b_{n-1}x^n-1+\ldots+b_1x+b_0$ ko'phadlarda $a_k=b_k$ ($k=o\ldots n$) bo'lgandagina bu ko'phadlar teng bo'ladi.

Ko'phadlar ustida ham amallar kiritish mumkin. Yig'indisi, ayirma, ko'paytma yana ko'phad bo'ladi. Bo'lish amali ham sonlardagiga o'xshab kiritiladi.

Ixtiyoriy P(x), Q(x) ko'phadlar uchun shunday q(x), r(x) ko'phad topish mumkinki

$P(x)=Q(x)\cdot q(x)+r(x)$ tenglik o'rinli bo'lib, r(x) darajasi Q(x)ning darajasidan kichik bo'ladi. q(x) ko'phad P(x)ni Q(x)ga bo'lishdagi bo'linma, r(x) esa qoldiq deyiladi.

Agar $r(x)\equiv0$ bo'lsa, P(x) ko'phad Q(x) ko'phadga bo'linadi deyiladi.

Biror a soni uchun $P(a)=0$ bo'lsa, a soni P(x) ko'phadning ildizi deyiladi.

Teorema. $P(x)$ кo'phadni $(x-a)$ ko'phadga bo'lishdagi qoldiq $P(a)$ ga teng bo'ladi.

Isboti. Bo'luvchi ko'phad 1-darajali bo'lganiligi uchun, qoldiq 0-darajali ko'phad, ya'ni o'zgarmas son bo'ladi, $r(x)=c$, U holda $P(x)=(x-a)q+c$ bo'ladi va $x=a$ da $P(a)=c$.

Natija. (Bezu teoremasi): a son $P(x)$ ko'phad ildizi bo'lishi uchun $P(x)$ning $(x-a)$ga bo'linishi zarur va yetarli.

Agar $P(x)$ ko'phad $(x-a)$, $(x-a)^2,\ldots,(x-a)^\kappa$larga qoldiqsiz bo'linsa, lekin $(x-a)^{\kappa+1}$ga bo'linmasa, a soni $P(x)$ кo'phadning *k* karrali ildizi deyiladi. Bu holda $P(x)=(x-a)^\kappa Q(x)$ bo'lib, $Q(x)$ ko'phad $(x-a)$ga bo'linmaydi.

Dastlab, $Pn(x)=a_nx^n+a_{n-1}x^{n-1}\ldots a_1x+a_0$ кo'phadni $Q(x)=x-\alpha$ ikkihadga bo'lishni ko'rib chiqamiz. Agar $P_n(x)=(x-2)q_{n-1}(x)+r$ bo'lsa, $a_nx^n+a_{n-1}x^{n-1}\ldots a_1x+a_0=(x-\alpha)[b_{n-1}x^{n-1}+b_{n-2}x^{n-2}+\ldots\ldots+b_1x+b_0]+r$ tenglik bajarilishi zarur. Bir xil darajalar oldidagi koeffitsiyentlarni tenglashtirib:

$$\begin{cases} a_n = b_{n-1} \\ a_{n-1} = b_{n-2} - \alpha b_{n-1} \\ a_{n-2} = b_{n-3} - \alpha b_{n-2} \\ \ldots\ldots\ldots\ldots\ldots \\ a_1 = b_0 - \alpha b_1 \\ a_n = r - \alpha b_n \end{cases} \text{yoki} \begin{cases} b_{n-1} = a_n \\ b_{n-2} = a_{n-1} + \alpha b_{n-1} \\ b_{n-3} = a_{n-2} + \alpha b_{n-2} \\ \ldots\ldots\ldots\ldots\ldots \\ b_0 = a_1 + \alpha b_1 \\ r = a_n + \alpha b_n \end{cases}$$

noma'lum koeffitsiyentli $q_{n-1}(x)$ ko'phadni, r-qoldiqni topishimiz mumkin.

Yuqoridagi hisoblashlarni **Gorner sxemasi** deb ataluvchi quyidagi jadval yordamida bajarish qulay:

	a_n	a_{n-1}	a_{n-2}	a_2	a_1	a_0
α	b_{n-1}	b_{n-2}	b_{n-3}	b_1	b_0	r

Masalan, $P_4(x)=x^4-x^3-7x^2+x+6$ ko'phad ratsional ildizlarini Gorner sxemasi bo'yicha izlaymiz: $\frac{p}{q}=\pm1;\pm2;\pm3;\pm6$.

	1	-1	-7	1	6
-1	1	-2	-5	6	0
1	1	-1	-6	0	
-2	1	-3	-0		
3	1	0			

Demak, $r=0$, $x^4-x^3-7x^2+x+6=(x+1)(x^3-2x^2-5x+6)$

$x^3-2x^2-5x+6=(x-1)(x^2-x-6)$, $x^2-x-6=(x+2)(x-3)$

Jadval $x^4-x^3-7x^2+x+6=(x+1)(x-1)(x+2)(x-3)$ ekanligini ko'rsatadi. $P_4(x)$ ratsional ildizlari $\pm1;-2$ ekan.

Dastlab, quyidagi teoremani isbotsiz keltiramiz.

Teorema. Darajasi birdan kichik bo'lmagan ixtiyoriy ko'pxad kamida bitta, umuman aytganda kompleks ildizga ega.

Agar biror $P_n(x)$ ko'pxad qarasak, oldingi teoremaga ko'ra, uning kamida bitta x_1 ildizi bor, ya'ni $P_n(x)=(x-x_1)\,P_{n-1}(x)$. O'z navbatida, $n-1\geq1$ bo'lsa, $P_{n-1}(x)$ ham biror x_2 ildizga ega: $P_n(x)=a_n(x-x_1)(x-x_2)\,P_{n-2}(x),\ldots$ Bu jarayonni davom ettirib: $P_n(x)=a_n(x-x_1)(x-x_2)\ldots\ldots(x-x_n)$ tenglikka kelamiz. Bunda $x_1,x_2,\ldots,x_n$ ildizlar orasida o'zaro tenglari (кarralilari) bo'lishini hisobga olsak, $P_n(x)=a_n(x-x_1)^{\kappa_1}(x-x_2)^{\kappa_2}\ldots\ldots(x-x_n)^{\kappa_n}$ bo'ladi, lekin, $\kappa_1+\kappa_2+\ldots+\kappa_n=n$, natijada quyidagi teorema o'rinliligi kelib chiqadi.

Teorema (algebraning asosiy teoremasi). Ixtiyoriy n-darajali ko'pxad n ta ildizga ega.

Agar haqiqiy koeffitsiyentli $P_n(x)=a_nx^n+\ldots+a_1x+a_0$ ko'phad $z=\alpha+\beta i$ kompleks ildizga ega bo'lsa, u holda $\bar{z}=\alpha-\beta i$ ham ildiz bo'ladi.

Haqiqatan, $P_n(\alpha+\beta i)=0$ bo'lsa, $P_n(\alpha-\beta i)=0$ bo'lishini tekshirish qiyin emas.

Natija. Agar $P_n(x)$ ko'phad darajasi n-toq bo'lsa, uning kamida bitta haqiqiy ildizi bor.

$\frac{Qm(x)}{Pn(x)}$ nisbat ratsional kasr deyiladi. Agar $m\geq n$ bo'lsa noto'g'ri, $m<n$ da esa to'g'ri kasr deyiladi.

Noto'g'ri kasr bo'lgan holda $Q_m(x)=P_n(x)\,q(x)+r(x)$ ekanligidan. $\frac{Qm(x)}{P_n(x)}=q(x)+\frac{r(x)}{P_n(x)}$ kelib chiqadi, ya'ni suratni maxrajga bo'lish yordamida noto'g'ri kasr butun qismi alohida, to'g'ri kasr qismi alohida yoziladi.

Umumiylikka ziyon keltirmagan holda $\frac{Qm(x)}{Pn(x)}$- to'g'ri kasr deb hisoblanishi mumkin ekan.

$X_i\in R$ uchun $(x-x_i)$ va $x^2-(z+\bar{z})x+z*\bar{z}$ ko'rinishdagi z, $\bar{z}$ ildizli ikkihad keltirilmas haqiqiy ko'phadlar deyiladi.

$X^2-(z+\bar{z})x+z\cdot\bar{z}=x^2+px+q$ ko'rinishda yozib olamiz. U holda $P_n(x)=a_n(x-x_1)^{k_1}\ldots(x-x_2)^{k_r}(x^2+px+q)^{l_1}\ldots(x^2+p_sx+q_s)^{l_s}$ ko'rinishda yoziladi, bunda $\kappa_1+\kappa_2+\ldots+\kappa_s+2(l_1+l_2+\ldots+l_s)=n$.

$\frac{A}{(x-x_i)^{k}}$, $\frac{Bx+C}{(x^2+px+q)^s}$ kasrlar sodda yoki elementar kasrlar deyiladi, bunda $x-x_i$, x^2+px+q- keltirilmas haqiqiy ko'phadlar.

Teorema. Har qanday to'g'ri ratsional kasr sodda kasrlar yig'indisi ko'rinishidagi yagona yoyilmaga ega.

Agar, masalan, $P_n(x)=(x-a)^k\ldots(x^2+px+q)^s\ldots$ bo'lsa,

$$\frac{Qm(x)}{Pn(x)}=\frac{Q_m(x)}{(x-a)^k\ldots(x^2+px+q)^s}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_k}{(x-a)^k}+\ldots+\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\ldots\frac{B_sx+C_s}{(x^2+px+q)^s}+\ldots$$

yoyilmaga ega bo‘ladi. Bu yerda A_i, B_i, C_i, $i\in N$ sonlar noma’lum sonlardir. Ularni aniqlash uchun tenglik umumiy maxrajga keltiriladi, suratlari tenglashtiriladi, so‘ngra noma’lumning bir xil darajalari oldida koeffitsiyentlar tenglashtirilib noma’lumlar soniga teng tenglamaga ega chiziqli sistemaga ega bo‘lamiz. Sistema yechilib, aniqmas koeffitsiyentlar topiladi.

Bu metod aniqmas koeffitsiyentlar metodi deyiladi.

Misol. $\frac{4x-2}{(x-1)^2(x^2+1)}$ kasrni sodda kasrlarga yoying.

$$\frac{4x-2}{(x-1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1},$$

Umumiy maxrajga keltirib, suratlarni tenglashtiramiz:

$4x-2=A(x^3-x^2+x-1)+B(x^2+1)+C(x^3-2x^2+x)+D(x^2-2x+1)$

$$\begin{array}{l|l} x^3 & 0=A+C \\ x^2 & 0=-A+B-2C+D \\ x & 4=A+C-2D \\ x^0 & -2=-A+B+D \end{array}$$

Barcha tenglamalarni hadma-had qo‘shsak, 2=2B kelib chiqadi, B=1.

Ikkinchi tenglamadan to‘rtinchisini ayirsak, 2=-2C bo‘ladi, C=-1. Demak, birinchi tenglamadan A=1. Uchinchi tenglamada A+C=0 ekanligidan 4=-2D, bo‘ladi, D=-2

Berilgan kasr sodda kasrlarga $\frac{4x-2}{(x-1)^2(x^2+1)}=\frac{1}{x-1}+\frac{1}{(x-1)^2}-\frac{x+2}{x^2+1}$ ko‘rinishida yoyilar ekan.

Mavzuga doir misollar

1. $x^3+2x-3=0$ uchun $x_1^2+x_2^2+x_3^2$ hisoblang.
2. $x^3-x^2-4x1=0$ uchun $x_1^3x_2+x_1x_2^3+x_2^3x_3+x_2x_3^3+x_3^3x_1+x_3x_1^3$ ni hisoblang.
3. Agar x_1, x_2, x_3 lar $x^3+px+q=0$ yechimlari bo‘lsa, quyidagilarni toping.

1) $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}$; 2) $x_1^4x_2^2+x_1^4x_2^2+x_1^2x_2^4+x_2^4x_3^2+x_3^4x_1^2+x_3^2x_1^4$;
3) $(x_1^2-x_2x_3)(x_2^2-x_1x_3)(x_2^3-x_2x_1)$; 4) $(x_1+x_2)^4(x_1+x_3)^4(x_2+x_3)^4$

4. Gorner sxemasi yordamida ratsional ildizlarini toping.

1) $x^4-2x^3-8x^2+13x-24=0$; 2) $x^3-6x^2+15x-14=0$; 3) $2x^3+3x^2+6x-4=0$;
4) $6x^4+19x^3-7x^2-26x+12=0$; 5) $4x^4-7x^2-5x-1=0$; 6) $x^4+4x^3-2x^2-12x+9=0$;
7) $24x^5+10x^4-x^3-19x^2-5x+6=0$

5. Sodda kasrlarga yoying.

1) $\frac{2x+3}{(x-2)(x+5)}$; 2) $\frac{x}{(x+1)(x+2)(x+3)}$; 3) $\frac{x}{(x+1)^2(x^2+2x+2)}$; 4) $\frac{1}{x^3+1}$; 5) $\frac{1}{x^3-1}$; 6) $\frac{1}{x^4-1}$;
7) $\frac{1}{x^4+1}$; 8) $\frac{1}{(x+1)(x+2)^2(x+3)^3}$; 9) $\frac{x}{(x+1)(x^2+1)(x^2+1)}$; 10) $\frac{x}{x^2-4x+4}$;
11) $\frac{1}{x^5-x^4+x^3-x^2+x-1}$; 12) $\frac{1}{(x^4-1)^2}$.

6-mavzu. Determinantlar. Matritsalar

6.1. Determinantlarning xossalari va ularni hisoblash

nxn ta elementdan tuzilgan, kvadrat jadval ko'rinishidagi, ikki vertikal kesma orasiga olingan

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{33} \\ \cdot & \cdot & \cdot \cdots \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}$$

ifoda $n-$ **tartibli determinant**, $a_{ij} \epsilon R$ $(i,j = \overline{1,n})$ sonlari esa **determinant elementlari** deyiladi.

Gorizontal qatorlar **yo'l** (satr), vertikal qatorlar esa **ustun** deyiladi.

Birinchi indeksi i bo'lgan elementlar i-yo'l (satr) elementlari, ikkichi indeksi j bo'lgan elementlar esa j-ustun elementlari deyiladi.

Masalan, a_{34} element 3-yo'l (satr), 4-ustunda joylashgan. a_{11}, a_{22},…, a_{nn} joylashgan diagonal determinant bosh diagonali, ikkinchi diagonal esa yordamchi diagonal deyiladi.

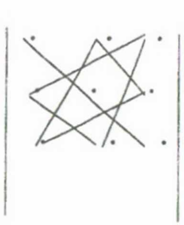
$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ifoda 2-tartibli determinant deyilib, qiymati $a_{11}a_{22} - a_{12}a_{21}$ ayirmaga teng hisoblanadi.

$\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix}$ ifoda 3-tartibli determinant, uning qiymati

$a_{11}\,a_{22}\,a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}\,a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}\,a_{23}a_{32}$ songa teng deyiladi.

3-tartibli determinant 6 ta had yig'indisidan iborat, uchtasi musbat, qolgan uchtasi manfiy ishoralidir. Hadlar yozilish tartibi, ishoralarni eslab qolish uchun "uchburchak qoidasi" deb ataluvchi sxemadan foydalaniladi.

+



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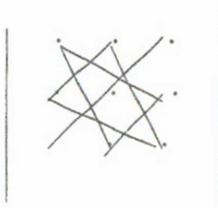

1) $\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = 4 \cdot 1 - (-2) \cdot 3 = 10,$

2) $\begin{vmatrix} 4 & -3 & 5 \\ 3 & -2 & 8 \\ 1 & -7 & -5 \end{vmatrix} = 4 \cdot (-2)(-5) + (-3) \cdot 8 \cdot 1 + 5 \cdot 3 \cdot (-7) - 5 \cdot (-2) \cdot 1 - 3 \cdot 3 \cdot (-5) -$
$4 \cdot 8 \cdot (-7) = 40 - 24 - 105 + 10 - 45 + +224 = 100$

$n-$ tartibli Δ determinantda a_{ij} element joylashgan yo'l va ustun o'chirilsa, $(n-1)$ tartibli determinant hosil bo'lib, u a_{ij} element minori deyiladi va M_{ij} harfi bilan belgilanadi.

$A_{ij} = (-1)^{i+j} \cdot M_{ij}$ soni esa a_{ij} element algebraik to'ldiruvchisi deyiladi.

Masalan, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ bo'lsa,

$A_{11} = (-1)^2 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$, $A_{21} = (-1)^3 \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 6$, $A_{32} = (-1)^5 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6$,..

kelgusida yo'l (satr) uchun o'rinli munosabatlarni ixtiyoriy qator uchun deb ataymiz.

Teorema.

1) Ixtiyoriy qator elemenlarini o'z algebraik to'ldiruvchilariga ko'paytmalari yig'indisi determinant qiymatiga teng.

2) Ixtiyoriy qator elementlari parallel qator elementlari algebraik to'ldiruvchilariga ko'paytmalari yig'indisi nolga teng.

$\Delta = \sum_{k=1}^{n} a_{i,k} A_{i,k}$, $\quad 0 = \sum_{k=1}^{n} a_{i,k} A_{i,s}$, $\quad$ bunda S=1…n, ≠k

Isbot. Soddalik uchun isbotni 3-tartibli determinantlar uchun keltiramiz (3-yo'l elementlarini tanladik).

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} =$$

$$= a_{31} \cdot (-1)^4 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32} \cdot (-1)^5 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{22} \end{vmatrix} a_{33} \cdot (-1)^6 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

$$= a_{12}\,a_{23}\,a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}\,a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33}$$

Masalan, $\quad a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} = 0, a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{33} = 0$

tengliklarni ham shunday tekshirish mumkin.

Misol.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 \\ -1 & 4 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 2 \cdot (-1)^6 \begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 1 \cdot (-1)^6 \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 2 \cdot (4+2) = 12$$

1-natija. Determinant biror qatori barcha elementlari nol bo'lsa, determinant qiymati nolga teng.

2-natija. Agar determinantda bosh diagonal bir tarafida turgan elementlar nol bo'lsa, determinant qiymati bosh diagonal elementlari ko'paytmasiga teng.

Isboti yoyish teoremasidan kelib chiqadi:

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} & \dots & d_{1n} \\ 0 & d_{22} & d_{23} & \dots & d_{2n} \\ 0 & 0 & d_{33} & \dots & d_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_{nn} \end{vmatrix} = d_{11} \begin{vmatrix} d_{22} & d_{23} & \dots & d_{2n} \\ 0 & d_{33} & \dots & d_{3n} \\ 0 & 0 & \dots & d_{4n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{nn} \end{vmatrix} = \dots = d_{11}\, d_{22} \dots d_{nn}$$

Determinant xossalarini 3-tartibli determinantlarda tushunishga harakat qilamiz.

$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ berilgan bo'lsin.

1°. *Determinant biror yo'li unga mos ustun bilan almashtirilsa determinant qiymati o'zgarmaydi, umuman, barcha yo'llari mos ustunlar bilan almashtirilsa (trasponirlansa) ham determinant qiymati o'zgarmaydi.*

2°. Determinant ikki parallel qatori o'rinlari almashtirilsa, determlinant qiymati ishorasi o'zgaradi.

Natija. Determinant ikki parallel qatori bir xil bo'lsa, determinant qiymati nolga teng.

3°. Determinant biror qatori o'zgarmas k songa ko'paytirilsa, uning qiymati ham k ga ko'payadi.

Isboti. K songa ko'paygan qator bo'yicha yoyib, xossa o'rinliligiga amin bo'lamiz.

Natija. Determinant ikki parallel qatori o'zaro proporsional bo'lsa, determinant qiymati nolga teng.

4°. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+\alpha_1 & a_{22}+\alpha_2 & a_{23}+\alpha_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Natija. Determinant biror qatori o'zgarmas k songa ko'paytirilib, o'ziga parallel qator elementlariga qo'shilsa va natijasi ular o'rniga yozilsa, determinant qiymati o'zgarmaydi.

Bu natijadan yuqori tartibli determinantlarni diagonal determinantga keltirishda foydalaniladi.

$$\Delta = \begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1+b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{vmatrix} = \boxed{\text{1-yo'lni (-1)ga ko'paytirib, qolgan barcha yo'llarga qo'shamiz}} = \begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 0 & b_1 & 0 & \dots & \\ 0 & 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_n \end{vmatrix}$$

$= b_1 b_2 b_3 \dots b_n$.

6.2. Matritsalar

m ta yo'l va n ta ustunga joylashgan, m n ta elementli, to'g'ri burchakli jadval mxn o'lchamli matritsa deyiladi va

$$\begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{Vmatrix} \quad yoki \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

ko'rinishda yoziladi. Determinantlardagi kabi a_{ij} element i -yo'l, j-ustunda joylashgan elementdir.

Matritsani $A = \|a_{ij}\|_{i=1,m\ ;j=n}$ ko'rinishda ham belgilash mumkin.

m=n bo'lsa, matritsa **kvadrat matritsa** deyiladi.

Bosh diagonaldagi elementlardan boshqa elementlar noldan iborat bo'lsa, matritsa **diagonal matritsa** deyiladi. Diagonal matritsa bosh diagonali elementlari 1 ga teng bo'lsa, birlik matritsa deb ataluvchi $E=\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ . & . & . & 1 \end{pmatrix}$ matritsa hosil bo'ladi.

$A = \|a_{ij}\|$ Kvadrat matritsaga mos determinant $|A|$ ko'rinishida belgilanadi.

Agar A matritsa uchun $|A| = 0$ bo'lsa, A matritsa xos, $|A| \neq 0$ bo'lsa, u holda A xosmas matritsa deyiladi.

Agar $\|a_{ij}\|$, $\|b_{ij}\|$ matritsalarda $a_{ij} = b_{ij}$ $(i = \overline{1,m}, j = \overline{1,n})$ *bo'lsa*, A va B matritsalar o'zaro teng deyiladi.

O‘lchamlari bir xil $A=\|a_{ij}\|$, $B=\|b_{ij}\|$ matritsalar ustida amallar quyidagicha kiritiladi:

1. Qo‘shish (ayirish). $A\pm B=\|a_{ij}+b_{ij}\|$
2. Songa ko‘paytrish. $\lambda\epsilon R$ soni uchun $\lambda\cdot A=\|\lambda a_{ij}\|$

Matritsani matritsaga ko‘paytirish birinchi matritsa ustunlari soni ikkinchi matritsa yo‘llari soniga teng bo‘lganda kiritiladi xolos.

$A\cdot B=C$ bo‘lsa, C matritsa elementlari $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$ $(i=\overline{1,m}, j=\overline{1,n})$ qoida yordamida topiladi, boshqacha aytganda,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ . & . & \cdots & . \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}=$$

$$=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1};\cdots;a_{11}b_{1k}+a_{12}b_{2k}+\cdots+a_{1n}b_{n1} \\ \\ \end{pmatrix}$$

Misol: $A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$, $C=\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

Matritsalar berilgan. 2A-3B, A·C ni toping.

1). $2A-3B=\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}-\begin{pmatrix} 0 & 3 & 6 \\ 9 & 12 & 15 \end{pmatrix}=\begin{pmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \end{pmatrix}$

2). $A\cdot C=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}=\begin{pmatrix} 1\cdot1+2\cdot2+3\cdot(-1); & 1\cdot0+2\cdot1+3\cdot3 \\ 4\cdot1+5\cdot2+6\cdot(-1); & 4\cdot0+5\cdot1+6\cdot3 \end{pmatrix}=\begin{pmatrix} 2 & 11 \\ 8 & 23 \end{pmatrix}$

Bu kiritilgan amallar quyidagi xossalarga ega:

1°. $A+0=0+A=A$ — 5°. $(\lambda+\mu)A=\lambda A+\mu A$

2°. $A+B=B+A$ — 6°. $(A+B)\cdot C=A\cdot C+B\cdot C$

3°. $\lambda(\mu\cdot A)=(\lambda\mu)\cdot A$ — 7°. $(A\cdot B)\cdot C=A\cdot(B\cdot C)$

4°. $\lambda(A+B)=\lambda A+\lambda B$

Shunisi qiziqki, $A\cdot B\neq B\cdot A$, lekin $|A\cdot B|=|B\cdot A|=|A|\cdot|B|$

Misol. $A=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 0 \\ 1 & -3 \end{pmatrix}$, bo‘lsa, $|A|=3$, $|B|=-3$

$A\cdot B=\begin{pmatrix} 3 & -6 \\ 3 & -9 \end{pmatrix}$, $B\cdot A=\begin{pmatrix} 1 & 2 \\ 1 & -7 \end{pmatrix}$, $|A\cdot B|=|B\cdot A|=-5$

Natija. Bittasi xos matritsalar ko‘paytmasi yana xos matritsa bo‘ladi.

Faraz qilaylik, A-kvadrat matritsa bo‘lsin, A matritsani n marta o‘zini-o‘ziga ko‘paytirsak, $A^n=A\cdot A\cdot A\cdots A$ hosil bo‘ladi va quyidagi xossalarga ega: $A^0=E$, $A'=A$, $A^m\cdot A^k=A^{m+k}$, $(A^m)^k=A^{mk}$

Eslatma. $A^n=0$ ekanligidan $A=0$ kelib chiqdi.

Matritsada yo‘l va ustunlar o‘rinlarini almashtirish transponirlash deyiladi va A^T ko‘rinishda belgilanishi mumkin. Agar *mxn* o‘lchovli bo‘lsa, A^T nxm o‘lchovli bo‘ladi.

Misol. $A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T=\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Transponirlash quyidagi xossalarga ega:

$(A^T)^T=A$, 2) $(\lambda A)^T=\lambda A^T$, 3) $(A+B)^T=A^T+B^T$, 4) $(A\cdot B)^T=B^T\cdot A^T$

Ikki nxn o‘lchamli A, B kvadrat matritsalar uchun $A\cdot B=B\cdot A=E$ bo‘lsa, B matritsa A matritsaga teskari deyiladi va A^{-1} ko‘rinishida belgilanadi.

Dastlab, $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ matritsa teskari $B=\begin{pmatrix} x & y \\ z & u \end{pmatrix}$ matritsani topamiz. $A\cdot B=E$ shartga ko’ra, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & u \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Demak,

$a_{11}x+a_{12}z=1$, $a_{11}y+a_{12}u=0$, $a_{21}x+a_{22}z=0$, $a_{21}y+a_{22}u=1$

Bu sistemalarni yechib, $x=\frac{A_{11}}{\Delta}$, $y=\frac{A_{21}}{\Delta}$, $z=\frac{A_{12}}{\Delta}$, $u=\frac{A_{22}}{\Delta}$, ekanligini topamiz.

Bunda, $\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, A_{ij} lar esa , a_{ij} – algebraik to‘ldiruvchilari.

$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ matritsaga teskari matritsa

$A^{-1}=\frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}=\begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} \end{pmatrix}$ ko‘rinishida bo‘lar ekan.

Teorema. Har qanday xosmas $A=\|a_{ij}\|$ kvadrat matritsaning teskarisi mavjud, yagona va

$$A^{-1}=\begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \cdots & \frac{A_{n1}}{\Delta} \\ . & . & . & . \\ \frac{A_{1n}}{\Delta} & \frac{A_{2n}}{\Delta} & \cdots & \frac{A_{nn}}{\Delta} \end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ . & . & \cdots & . \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}, \text{ bunda } \Delta=|A|$$

Isbot.

$$A\cdot A^{-1}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ . & . & \cdots & . \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}\cdot\frac{1}{\Delta}=\frac{1}{\Delta}\begin{pmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & \cdots & \Delta \end{pmatrix}=E$$

$A^{-1}\cdot A$ ni ham E ga tengligini tekshirishimiz mumkin.

Agar A^{-1} dan farqli C ham teskari bo‘lsa, ya’ni

$AC=CA=E$. $C\cdot A\cdot A^{-1}=C(A\cdot A^{-1})=C\cdot E=C$, $CA\cdot A^{-1}=(CA)A^{-1}=E\cdot A^{-1}=A^{-1}$

Bu tengliklardan $C=A^{-1}$ kelib chiqadi.

$|A|=0$ bo‘lsa, A^{-1} mavjud bo‘lmasligi ravshan.

$|A\cdot A^{-1}|=|E|=1$ dan $|A^{-1}|=\frac{1}{|A|}$ kelib chiqadi .

Misol. 1) $A=\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ ga teskari matritsa toping.

$|A|=\cos^2\alpha+\sin^2\alpha=1$ ekanligidan teskari matritsa mavjud va yagona.

$A_{11} = cos\alpha,\ A_{21} = \sin\alpha\ A_{12} = -\sin\alpha,\ \ A_{22} = \cos\alpha$ bo‘lganligi uchun

$$A^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

2) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ ga teskari matritsani toping.

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix} \begin{vmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{vmatrix} = -16$$

$A_{11} = -4,\quad A_{21} = -4,\quad A_{31} = -4,\quad A_{41} = -4$

$A_{12} = -4,\quad A_{22} = -4,\quad A_{32} = 4,\quad A_{42} = 4$

$A_{13} = -4,\quad A_{23} = 4,\quad A_{33} = -4,\quad A_{43} = 4$

$A_{14} = -4,\quad A_{24} = 4,\quad A_{34} = 4,\quad A_{44} = -4$

Demak, $A^{-1} = \frac{1}{-16}\begin{pmatrix} -4 & -4 & -4 & -4 \\ -4 & -4 & 4 & 4 \\ -4 & 4 & -4 & 4 \\ -4 & 4 & 4 & -4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

Biror mxn tartibli $A = \|a_{ij}\|$...matritsaning k ta yo‘li va k ustunini olib, *kxk* tartibli kvadrat matritsa tuzamiz. Bu kvadrat matritsa determinanti A matritsaning k tartibli minori deyiladi.

Bunday k tartibli minorlar bir nechta bo‘lib, ular turli xil qiymat qabul qilishi mumkin. Ular orasida noldan farqli bo‘lgan yuqori tartibli minorni topish muhimdir.

A matritsaning noldan farqli minorlarining eng yuqori tartibi uning rangi deyiladi va rang A ko‘rinishda belgilanadi.

Misol. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 1 & -1 \end{pmatrix}$ rangini toping.

$|1| = 1,\ \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = -10 \neq 0,\ \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 1 & -1 \end{vmatrix} = 0$ bo‘lganligi uchun rang A= 2

Rang hisoblashda turli xil deteminantlarni hisoblashga to‘g‘ri keladi. Shuning uchun rang hisoblashning oson usullaridan birini keltiramiz.

Berilgan matritsada:

1) ikki parallel qator o‘rinlarini almashtirish;

2) biror qatorni o‘zgarmas songa ko‘paytirish;

3) biror qatorga o‘zgarmas songa ko‘paytirilgan boshqa parallel qatorni qo‘shish shu matritsaning elementar almashtirishlari deyiladi.

Elementar almashtirishlar matritsa rangini o‘zgartirmaydi.

Demak, matritsa diagonal ko‘rinishga keltiriladi va rangi oson topiladi.

Misol. $A = \begin{pmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{pmatrix}$ matritsani rangini toping.

Dastlab, 1-yo‘lni (-1) ga ko‘paytrib, 4-yo‘lga, (-3) ga ko‘paytrib 2, 3–yo‘llarga qo‘shamiz:

$$\begin{pmatrix} 25 & 31 & 17 & 43 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

2–yo‘lini (-1) ga ko‘paytrib, 3, 4-yo‘llarga qo‘shamiz:

$$\begin{pmatrix} 25 & 31 & 17 & 43 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

3-yo‘lini (-1) ga ko‘paytrib , 4- yo‘lga qo‘shamiz:

$$\begin{pmatrix} 25 & 31 & 17 & 43 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Bu matritsaning noldan farqli eng katta minorlaridan biri $\begin{vmatrix} 25 & 31 & 17 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 25$ bo‘ladi va $|A| = 0$ ekanligidan rang A= 3.

7-mavzu. Chiziqli tenglamalar sistemasi

n ta $x_1, x_2, x_3, \dots, x_n$ noma'lumli, chiziqli, n ta

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad a_{ij},\ b_i\ \in R.$$ tenglamalar sistemasini yechish usullari, yechimi qanday bo'lishi masalalarini ko'rib chiqamiz.

7.1. Kramer formulasi

Noma'lumlar koeffitsiyentlaridan tuzilgan $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$ determinant tenglamalar sistemasining asosiy determinanti, undagi j-ustun o'rniga ozod b_i hadlardan iborat ustun qo'yilgan determinant esa j-yordamchi determinant deyiladi va Δ_j ko'rinishida belgilanadi.

$$\Delta_j = \begin{vmatrix} a_{11}\dots a_{1j-1} b_1\ a_{1j+1}\dots a_{1n} \\ a_{21}\dots a_{2j-1} b_2\ a_{2j+1}\dots a_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\dots a_{nj-1} b_n\ a_{nj+1}\dots a_{nn} \end{vmatrix}$$

Dastlab, berilgan tenglamalar sistemasidan har bir i-tenglamani A_{i1} *ga* ko'paytiramiz va hosil bo'lgan tenglamalarni qo'shamiz:

$$(a_{11}A_{11} + a_{21}A_{21} + \dots + a_{n1}A_{n1})x_1 + ((a_{12}A_{11} + a_{22}A_{21} + \dots + a_{n2}A_{n1})x_2 + \dots + (a_{1n}A_{11} + a_{2n}A_{21} + \dots + a_{nn}A_{n1})x_n = b_1A_{11} + b_2A_{21} + \dots + b_nA_{n1}$$

Determinantni yoyish haqidagi teoremaga ko'ra: $\Delta \cdot x_1 = \Delta_1$. Endi sistemadagi har bir i-tenglama A_{i2} ga ko'paytirib qo'shilsa, $\Delta \cdot x_2 = \Delta_2, \dots,$ A_{in} ga ko'paytirib qo'shilsa, $\Delta x_n = \Delta_n$ tenglik hosil bo'ladi.

Demak, sistemadagi noma'lumlar $x_j = \frac{\Delta_j}{\Delta}$ formula yordamida hisoblanar ekan. Bu Kramer formulasidir.

$\Delta \cdot x_j = \Delta_j$ tenglikdan quyidagilar kelib chiqadi:

1) $\Delta \neq 0$ da sistema yagona yechimga ega, uni birgalikda deyiladi.
2) $\Delta = 0$, $\Delta_j = 0$ bo'lsa, sistema cheksiz ko'p yechimga ega.
3) $\Delta = 0$, Δ_j lardan birortasi noldan farqli bo'lsa, sistema yechimga ega emas.

Misol. Kramer formulasi yordamida yeching:

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 2 \\ x_1 + 2x_2 + 3x_3 - 4x_4 = -2 \\ 2x_1 + x_2 - x_3 + x_4 = 5 \\ 4x_1 + 3x_2 + 2x_3 - 4x_4 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & -4 \\ 2 & 1 & -1 & 1 \\ 4 & 3 & 2 & -4 \end{vmatrix} = 3,$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 1 & -1 \\ -2 & 2 & 3 & -4 \\ 5 & 1 & -1 & 1 \\ 0 & 3 & 2 & -4 \end{vmatrix} = 3,\ \Delta_2 = \begin{vmatrix} 1 & 2 & 1 & -1 \\ 1 & -2 & 3 & -4 \\ 2 & 2 & -1 & 1 \\ 4 & 0 & 2 & -4 \end{vmatrix} = 6,$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -2 & -4 \\ 2 & 1 & 5 & 1 \\ 4 & 3 & 0 & -4 \end{vmatrix} = 9,\ \Delta_4 = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & -2 \\ 2 & 1 & -1 & 5 \\ 4 & 3 & 2 & 0 \end{vmatrix} = 12$$ bo'lganligi uchun

$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4.$

7.2. Matritsaviy usulda yechish

Berilgan tenglamalar sistemasini matritsaviy

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$ yoki $A \cdot X = B$ ko'rinishida yozish mumkin.

Agar $|A| \neq 0$ bo'lsa, A^{-1} matritsa mavjud va yagona bo'lishidan $A^{-1} \cdot A \cdot X = A^{-1} \cdot B$ yoki $X = A^{-1} \cdot B$.

Noma'lumlardan iborat X-ustun matritsani bunday topish matritsaviy usul deyiladi.

Misol. Yuqoridagi sistemani shu usul yordamida qayta yechamiz.

$|A| = \Delta = 3$ ekanligini hisoblaganmiz.

$A = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & -4 \\ 2 & 1 & -1 & 1 \\ 4 & 3 & 2 & -4 \end{vmatrix}$ matritsaga teskari A^{-1} ni topamiz.

$$A_{11} = \begin{vmatrix} 2 & 3 & -4 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = 5;\ A_{21} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = -4;\ A_{31} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & 2 & -4 \end{vmatrix} = -3;\ A_{41} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 2 & -1 & 1 \end{vmatrix} = 2$$

$$A_{12} = -\begin{vmatrix} 1 & 3 & -4 \\ 2 & -1 & 1 \\ 4 & 2 & -4 \end{vmatrix} = -6;\ A_{22} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 4 & 2 & -4 \end{vmatrix} = 6;\ A_{32} = -\begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -4 \\ 4 & 2 & -4 \end{vmatrix} = 6;\ A_{42} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -4 \\ 2 & -1 & 1 \end{vmatrix} = -3$$

$$A_{13} = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ 4 & 3 & -4 \end{vmatrix} = 9;\ A_{23} = -\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 4 & 3 & -4 \end{vmatrix} = -3;\ A_{33} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -4 \\ 4 & 3 & -4 \end{vmatrix} = -3;\ A_{43} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -4 \\ 2 & 1 & 1 \end{vmatrix} = -2;$$

$$A_{14} = -\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 4 & 3 & 2 \end{vmatrix} = 5;\ A_{24} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 3 & 2 \end{vmatrix} = -1;\ A_{34} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 0;\ A_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = -1$$

Demak, $A^{-1} = \frac{1}{3}\begin{vmatrix} 5 & -4 & -3 & 2 \\ -6 & 6 & 6 & -3 \\ 9 & -3 & -3 & -2 \\ 5 & -1 & 0 & -1 \end{vmatrix}$

$$X = A^{-1} \cdot B = \frac{1}{3}\begin{pmatrix} 5 & -4 & -3 & 2 \\ -6 & 6 & 6 & -3 \\ 9 & -3 & -3 & -2 \\ 5 & -1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 5 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 6 \\ 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix},$$ ya'ni $x_1=1, x_2=2, x_3=3, x_4=4$.

7.3. Noma'lumlarni ketma-ket yo'qotish (Gauss) usuli

Berilgan chiziqli tenglamalar sistemasi koeffitsiyentlari orqali quyidagi jadvalni tuzib olamiz.

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot\cdot \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array}\right)$$

Bu jadval berilgan sistema kengaytirilgan matritsasi deyiladi.

Har bir satrda bittadan tenglama turibdi, faqat tenglik o'rniga chiziqcha tortilgan.

Bu matritsa ustida o'tkaziladigan har bir elementar almashtirish berilgan sistemaga ekvivalent sistema hosil qiladi. Shu sababli, elementar almashtirishlar , yordamida kengaytirilgan matritsani uchburchak ko'rinishiga keltirib olamiz, buning uchun $a_{11}\neq 0$ bo'lishi kifoya agar $a_{11}=0$ bo'lsa, birinchi tenglamani boshqa yo'ldagi tenglama bilan almashtirish orqali bunga erishish mumkin.

Faraz qilaylik, elementar almashtirishlar yordamida kengaytirilgan matritsa $\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & C_{22} & C_{23} & \dots & C_{2n} & C_2 \\ 0 & 0 & C_{33} & \dots & C_{3n} & C_3 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot\cdot \\ 0 & 0 & 0 & \dots & C_{nn} & C_n \end{array}\right)$ ko'rinishga kelsin.

Unga mos sistema: $$\begin{array}{r} a_{11}x_1+a_{12}x_2+a_{13}x_3+\dots+a_{1n}x_n=b_1 \\ C_{22}x_2+C_{23}x_3+\dots+C_{2n}x_n=C_2 \\ C_{33}x_3+\dots+C_{3n}x_n=C_3 \\ \dots\dots\dots\dots\dots\dots \\ C_{nn}x_n=C_n \end{array}$$ ko'rinishida bo'ladi.

Bu sistemadan dastlab x_n , so'ngra x_{n-1}, ... , va nihoyat x_1 topiladi.

Bu usulda 2-tenglamadan x_1,ni 3-tenglamadan x_1 ва x_2, ..., n-tenglamadan $x_1, x_2, \dots, x_{n-1}$ ketma-ket yo'qotilayotganligi uchun **noma'lumlarni ketma-ket yo'qotish usuli** deyiladi. Bu usul Gauss nomi bilan bog'liq bo'lib, talabalarga elementar matematikadan ma'lum.

Misol. Avvalgi usullarda yechilgan sistemani olaylik. Uning kengaytirilgan matritsasi $\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 1 & 2 & 3 & -4 & -2 \\ 2 & 1 & -1 & 1 & 5 \\ 4 & 3 & 2 & -4 & 0 \end{array}\right)$ ko'rinishda bo'ladi. 1-yo'l elementlarini (-1)ga ko'paytirib 2-yo'lga (-2)ga ko'paytirib 3-yo'lga, (-4)

ga ko'paytirib 4-yo'lga qo'shamiz, natijada, kengaytirilgan matritsa. $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -3 & -4 \\ 0 & -1 & -3 & 3 & 1 \\ 0 & 1 & 2 & 0 & -8 \end{array}\right)$ ko'rinishiga keladi. 2 yo'lni 3 ,4 -yo'l elementlariga qo'shamiz.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -3 & -4 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & -3 & -12 \end{array}\right)$$

Bu matritsaga mos sistema. $$\begin{array}{r} x_1+x_2+x_3-x_4=2 \\ x_2+2x_3-3x_4=-4 \\ x_3=3 \\ -3x_4=-12 \end{array}$$

ko'rinishida bo'ladi. Ketma-ket $x_4=4$; $x_3=3$ larni topib, 2-tenglamaga qo'yamiz.

$$x_2+2\cdot 3-3\cdot 4=-4.$$

Bu yerdan $x_2=2$ ekanligini topib, 1-tenglamaga o'tamiz.

$x_1+2+3-4=2$. Demak, $x_1=1$.

7.4. Bir jinsli sistemalar

Agar qaralayotgan chiziqli tenglamalar sistemasida barcha ozod hadlar nol bo'lsa $b_i=0$,$(i=\overline{1,n})$, bunday sistema bir jinsli deyiladi.

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=0 \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=0 \end{cases}$$

Bu holda $x_1=x_2=x_3=\dots=x_n=0$ sonlar har bir tenglamani qanoatlantirib, **sistemaning trivial yechimi** deyiladi.

Bir jinsli sistemaning trivial bo'lmagan notrivial yechimlarini qidiramiz.

Kramer formulasiga ko'ra $\Delta_1=\Delta_2=\dots=\Delta_n=0$ notrivial yechim mavjud bo'lishi uchun $\Delta=0$ bo'lishi zarur. Unda sistema cheksiz ko'p yechimga ega bo'ladi.

Notrivial yechimlarni topish uchun sistema uchburchak ko'rinishga keltiriladi.

$\Delta=0$ekanligidan sistema oxirgi tenglamasida ikki noma'lum qoladi. Ulardan birini ozod parametr deb olib, qolgan noma'lumlar u orqali yoziladi.

Parametr cheksiz ko'p qiymat qabul qilgani uchun notrivial cheksiz ko'p yechimlarni topamiz.

Misol.

$\begin{cases} 2x_1+x_2-4x_3=0 \\ 3x_1+5x_2-7x_3=0 \\ 4x_1-5x_2-6x_3=0 \end{cases}$ sistema notrivial yechimlarini toping

$$\Delta=\begin{vmatrix}2&1&-4\\3&5&-7\\4&-5&-6\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2&1&-4\\0&7&-2\\0&-7&2\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2&1&-4\\0&7&-2\\0&0&0\end{vmatrix}=0$$ bo‘lgani uchun trivial bo‘lmagan yechimlar mavjud.

Sistemaning oxirgi tengligi $-7x_2+2x_3=0$ ko‘rinishda bo‘ladi. Agar $x_3=7\lambda$ desak, $x_2=2\lambda$ bo‘ladi. Ularni birinchi tenglamaga qo‘yib:

$$2x_1+2\lambda-4\cdot 7\lambda=0 \quad va \quad x_2=13\lambda$$

Demak, $(13\lambda;\ 2\lambda;\ 7\lambda)$, $\lambda\in R$ ko‘rinishdagi uchlik sistemaning yechimidir. Bu yechimlar oilasi trivial yechim (0; 0; 0)ni o‘zida saqlaydi.

Shu paytgacha qaralgan sistemalarda noma’lumlar soni tenglamalar soniga teng edi. Umuman, $\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m\end{cases}$ $m\neq n$, sistemalarni ham qarash mumkin. Bunday sistemalar birgalikda bo‘lishi asosiy va kengaytirilgan quyidagi matritsalar rangiga bog‘liq bo‘ladi.

$$A=\begin{pmatrix}a_{11}&a_{12}&\dots a_{1n}\\a_{21}&a_{22}&\dots a_{2n}\\ \cdot&\cdot&\dots\cdot\\a_{m1}&a_{m2}&\dots a_{mn}\end{pmatrix},\quad \overline{A}=\begin{pmatrix}a_{11}&a_{12}&\dots a_{1n}b_1\\a_{21}&a_{22}&\dots a_{2n}b_2\\ \cdot&\cdot&\dots\cdot\\a_{m1}&a_{m2}&\dots a_{mn}b_m\end{pmatrix}$$

Teorema. (Kroneker-Kapelli): Tenglamalar sistemasi birgalikda bo‘lishi uchun A va $\overline{A}$ matritsalar ranglari teng rangA=rang$\overline{A}$ bo‘lishi zarur.

Mavzuga doir misol va masalalar

1. Determinantlarni hisoblang.

1) $\begin{vmatrix}5&4\\3&-2\end{vmatrix}$ 2) $\begin{vmatrix}sin\alpha&-cos\alpha\\cos\alpha&sin\alpha\end{vmatrix}$ 3) $\begin{vmatrix}sin^2\alpha&cos^2\alpha\\sin^2\beta&cos^2\beta\end{vmatrix}$ 4) $\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$

5) $\begin{vmatrix}3&4&-5\\8&7&-2\\2&-1&8\end{vmatrix}$

2. Nollari ko‘p qator elementlari bo‘yicha yoyib hisoblang:

1). $\begin{vmatrix}1&b&1\\0&b&0\\b&0&-b\end{vmatrix}$ 2). $\begin{vmatrix}-x&1&x\\0&-x&-1\\x&1&-x\end{vmatrix}$ 3). $\begin{vmatrix}1&0&0\\-2&-3&1\\3&8&-2\end{vmatrix}$

3. Tenglamalarni yeching:

1). $\begin{vmatrix}3&x&-x\\2&-1&3\\x+10&1&1\end{vmatrix}=0$; 2). $\begin{vmatrix}x&x+1&x+2\\x+3&x+4&x+5\\x+6&x+7&x+8\end{vmatrix}=0$;

3). $\begin{vmatrix}cos8x&-sin5x\\sin8x&cos5x\end{vmatrix}=0$

4. Determinant xossalaridan foydalanib hisoblang:

1). $\begin{vmatrix}sin^2\alpha&1&cos^2\alpha\\sin^2\beta&1&cos^2\beta\\sin^2\gamma&1&cos^2\gamma\end{vmatrix}$ 2). $\begin{vmatrix}sin^2\alpha&cos2\alpha&cos^2\alpha\\sin^2\beta&cos2\beta&cos^2\beta\\sin^2\gamma&cos2\gamma&cos^2\gamma\end{vmatrix}$

3). $\begin{vmatrix}(a_1+b_1)^2&a_1^2+b_1^2&a_1b_1\\(a_2+b_2)^2&a_2^2+b_2^2&a_2b_2\\(a_3+b_3)^2&a_3^2+b_3^2&a_3b_3\end{vmatrix}$

5. Determinant xossalaridan foydalanib hisoblang:

1). $\begin{vmatrix}1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{vmatrix}$ 2). $\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}$

3). $\begin{vmatrix}1&2&3&4\\1&4&9&16\\1&8&27&64\\1&16&81&256\end{vmatrix}$ 4). $\begin{vmatrix}35&59&71&52\\42&70&77&54\\43&68&72&52\\29&49&65&50\end{vmatrix}$

6. Uchburchak ko‘rinishiga keltirib hisoblang.

1). $\begin{vmatrix}1&2&3&\cdots&n\\-1&0&3&\cdots&n\\-1&-2&0&\cdots&n\\ \cdots&\cdots&\cdots&\cdots&\cdots\\-1&-2&-3&&0\end{vmatrix}$ 2). $\begin{vmatrix}1&2&2&\cdots&2\\2&2&2&\cdots&2\\2&2&3&\cdots&2\\ \cdots&\cdots&\cdots&\cdots&\cdots\\2&2&2&\cdots&n\end{vmatrix}$

3). $\begin{vmatrix}0&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\ \cdots&\cdots&\cdots&\cdots&\cdots\\1&1&1&\cdots&0\end{vmatrix}$ 4). $\begin{vmatrix}n&1&1&\cdots&1\\1&n&1&\cdots&1\\1&1&n&\cdots&1\\ \cdots&\cdots&\cdots&\cdots&\cdots\\1&1&1&\cdots&n\end{vmatrix}$

7. $A=\begin{pmatrix}3&-2\\5&-4\end{pmatrix}$, $B=\begin{pmatrix}3&4\\2&5\end{pmatrix}$, $C=\begin{pmatrix}4&2\\7&-1\end{pmatrix}$ bo‘lsa, $A\cdot B-2C$ ni hisoblang.

8. $A=\begin{pmatrix}1&i\\1&-i\end{pmatrix}$, $B=\begin{pmatrix}i&1\\-i&1\end{pmatrix}$ bo‘lsa, $(i+1)\cdot A+(i-1)\cdot B$ ni hisoblang.

9. Hisoblang.

1) $\begin{pmatrix}1&-2\\3&-4\end{pmatrix}^2$, 2) $\begin{pmatrix}1&a\\0&1\end{pmatrix}^n$, 3) $\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}^n$, 4) $\begin{pmatrix}2&-1\\3&-2\end{pmatrix}^n$, 5) $\begin{pmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{pmatrix}^n$

10. Kvadrati nol matritsa bo‘lgan barcha kvadrat matritsalarni toping.

11. Kvadrati birlik matritsa bo‘lgan barcha kvadrat matritsalarni toping.

12. Quyidagi matritsalarga teskari matritsani toping.

1) $\begin{pmatrix}2&4\\1&3\end{pmatrix}$ 2) $\begin{pmatrix}1&2&-3\\0&1&2\\0&0&1\end{pmatrix}$ 3) $\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}$

13. Matritsaviy tenglamalarni yeching.

1) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\cdot X=\begin{pmatrix}3 & 5\\ 5 & 9\end{pmatrix}$ 2) $X\cdot\begin{pmatrix}3 & -2\\ 5 & -4\end{pmatrix}=\begin{pmatrix}-1 & 2\\ -5 & 6\end{pmatrix}$

3) $\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}\cdot X\cdot\begin{pmatrix}-3 & 2\\ 5 & -3\end{pmatrix}=\begin{pmatrix}-2 & 4\\ 3 & -1\end{pmatrix}$

4) $\begin{pmatrix}1 & 2 & -3\\ 3 & 2 & -4\\ 2 & -1 & 0\end{pmatrix}\cdot X=\begin{pmatrix}1 & -3 & 0\\ 10 & 2 & 7\\ 10 & 7 & 8\end{pmatrix}$

14. Matritsa rangini hisoblang.

1) $\begin{pmatrix}0 & 4 & 10 & 1\\ 4 & 8 & 18 & 7\\ 10 & 18 & 40 & 17\\ 1 & 7 & 17 & 3\end{pmatrix}$ 2) $\begin{pmatrix}1 & -1 & 2 & 3 & 4\\ 2 & 1 & -1 & 2 & 0\\ -1 & 2 & 1 & 1 & 3\\ 1 & 5 & -8 & -5 & -12\\ 3 & -7 & 8 & 9 & 13\end{pmatrix}$

15. Tenglamalar sistemasini 1) Kramer formulasi 2) Matritsaviy 3) Gauss usullari yordamida yeching.

1) $\begin{cases}3x+2y=7\\ 4x-5y=40\end{cases}$ 2) $\begin{cases}5x+2y=4\\ 7x+4y=8\end{cases}$ 3) $\begin{cases}x_1+x_2+2x_3=-1\\ 2x_1-x_2+2x_3=-4\\ 4x_1+x_2+4x_3=-2\end{cases}$

4) $\begin{cases}x_1+2x_2+4x_3=31\\ 5x_1+x_2+2x_3=29\\ 3x_1-x_2+x_3=10\end{cases}$

5) $\begin{cases}x_1+2x_2+3x_3-2x_4=6\\ 2x_1-x_2-2x_3-3x_4=8\\ 3x_1+2x_2-x_3+2x_4=4\\ 2x_1-3x_2+2x_3+x_4=-8\end{cases}$ 6) $\begin{cases}x_1+2x_2+3x_3+4x_4=5\\ 2x_1+x_2+2x_3+3x_4=1\\ 3x_1+2x_2+x_3+2x_4=1\\ 4x_1+3x_2+2x_3+x_4=-5\end{cases}$

7) $\begin{cases}\lambda x+y+z=1\\ x+\lambda y+z=\lambda\\ x+y+\lambda z=\lambda^2\end{cases}$ 8) $\begin{cases}x+ay+a^2z=a^3\\ x+by+b^2z=b^3\\ x+cy+c^2z=c^3\end{cases}$ 9) $\begin{cases}ax+by+z=1\\ x+aby+z=b\\ x+by+az=1\end{cases}$

16. Bir jinsli tenglamalar sistemasini yeching:

1) $\begin{cases}x_1+2x_2+3x_3+4x_4=0\\ x_1+x_2+2x_3+3x_4=0\\ x_1+5x_2+x_3+2x_4=0\\ x_1+5x_2+5x_3+2x_4=0\end{cases}$ 2) $\begin{cases}x_1+x_2+x_3+x_4=0\\ x_2+x_3+x_4+x_5=0\\ x_1+2x_2+3x_3=0\\ x_2+2x_3+3x_4=0\\ x_3+2x_4+3x_5=0\end{cases}$

Oliy algebra elementlariga doir joriy nazorat uchun uy vazifalari

1. $z=\frac{1}{(-1)^N+\sqrt{(3)^k}\cdot i}$ kompleks son berilgan, bunda $k=\begin{cases}1, agar\ n=3k-2\\ 0, agar\ n=3k-1\\ -1, agar\ n=3k\end{cases}$

a) z ni algebraik formada yozing va z^2 ni hisoblang.

b) z ni trigonometrik formada yozing va z^{N+20}, $\sqrt[4]{z}$ larni hisoblang.

c) z ni ko'rsatkichli formada yozing.

2. $x^3+Nx+1=0$ tenglama yechimlari x_1, x_2, x_3 bo'lsa, quyidagilarni hisoblang

1) $x_1^2+x_2^2+x_3^2$; 2) $x_1^2x_2+x_1x_2^2+x_2^2x_3+x_2x_3^2+x_3^2x_1+x_3x_1^2$;

3) $x_1^4x_2^2+x_1^2\cdot x_2^4+x_2^4x_3^2+x_2^2x_3^4+x_3^4x_1^2+x_3^2\cdot x_1^4$

4) $\frac{x_1^2}{(x_1+1)^2}+\frac{x_2^2}{(x_2+1)^2}+\frac{x_3^2}{(x_3+1)^2}$

3. Ferrari usuli bilan yeching;

$x^4-x^3+\left(4-\frac{N}{2}-\frac{N^2}{4}\right)x^2+2Nx+\left(\frac{N^2}{16}-1\right)-0$, yordamchi kubik tenglamaning bitta yechimi $-\frac{N}{4}$

4. Gorner sxemasi yordamida

$P_6(x)=x^6+(1-N)x^5-Nx^4-x^2+(N-1)x+N$ va

$P_4(x)=x^4-(N+1)x^3+(N+x)x^2-(N+1)x+N$ ko'phadlar EKUB va EKUK larni toping.

5. $\frac{x+N}{x^5+2x^4+(N-1)x^3+2(N-1)x^2-Nx-2N}$ kasrni sodda kasrlarga yoying.

6. Nollari ko'p qator elementlari bo'yicha yoyib hisoblang.

$\begin{vmatrix}1 & 2 & 3 & -5\\ 2 & 0 & -1 & 0\\ 3 & 0 & 2 & N\\ 4 & 0 & 5 & 0\end{vmatrix}$

7. $\begin{cases}x_1-Nx_2-x_3-x_4=3\\ x_1+x_2+Nx_3+2x_4=0\\ 2x_1+Nx_2+x_3+x_4=0\\ N\cdot x_1-3x_2+2x_3-x_4=-1\end{cases}$ sistemani

1) Kramer qoidasi, 2)Matritsaviy, 3) Gauss usuli yordamida yeching.

Fazoda analitik geometriya

8-mavzu. Fazoda analitik geometriya

8.1. Fazoda Dekart va yarim qutbiy koordinatalar sistemasi

1. To'g'ri burchakli 0xyz Dekart koordinatalar sistemasi o'lchov birligi aniqlangandan so'ng o'zaro perpendikulyar, bitta 0 nuqtada kesishuvchi 0x, 0y, 0z o'qlari yordamida kiritiladi. Bunda 0-koordinata boshi, 0x-abssissa, 0y-ordinata, 0z-oplikata o'qlari deyiladi.

Biror C nuqta berilsa, undan 0x, 0y, 0z o'qlariga perpendikulyar tekisliklar o'tkazamiz. Bu tekisliklarning son o'qlari bilan kesishgan nuqtalari C nuqtaning **to'g'ri burchakli** yoki **Dekart koordinatalari** deyiladi. C(x;y;z), x=$0C_x$, y=$0C_y$, z=$0C_z$.

Bu kattaliklar, mos ravishda C nuqta abssissasi, ordinatasi, oplikatasi deyiladi.

0xy, 0yz, 0xz tekisliklari koordinata tekisliklari deyiladi. Ular fazoni 8ta bo'lak - oktantlarga ajratadi. Masalan, *I* oktantda x>0, y>0, z>0 bo'lsa, oxirgi VIII oktantda x<0,y<0, z<0 bo'ladi.

2. Fazodagi C nuqta holatini qutb **koordinatalari** va oplikata yordamida aniqlash mumkin. Buning uchun Dekart koordinatalari boshi va qutb boshini bitta nuqtaga, boshlang'ich nurni abssissaga ustma-ust qo'yamiz. C nuqtaning 0xy tekislikdagi proyeksiyasi C' bo'lsa, r=I$0C'$I, φ=<x$0C'$, z=$C'C$ kattaliklar yordamida C ning fazodagi holati C (r, φ, z) tarzida aniqlanadi. Bunda r, φ, z – silindrik koordinatalari, kiritilgan sistema esa silindrik koordinatalar sistemasi deyiladi. Silindrik va Dekart koordinatalari o'zaro bog'lanishi qutb koordinatalar yordamida

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \quad r=\sqrt{x^2+y^2}, \quad \text{tg}\varphi=\frac{y}{x}$$ ko'rinishida bo'lishi avvaldan ma'lum.

3. Fazodagi C nuqtani ko'ramiz. $0C$=ρ, <C0z=θ bo'lsin. Bundan tashqari C nuqtaning qutbiy φ koordinatasini ham ko'ramiz.

ρ, φ, θ kattaliklar C nuqtaning **sferik koordinatalari**, kiritilgan sistema esa, **sferik koordinatalar sistemasi** deyiladi. Yordamchi kattalik sifatida C ning qutbiy r koordinatasi ma'lum desak,

r=ρcos(90^0-φ)=ρsinθ o'rinli ekanligidan,

$$\begin{cases} x = r\cos\varphi = \rho\sin\theta\cos\varphi \\ y = r\sin\varphi = \rho\sin\theta\sin\varphi \\ z = \rho\cos\theta \end{cases}$$ o'zaro bog'lanishni keltirib chiqaramiz.

Aksincha, $\cos\varphi=\frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi=\frac{y}{\sqrt{x^2+y^2}}$, $r=\sqrt{x^2+y^2}$, $\text{tg}\varphi=\frac{y}{x}$,

$\rho=\sqrt{x^2+y^2+z^2}$, $\cos\theta=\frac{z}{\sqrt{x^2+y^2+z^2}}$, $\sin\theta=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$.

bog'lanishlarni keltirib chiqarish talabaga qiyinchilik tug'dirmaydi.

Silindrik, sferik koordinatalar sistemasida ba'zi qutbiy koordinatalar qatnashganligi uchun ular **yarim qutbiy koordinatalar sistemasi** deyiladi.

8.2. Fazoda masofa, kesmani berilgan nisbatda bo'lishi, koordinatalarni almashtirish

Fazoda Dekart koordinatalari kiritilgan, A(x_1,y_1,z_1), B(x_2,y_2,z_2) nuqtalar berilgan bo'lsin. Agar A',B' nuqtalar A va B ning 0xy tekislikdagi proyeksiyasi bo'lsa, I$A'B'$I=$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

A nuqtadan $A'B'$kesmaga parallel chiziq o'tkazib, uni BB' bilan kesishgan nuqtasini B'' bilan belgilaymiz. U holda IBB'I=z_2-z_1. Pifagor teoremasiga ko'ra: IABI=$\sqrt{|A'B'|^2+|BB''|^2}$.

Demak, $|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ bu **ikki nuqta orasidagi masofani hisoblash formulasi** deyiladi.

Agar A va B tutashtirilib, kesma hosil qilinsa va bu kesmada C(x;y;z) nuqta olinib, $\frac{|AC|}{|CB|}$=λ munosabat o'rinli bo'lsa, $x=\frac{x_1+\lambda x_2}{1+\lambda}$, $y=\frac{y_1+\lambda y_2}{1+\lambda}$, $z=\frac{z_1+\lambda z_2}{1+\lambda}$ formulalarni keltirib chiqarish mumkin. Xususan, IACI=ICBI, λ=1 bo'lsa, $x=\frac{x_1+y_2}{2}$, $y=\frac{y_1+y_2}{2}$, $z=\frac{z_1+z_2}{2}$ kelib chiqadi.

Agar koordinatata boshi O(0;0;0) dan biror-bir O' (a;b;c) nuqtaga ko'chirilsa, A(x;y;z) nuqtaning yangi $x'o'y'z'$, sistemadagi koordinatalari mos ravishda $A'(x',y',z')$ bo'ladi. Eski va yangi koordinatalar

$$\begin{cases} x = x' + a \\ y = y' + b \\ z = z' + c \end{cases}$$ formulalar yordamida o'zaro bog'lanadi.

Agar x,y o'qlari 0z atrofida biror α burchakka burilsa, eski va yangi koordinatalar bog'lanishi

$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha \\ y = x'\sin\alpha + y'\cos\alpha \\ z = z' \end{cases}$$

ko'rinishda, x,z o'qlari 0y atrofida biror β burchakka burilsa,

$$\begin{cases} x = x'\cos\beta - z'\sin\beta \\ y = y' \\ z = y'\sin\beta + z'\cos\beta \end{cases}$$

ko'rinishda, y,z o'qlari 0x atrofida biror-bir γ burchakka burilsa,

$$\begin{cases} x = x' \\ y = y'cos\gamma - z'\ sin\gamma \\ z = y'sin\gamma + z'\ cos\gamma \end{cases}$$

bog'lanishlar o'rinli bo'ladi. Bunda α, β, γ – burchaklar **Eyler burchaklari** deyiladi.

8.3. Vektorlar, amallar, xossalari

Ko'pgina miqdorlar (hajm, massa, zichlik, temperatura) faqatgina son orqali aniqlanadi. Shuning uchun ular **skalyar miqdorlar** deyiladi. Ba'zi miqdorlar esa ham son qiymati, ham yo'nalishi bilan aniqlanadi (kuch, tezlik) bunday miqdorlar **vektor miqdorlar** deyiladi. Ularni o'rganish uchun "vector" tushunchasi kiritiladi.

Yo'naltirilgan kesma vektor deyiladi. Kesma boshi vektor boshi, oxiri esa vector oxiri deyiladi. Agar nuqta A nuqtada boshlanib, B nuqtada tugasa, $\overrightarrow{AB}$ yoki $\overrightarrow{a}$ kabi belgilanadi.

Agar ikki vektordan birini parallel ko'chirish natijasida ikkinchisini hosil qilish mumkin bo'lsa, ular teng bo'ladi, ya'ni yo'nalishdosh, uzunligi teng vektorlar o'zaro tengdir.

Parallel to'g'ri chiziqlarda yotuvchi vektorlar **kolleniar**, bir tekislikda yotuvchi vektorlar o'zaro **komplanar** deyiladi.

Boshi va oxiri ustma-ust tushgan vektor **nol vektor** deyiladi va $\overline{0}$ tarzida yoziladi, uning yo'nalishi ixtiyoriy deb qabul qilinadi.

8.4. Chiziqli amallar

Ikki $\vec{a}$ va $\vec{b}$ vektorlar yig'indisi deb shunday $\vec{c}$ vektorga aytiladiki, bu vektor $\vec{a}$ ning oxiriga $\vec{b}$ parallel ko'chirib keltirilganda, $\vec{a}$ ning boshi va $\vec{b}$ ning oxirini tutashtiruvchi vektordir. $\vec{c}=\vec{a}+\vec{b}$

Agar vektorlar boshi bir nuqtaga ko'chirilib, tomonlari shu vektolar bo'lgan vektor yasasak, umumiy uchdan chiquvchi diagonal yig'indi vektor bo'ladi. Qo'shishning bu usullari **uchburchak va parallelogramm qoidalari** deyiladi .

$\vec{a}$ va $\vec{b}$ vektorlar ayirmasi deb, shunday $\vec{c}$ vektorga aytiladiki, $\vec{a}=\vec{c}+\vec{b}$ o'rinli bo'ladi. Parallelogramm usulida $\vec{c}$-ayirma vektor berilgan vektorlar uchlarini tutashtiruvchi, $\vec{a}$ tomon yo'nalgan diagonal vektordir.

$\vec{a}$ vektorning haqiqiy λ songa ko'paytmasi deb shunday vektorga aytiladiki, bu vektor uzunligi IλI.I$\vec{a}$I ga, yo'nalishi λ>0 da $\vec{a}$ bilan bir xil, λ<0 da esa $\vec{a}$ ga qarama-qarshi yo'nalgan vektordir.

Fazoda boshi A(x_1 ; y_1; z_1), oxiri B(x_2; y_2 ;z_2) nuqtada bo'lgan vector $\vec{a}=\overrightarrow{AB}=(x_2 - x_1; y_2 - y_1; z_2 - z_1)$, vektorga teng. Demak, ixtiyoriy vektor boshini koordinata boshiga ko'chirish mumkin, ya'ni fazoda qancha nuqta bo'lsa, shuncha vektor mavjud va aksincha. Qolgan vektorlar "aylangani chiqqan" xolos.

$\vec{a}$ vektorning 0x,0y,0z o'qlariga proyeksiyalari mos ravishda x,y,z bo'lsa, ular vektorning koordinatalari deyiladi va $\vec{a}$ (x;y;z) tarzida yoziladi.

Ikki nuqta orasidagi masofa formulasidan: $\vec{a}$ $=\sqrt{x^2+y^2+z^2}$, $|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ ekanligi kelib chiqadi.

Koordinatalari bilan berilgan $\vec{a}$ (x_1 ; y_1; z_1), $\vec{b}$ (x_2; y_2 ;z_2) C ustida arifmetik amallar quyidagicha kiritiladi:

$$\vec{a}\pm\vec{b}=(x_2 \pm x_1; y_2 \pm y_1; z_2 \pm z_1),\ \lambda.\ \vec{a}=(\lambda x_1 ; \lambda y_1; \lambda z_1)$$

Agar $\vec{a}$, $\vec{b}$ vektorlar o'zaro kolleniar bo'lsa, shunday haqiqiy λ topish mumkinki, $\vec{b}$=λ $\vec{a}$ o'rinli bo'ladi, ya'ni $\frac{x_2}{x_1}=\frac{y_2}{y_1}=\frac{z_2}{z_1}$=λ.

Agar $\vec{a}$ (x;y;z) vektorning 0x;0y;0z o'qlariga og'ish burchaklari mos ravishda α,β,γ bo'lsa, bu burchaklar kosinuslari-cosα, cosβ, cosγ lar vektorning yo'naltiruvchi kosinuslari deyiladi.

x= | $\vec{a}$ I.cosα, y= | $\vec{a}$ I.cosβ z= | $\vec{a}$ I. cosγ ekanligidan doimo

$cos^2\alpha + cos^2\beta + cos^2\gamma=1$ o'rinli bo'ladi va

$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}},\ \cos\beta=\frac{y}{\sqrt{x^2+y^2+z^2}},\ \cos\gamma=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Vektorni qo'shish, ayirish, songa ko'paytirish amallari quyidagicha xossalarga ega:

1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
3) $\lambda(\mu\vec{a})=(\lambda\mu)\vec{a}$
4) $(\lambda+\beta)\vec{a}=\lambda\vec{a}+\mu\vec{a}$
5) $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$

Bir necha $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vektorni qo'shish uchun, birining oxiriga ikkinchisini parallel ko'chiramiz. $\vec{a}_1$ ning boshi va $\vec{a}_n$ning oxirini tutashtiruvchi vektor **yig'indi vektor** deyiladi. Bu esa qo'shishning **ko'pburchak usuli** deyiladi. $\vec{a} = \vec{a}_1 + \vec{a}_2 + ... + \vec{a}_n$

Fazoda koordinata boshidan son o'qlari musbat yo'nalishi bo'yicha $\vec{i}, \vec{j}, \vec{k}$ ko'rinishda belgilanuvchi birlik vektorlarni ko'ramiz: $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

Bu uchlik bazis deb ataladi, chunki fazodagi ixtiyoriy $\vec{a}$ vektor $\vec{i}, \vec{j}, \vec{k}$ bazis orqali yagona ko'rinishda yoyiladi: $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$

x,y,z lar $\vec{a}$ ning koordinatalaridir, ya'ni $\vec{a}$ (x;y;z). Qaralgan $\vec{i}, \vec{j}, \vec{k}$ vektorlar **ortlar** deyiladi.

8.5. Skalyar ko'paytma

Nolga teng bo'lmagan $\vec{a}$ va $\vec{b}$ vektorlarning skalyar ko'paytmasi deb, shu vektorlar uzunliklari bilan ular orasidagi burchak kosinusi ko'paytmasidan iborat songa aytiladi, $\vec{a} \cdot \vec{b}$ yoki $|\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$

Skalyar ko'paytma quyidagi xossalarga ega.

1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2) $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b})$
3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} * \vec{b} + \vec{a} * \vec{c}$
4) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
5) $\vec{a} * \vec{b} = 0 \leftrightarrow \vec{a} \perp \vec{b}$

Oxirgi xossalardan ortlar uchun $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = 1$, $\vec{i} * \vec{j} = \vec{i} * \vec{k} = \vec{j} * \vec{i} = \vec{j} * \vec{k} = \vec{k} * \vec{i} = \vec{k} * \vec{j} = 0$ ekanligi kelib chiqadi.

Fazoda kordinatalari bilan berilgan $\vec{a}$ $(x_1; y_1; z_1)$, $\vec{b}$ $(x_2; y_2; z_2)$ vektorlar skalyar ko'paymasini topamiz.

Kosinuslar teoremasiga ko'ra;

$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}| \cdot |\vec{b}| \cos\varphi = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} * \vec{b}$

Ikkinchi tenglamadan

$|\vec{b} - \vec{a}|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2 + z_2^2 + 2z_1z_2 + z_1^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)$

Demak, $\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2$.

Bu formulani vektorlarning ortlar bo'yicha yoyilmasi yordamida ham olish mumkin.

$\vec{a} \cdot \vec{b} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) \cdot (x_2\vec{i} + y_2\vec{j} + z_2\vec{k}) = x_1x_2 + y_1y_2 + z_1z_2$

Bu ikki vektor orasidagi burchak quyidagicha topiladi:

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} * \sqrt{x_2^2 + y_2^2 + z_2^2}}$$

Misol. A(1;1;1), B(2;2;1), C(2;1;2) nuqtalar berilgan. φ=<BAC ni toping.

$\overrightarrow{AB} = (1;1;0)$, $\overrightarrow{AC} = (1;0;1)$ ekanligidan, $\cos\varphi = \frac{\vec{a} * \vec{b}}{|\vec{a}| * |\vec{b}|} = \frac{1*1 + 1*0 + 0*1}{\sqrt{1^2+1^2+0^2}\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$. Demak, <BAC=$60^0$.

8.6. Vektor ko'paytma

$\vec{a}$ vektorning $\vec{b}$ vektorga vektor ko'paytmasi deb, shunday $\vec{c}$ vektorga aytiladiki, u quyidagi shartlarga bo'ysunadi:

1) $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$,

2) $|\vec{c}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$

3) $|\vec{c}|$ uchidan qaralganda, $\vec{a}$ *dan* $\vec{b}$ ga yo'nalish soat strelkasi yo'nalishiga qarama-qarshi bo'lishi kerak.

Vektor ko'paytma $\vec{c} = \vec{a}x\vec{b} = [\vec{a}; \vec{b}]$ tarzida belgilanadi.

Ta'rifdan ko'rinadiki, $\vec{c}$ ning uzunligi $\vec{a}$ *va* $\vec{b}$ vektorlarga qurilgan parallelogramm yuzasini ifodalovchi songa teng.

Vektorlar vektor ko'paytmasi quyidagi xossalarga ega;

1) $\vec{a}\vec{b}$ bo'lsa $\overline{ax}\vec{b} = 0$
2) $\vec{a}x\vec{b} = -\vec{b}x\vec{a}$
3) $\lambda\vec{a}x\vec{b} = \lambda(\vec{a}x\vec{b}) = \vec{a}x\lambda\vec{b}$
4) $(\vec{a} + \vec{b})x\vec{c} = \vec{a}x\vec{c} + \vec{b}x\vec{c}$
5) $\vec{i}x\vec{i} = \vec{j}x\vec{j} = \vec{k}x\vec{k} = 0$ $\vec{i}x\vec{j} = \vec{k}$, $\vec{j}x\vec{i} = -\vec{k}$, $\vec{j}x\vec{k} = \vec{i}$, $\vec{k}x\vec{j} = -\vec{i}$, $\vec{k}x\vec{i} = \vec{j}$, $\vec{i}x\vec{k} = -\vec{j}$

Koordinatalari bilan berilgan $\vec{a}(x_1; y_1; z_1)$, $\vec{b}(x_2; y_2; z_2)$ vektorlar vektor ko'paytmasini hisoblab topish masalasini ko'ramiz.

$\vec{c} = \vec{a}x\vec{b} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k})x(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}) = -y_1x_2\vec{k} + z_1x_2\vec{j} + x_1y_2\vec{k} - z_1y_2\vec{i} - x_1z_2\vec{j} + y_1z_2\vec{i} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

Demak, $\vec{a}x\vec{b} = \vec{c}$ bo'lsa,

$$\vec{c}=\vec{c}\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Natijalar. 1) $\vec{a}$ va $\vec{b}$ vektorlar perpendikulyar bo'lishi uchun $\vec{a} * \vec{b}=0$ bo'lishi zarur va yetarlidir.

2) $\vec{a}$ va $\vec{b}$ vektorlarga yasalgan uchburchak yuzi

$$S_\Delta=\frac{1}{2}\ |\vec{a} * \vec{b}|=\frac{1}{2}\left\|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}\right\|=$$

$$\frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}^2+\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}^2+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^2}$$

formula yordamida topiladi.

Agar $\vec{a}, \vec{b}$ vektorlar x0y tekisligida yotsa, $z_1 = z_2 = 0$ ekanligidan,

$$S_\Delta=\frac{1}{2}\left\|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right\|=\frac{1}{2}\ |x_1 y_2 - y_1 x_2|$$

formula bilan topilishi kelib chiadi.

Misol. A(1;1;1), B(2;2;1), C(2;1;2) nuqtalar hosil qilgan uchburchak yuzini toping.

$\overline{AB}$=(1;0;1)$\overline{AC}$ =(1;0;1) ekanligidan,

$$S_{\Delta ABC}=\frac{1}{2}\left\|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\right\|=\frac{1}{2}\sqrt{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2+\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}^2+\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^2}=$$

$$=\frac{1}{2}\sqrt{1^2+1^2+(-1)^2}=\frac{\sqrt{3}}{2}$$

8.7. Aralash ko'paytma

$\vec{a}, \vec{b}$ va $\vec{d}$ vektorlar aralash ko'paytmasi deb, $\vec{a}x\vec{b}$ va $\vec{d}$ vektorlar skalyar ko'paytmasiga teng songa aytiladi va $(\vec{a}x\vec{b}) * \vec{d}$ yoki $(\vec{a};\vec{b};\vec{d})$ ko'rinishda belgilanadi.

Agar $\vec{a}, \vec{b}$ vektorlar x0y tekisligida joylashgan bo'lsa, $\vec{c}=\vec{a}x\vec{b}$ vektor 0z o'qiga parallel yo'naladi. Agar $\vec{d}$ vektor 0z o'qi bilan biror α burchak hosil qilsa, u holda $h=|\vec{d}|.\cos\alpha$ kattalik, asosi $\vec{a}$ va $\vec{b}$ ga qurilgan parallelogramm, yon qirrasi $\vec{d}$ bo'lgan parallelepiped balandligidir. Demak,

$$(\overline{a x b})\,\vec{d}=\vec{c}.\vec{d}=|\vec{c}\vec{d}|.\cos\alpha=S\ .h=V_{par}$$

$V_{par}=|(\overline{a x b}).\vec{d}|$, chunki $(\vec{b}x\vec{a}).\vec{c}=-V_{par}$ bo'lishi mumkin.

$\vec{a}, \vec{b}, \vec{d}$ vektorlarga qurilgan piramida hajmi esa,

$$V_{pir}=\frac{1}{6}\ |(\vec{a}x\vec{b}).\vec{d}|$$

chunki bu piramida uchburchakli prizmaning $\frac{1}{3}$ qismidir, parallelepipedning $\frac{1}{6}$ qismi bo'ladi.

Vektorlar koordinatalari yordamida aralash ko'paytmani hisoblash masalasini ko'ramiz.

$\vec{c}=\vec{a}x\vec{b}=\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$, $\vec{d}=(x_3, y_3, z_3)$ bo'lsa,

$$(\vec{a}x\vec{b}) * \vec{d}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}.x_3-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}.y_3+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.z_3=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

kelib chiqadi.

Natijalar. 1) Agar $\vec{a}, \vec{b}, \vec{d}$ vektorlar komplanar bo'lsa,

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$$

bo'ladi va aksincha. 2) $(\vec{a}x\vec{b}, \vec{d})=(\vec{a}, \vec{b}x\vec{d})$ 3) $V_{pir}=\frac{1}{6}\left\|\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\right\|$.

8.8. Qo'sh vektor ko'paytma

$(\vec{a}x\vec{b})x\vec{d}$ vektor **qo'sh vektor ko'paytma** deyiladi.

$$(\vec{a}x\vec{b})x\vec{d}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ x_3 & y_3 & z_3 \end{vmatrix}$$

tarzida bu vektorni topish mumkin.

1.$\vec{a}(1;-2;5)$, $\vec{b}(2;3;-4)$, $\vec{c}(1;-2;4)$ vektorlar berilgan. Quyidagilarni toping:

$2\vec{a}-3\vec{b}+\vec{c}$; $\vec{a}.\vec{b}$, $\vec{a}x\vec{b}$, $(\vec{a}x\vec{b}).\vec{c}$

1) $2\vec{a}-3\vec{b}+\vec{c}$=2(1;-2;5)–3(2;3;-4)+(1;-2;4)=(2;-4;10)–(6;9;-12)+ +(1;-2;4)=(-3;-15;26)

2) $\vec{a}.\vec{b}$ =1.2+ (-2).3+5.(-4)=2-6-20=-24.

$3)\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 5 \\ 2 & 3 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 3 & -4 \end{vmatrix} \cdot \vec{i} - \begin{vmatrix} 1 & 5 \\ 2 & -4 \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \cdot \vec{k} = -7\vec{i} + 14\vec{j} + 7\vec{k};$

$4)\ (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 1 & -2 & 5 \\ 2 & 3 & -4 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 5 \\ 0 & 7 & -14 \\ 0 & 0 & -1 \end{vmatrix} = -7$

Mavzuga doir misol va masalalar

1. Uchlari A(3;-1;2), B(0;-4;2), C(-3;2;1) bo‘lgan uchburchak teng yonli ekanligini isbotlang.
2. Uchlari A(3;-1;6), B(-1;7;-2), C(1;-3;2) bo‘lgan uchburchak to‘g‘ri burchakli ekanligini isbotlang.
3. Uchlari A(3;-2;5), B(-2;1;-3), C(5;1;-1) bo‘lgan uchburchak o‘tkir burchakli ekanligini isbotlang.
4. Abssissa o‘qida A(-3;4;8) nuqtadan 12 birlik uzoqlikdagi nuqtani toping.
5. Ordinatalar o‘qida A(1;-3;7), B(5;7;-5) nuqtalardan bir hil uzoqlikdagi nuqtani toping.
6. Uchlari A(2;-1;4), B(3;2;-6), C(-5;0;2) bo‘lgan uchburchak A uchidan tushirilgan medianasi uzunligini toping.

ABCD parallelogramm ikki uchi A(2;-3;-5), B(-1;3;2) diagonallari kesishishi nuqtasi E 6-bobga doir masalalar.

7. (4;-1;7) bo‘lsa, qolgan ikki uchini toping.
8. Parallelogrammning uchta uchi A(3;-1;2), B(1;2;-4), C(-1;1;2) bo‘lsa, to‘rtinchi uchini toping.
9. Kesma C(2;0;2), D(5;-2;0) nuqtalalri bilan uchta teng bo‘lakka ajratilgan bo‘lsa, kesma uchlari koordinatalarini toping.
10. Parallel ko‘chirishda A(1;2;3) nuqta A'(2;-1;-4) nuqtaga o‘tsa, B(1;1;1) qanday nuqtaga o‘tadi.
11. z=xy sirtda 0x,0y o‘qlari 0z atrofida α =45⁰ burilsa, tenglama qanday ko‘rinishga keladi.
12. Silindrik koordinatalarni toping.

a) (2;-2;-3) b) $(-\sqrt{2};\ \sqrt{2};1)$ c) $(2;-\frac{2}{\sqrt{3}}, 2)$ d) $(-\frac{\sqrt{3}}{2};\ \frac{1}{2};\frac{1}{4})$

e) $(4\cos 15^\circ;\ -4\sin 15^\circ;1)$ f) $(\frac{1}{2}\sin\frac{\pi}{8};\ \frac{1}{2}\cos\frac{\pi}{8};\ \frac{\sqrt{3}}{2})$ k) $(3;4;\ \frac{1}{2})$

13. Silindrik koordinatalarda tenglamalarni yozing.

a) $x^2+y^2+z^2=1$ b) $x^2+y^2+2z^2+2z-5=0$ c) x-2y+3z-5=0

14. Silindrik koordinatalarda tekislik tenglamasi

Rrcos(φ-α)+Cz+D=0 bo‘lishini isbotlang, bunda R,r,C,D haqiqiy sonlar.

15. Sferik koordinatalarini toping .

a) (1;1;1) b) (7;-7;5) c) $(\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}, \frac{5\sqrt{3}}{\sqrt{2}},)$ d) (0;0;-π) e) (1;2;3)

f) (cos77;sin77;0) g) (0;1;0)

16. Sferik koordinatalarda tenglamalarni yozing.

a) y=0 b) z=1 c) $x^2+(y-1)^2+z^2=1$

17. $\vec{a}$=(6;3;-2) vektor modulini hisoblang.
18. A(3;-2;1), B(5;4;-3) bo‘lsa, $\overrightarrow{AB}$, $\overrightarrow{BA}$ koordinatalarini yozing.
19. Agar $\vec{a}$=(2;-3;-1) oxiri B(1;-1;2) bo‘lsa, boshini toping.
20. Agar $\vec{a}$=(12;-15;-16) vektor yo‘naltiruvchi kosinuslarini toping.
21. $\vec{a}$ vektor 0x,0y o‘qlari bilan mos ravishda $60^0, 120^0$ burchak hosil qilsa va $|\vec{a}|=2$ bo‘lsa, koordinatalarini toping.
22. $|\vec{a}|=13$, $|\vec{b}|=19$ $|\vec{a}+\vec{b}|=24$ bo‘lsa, $|\vec{a}-\vec{b}|$ ni toping.
23. $|\vec{a}|=11$, $|\vec{b}|=23$, $|\vec{a}-\vec{b}|=30$ bo‘lsa, $|\vec{a}+\vec{b}|$ ni toping.
24. Agar ABC uchburchak og‘irlik markazi 0 bo‘lsa, $\overrightarrow{0A}+\overrightarrow{0B}+\overrightarrow{0C}=0$ ekanligini isbotlang.
25. Muntazam beshburchak ABCDE va $\overrightarrow{AB}=\vec{m}$, $\overrightarrow{BC}=\vec{n}$, $\overrightarrow{CD}=\vec{p}$, $\overrightarrow{DE}=\vec{q}$, $\overrightarrow{EA}=\vec{r}$ bo‘lsa, quyidagi vektorlarni yasang.

a) $\vec{m}-\vec{n}+\vec{p}-\vec{q}+\vec{r}$ b) $\vec{m}+2\vec{p}+\frac{\vec{r}}{2}$ c) $2\vec{m}+\frac{1}{2}\vec{n}-3\vec{p}-3\vec{q}+2\vec{r}$

26. α, β larning qanday qiymatlarida $\vec{a}$=(-2;3;β), $\vec{b}$=(α;-6;2) vektorlar kolleniar bo‘ladi.
27. $\vec{a}$=(9;4) vektorni $\vec{p}$=(2;-3), $\vec{q}$=(1;2) lar bo‘yicha yoying.
28. $\vec{c}$=(11;-6;5), vektorni $\vec{p}$=(3;-2;1), $\vec{q}$=(-1;1;2), $\vec{r}$=(2;1;-3) lar bo‘yicha yoying.
29. $(\vec{a}+\vec{b})^2+(\vec{a}-\vec{b})^2=2(a^2+b^2)$ ayniyatni isbotlang.
30. $\vec{a}$=(2;-4;4), $\vec{b}$ =(-3;2;6) vektorlar orasidagi burchak kosinusini toping.
31. Uchlari A(-1;-2;4), B(-4;-2;0), C(3;-2;1) bo‘lgan uchburchak ichki B burchagini toping.
32. Uchlari A(3;2;-3), B(5;1;-1), C(1;-2;1) bo‘lgan uchburchak A uchidagi tashqi burchagini toping.
33. A(1;2;1), B(3;-1;7), C(7;4;-2) uchli uchburchak ichki burchaklarini toping.
34. Shunday $\vec{x}$ vektor topingki, $\vec{a}$=(2;1;-1) uchun $\vec{x}\vec{a}$=3 bo‘lsin. Bunda $\vec{x}$ va $\vec{a}$ o‘zaro kolleniar.

35. Shunday $\vec{x}$ vektor topingki, y $\vec{a}$=(2;3;-1), $\vec{b}$ =(1;-2;3) larga perpendikulyar $\vec{x}*(2\vec{i}-\vec{j}+\vec{k})$=-6 bo'lsin.

36. $|\vec{a}|=10$, $|\vec{b}|=2$ $\vec{a}*\vec{b}$=12 bo'lsa, $\vec{a}\ x\ \vec{b}$ ni toping.

37. $|\vec{a}|=3$, $|\vec{b}|=26$ $|\vec{a}\ x\ \vec{b}|$=72 bo'lsa, $\vec{a}*\vec{b}$ ni toping.

38. Uchlari A(1;2;0), B(3;0;-3), C(5;2;6) bo'lgan uchburchak yuzini toping.

39. Uchlari A(1;-1;2), B(5;-6;2), C(1;3;-1) bo'lgan uchburchakning B uchidan tushirilgan balandlik uzunligini toping.

40. $\vec{a}$=(1;-1;3), $\vec{b}$ =(-2;-2;1), $\vec{c}$(3;-2;5) bo'lsa, $(\vec{a}\ x\vec{b})*\ \vec{c}$ ni hisoblang.

41. A(1;2;-1), B(0;1;5), C(-1;2;1), D(2;1;3) nuqtalar bir tekislikda yotishini isbotlang.

42. Uchlari A(2;-1;1), B(5;5;4), C(3;2;-1), D(4;1;3) bo'lgan piramida hajmini toping.

9-mavzu. Fazoda tekislik tenglamalari

Fazoda biror S sirtning tenglamasi F(x;y;z)=0 deyiladi, agar S ning har bir nuqtasi koordinatalari bu tenglamani qanoatlantirsa, va aksincha, S ga tegishli bo'lmagan nuqtalar koordinatalari bu tenglamani qanoatlantirmasa. Masalan. $x^2+y^2+z^2=R^2$ tenglama markazi koordinata boshida bo'lsa, radiusi R bo'lgan sfera tenglamasidir.

Fazoda chiziqlar ikki sirt kesishmasi sifatida berilishi mumkin, ya'ni

$$\begin{cases} F(x;y;z)=0 \\ F_2(x_2;y_2;z_2)=0 \end{cases}$$

Masalan, $\begin{cases} x^2+y^2+z^2=1 \\ x^2+y^2+(z-3)^2=10 \end{cases}$ kesishma oxy tekisligida, markazi koordinata boshidagi, radiusi birga teng aylanani bildiradi.

9.1. Fazoda tekislik tenglamalari, asosiy masalalar

Normal vektori va nuqtasi ma'lum tekislik tenglamasi

Nol bo'lmagan, tekislikka perpendikulyar bo'lgan ixtiyoriy vektor **tekislikning normal vektori** deyiladi.

Tekislikning, masalan, koordinata boshidan o'tkazilgan normal vektori $\vec{N}$(A;B;C) va E$(x_0;y_0;z_0)$ nuqtasi ma'lum bo'lsin. Tekislikninq ixtiyoriy F(x; y; z) nuqtasini olamiz.

$\overrightarrow{EF}=(x-x_0;y-y_0;z-z_0)$ vektor tekislikda yotganligi uchun $\vec{N}$ vektorga perpendikulyar, ya'ni $\overrightarrow{EF}\ \vec{N}$=0, koordinatalar bo'yicha yozsak,

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \qquad (1)$$

Hosil bo'lgan tenglama tekislikning biz qidirayotgan tenglamasidir.

9.2. Tekislikning umumiy tenglamasi

Fazoda E$(x_0;y_0;z_0)$, F$(x_1;y_1;z_1)$ nuqtalardan bir xil uzoqlikda yotgan nuqtalar to'plami tekislikdir, agar C(x; y; z) tekislik nuqtasi bo'lsa, $|EC|=|CF|$. Bundan $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ $=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$

Tomonlarni kvadratga ko'tarib, guruhlaymiz:

$(2x_1-2x_0)x+(2y_1-2y_0).y+(2z_1-2z_0)z+(x_0^2+y_0^2+z_0^2-x_1^2-y_1^2-z_1^2)=0$

Qavslarni mos ravishda A,B,C,D lar bilan belgilasak, tekislikning umumiy tenglamasi:

$$Ax+By+Cz+D=0 \qquad (2)$$

hosil bo'ladi. Bu tenglamani tekislikning oldingi tenglamasida:

$$D=-Ax_0-By_0-Cz_0$$

belgilash yordamida ham olish mumkin edi, demak, (2)-tenglamadagi noma'lumlar koeffitsiyentlari normal vektor koordinatalari ekan.

A,B,C,D sonlariga bog'liq holda quyidagi xususiy holatlar bo'lishi mumkin 1) D=0. U holda tekislik tenglamasi Ax+By+Cz=0 ko'rinish oladi.

Bu tenglama tekislikning koordinatalar boshidan o'tuvchi ekanligini bildiradi.

2) C=0. Bunda tekislik Ax+By+D=0 tenglamaga ega bo'lib, 0z o'qiga parallel tekislikni bildiradi, x0z tekisligida Ax+By+D=0 to'g'ri chizig'i bo'yicha o'tadi.

3) B=0. Tekislik Ax+Cz+D=0 tenglamaga ega va 0y o'qiga parallel.

4) A=0. Tekislik By+Cz+D=0 tenlamaga ega va 0x o'qiga parallel.

5) A=B=0. Tekislik Cz+D=0 tenglamaga ega. Undan $z=-\frac{D}{C}$ kelib chiqib, 0xy koordinatalar tekisligiga parallel tekislik ekanligini bildiradi.

6) A=C=0. Tekislik By+D=0 tenlamaga ega va 0xz tekisligiga parallel.

7) B=C =0. Tekislik Ax+D=0 tenlamaga ega va 0yz tekisligiga parallel.

8) A=B=D=0 bo'lsa, tekislik Cz=0, ya'ni z=0 tenglamaga ega bo'lib, u 0xy tekisligidir.

9) B=C=D= 0 bo'lsa, By=0, ya'ni y=0 bo'lib, 0xz tekisligini bildiradi.

10) B=C=D=0 bo'lsa, Ax=0 dan x=0 bo'lib, 0yz koordinati tekisligini bildiradi.

9.3. Tekislikning kesmalar bo'yicha tenglamasi

Koordinatalar boshi 0(0;0;0) dan o'tmaydigan biror Ax+By+Cz+D=0 tekislikni ko'ramiz. Uni $\frac{x}{-\frac{D}{A}}+\frac{y}{-\frac{D}{B}}+\frac{z}{-\frac{D}{C}}=1$ ko'rinishda yozish mumkin. Agar $a=-\frac{D}{A}$ $b=-\frac{D}{B}$, $c=-\frac{D}{C}$ belgilashlar kiritsak,tekislik tenglamasi:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (3)$$

ko'rinishga keladi. Bu tenglama tekislikning son o'qlaridan ajratgan kesmalari bo'yicha tenglamasidir.

Haqiqatdan, tekislik 0x o'qidan a kesma, 0y o'qidan b kesma va 0z o'qidan c kesma ajratadi. Bu tekislik chizmada uchburchak shaklida ko'rinadi, ular a,b,c lar ishoralariga qarab, 8ta oktantdan birida joylashishi mumkin.

9.4. Uchta nuqtadan o'tgan tekislik tenglamasi

Fazoda $A_1(x_1;y_1;z_1)$, $A_2(x_2;y_2;z_2)$, $A_3(x_3;y_3;z_3)$ nuqtalar bir to'g'ri chiziqda yotmasa, ulardan yagona tekislik o'tishi ma'lum.

A (x ; y ; z) nuqta o'sha tekislik ixtiyoriy nuqtasi bo'lsin.

$$\overrightarrow{A_1A}=(x-x_1;\ y-y_1;\ z-z_1),$$
$$\overrightarrow{A_1A_2}=(x_2-x_1;\ y_2-y_1;\ z_2-z_1)$$
$$\overrightarrow{A_1A_3}=(x_3-x_1;\ y_3-y_1;\ z_3-z_1)$$

vektorlar o'zaro kolleniar bo'lganligi uchun, ularning aralash ko'paytmasi nolga teng, ya'ni $(\overrightarrow{A_1A}x\,\overrightarrow{A_1A_2}).(\overrightarrow{A_1A_3})=0$

Koordinatlar bo'yicha bu shart

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \qquad (4)$$

tenglamani hosil qilib, izlanayotgan tekislik tenglamasidir.

9.5. Tekislikning normal tenglamasi

Tekislikka koordinata boshidan tushirilgan normal vektor uzunligi p, yo'naltiruvchi kosinuslari cosα, cosβ, cosγ bo'lsin.

Normal bo'yicha yo'nalgan birlik $\vec{n}$ (cosα;cosβ;cosγ) vektorlarni kiritamiz.

Agar C(x ; y ; z) tekislikning ixtiyoriy nuqtasi bo'lsa, $\overrightarrow{0C}$ vektorning normalga proyeksiyasi p bo'ladi.

$\vec{n}.\overrightarrow{0C}=x\cos\alpha+y\cos\beta+z\cos\gamma$ va C nuqta tekislikda yotishi uchun, uning koordinatalari

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0 \qquad (5)$$

tenglamani qanoatlantirishi kerak. Hosil bo'lgan tenglama **tekislikning normal tenglamasi** deyiladi.

Bu tenglama umumuiy Ax+By+Cz+D=0 tenglamadan quyidagicha chiqariladi.

Umumiy tenglama ikkala tomonini normallovchi ko'paytuvchi deb ataluvchi $\mu=\pm\frac{1}{\sqrt{A^2+B^2+C^2}}$ soniga ko'paytiriladi.

$$\pm\frac{A}{\sqrt{A^2+B^2+C^2}}x\pm\frac{B}{\sqrt{A^2+B^2+C^2}}y\pm\frac{C}{\sqrt{A^2+B^2+C^2}}z\pm\frac{D}{\sqrt{A^2+B^2+C^2}}=0$$

Agar bu tenglamadagi to'g'ri kasrlar mos ravishda cosα; cosβ; cosγ va p deb belgilansa, tekislikning normal tenglamasi hosil bo'ladi. Demak, normallanuvchi ko‘paytuvchi ishorasi D ishorasiga qarama-qarshi bo‘lishi kerak ekan.

Umuman, μ-normallovchi ko‘paytuvchi bo‘lsa,

$\mu Ax+\mu By+\mu Cz+\mu D=0$

normal tenglama bo‘ladi. Ya'ni, $(\mu A)^2+(\mu B)^2+(\mu C)^2=1$

Bundan $\mu=\pm\frac{1}{\sqrt{A^2+B^2+C^2}}$ ekanligi kelib chiqadi.

9.6. Fazoda tekislikka doir asosiy masalalar

Ikki tekislik orasidagi burchak

Umumiy tenglamalari bilan berilgan ikki

$$A_1x+B_1y+C_1z+D_1=0 \qquad A_2x+B_2y+C_2z+D_2=0$$

tekislik orasidagi burchak, ularning normal $\overrightarrow{N_1}=(A_1,B_1,C_1)$ va $\overrightarrow{N_2}=(A_2,B_2,C_2)$ vektorlari orasidagi burchakka tengdir.

Demak, ikki tekislik orasidagi φ burchak

$$\cos\varphi=\frac{\overrightarrow{N_1}.\overrightarrow{N_2}}{|\overrightarrow{N_1}|.|\overrightarrow{N_2}|}=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}*\sqrt{A_2^2+B_2^2+C_2^2}}$$

formulasi yordamida topiladi.

Tekisliklar parallellik sharti $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$, perpendikulyarlik sharti esa $\overrightarrow{N_1}.\overrightarrow{N_2}=0$ yoki $A_1A_2+B_1B_2+C_1C_2=0$ bo‘ladi.

9.7. Nuqtadan tekislikkacha bo‘lgan masofa

Normal tenglamasi bilan berilgan xcosα+ycosβ+zcosγ–p=0 tekislik va biror $E(x_o;\, y_o;\, z_0)$ nuqtasi berilgan bo‘lsin. Berilgan tekislikka parallel, $E(x_o;\, y_o;\, z_0)$ dan o‘tuvchi tekislik xcosα+ycosβ+zcosγ–q=0 tenglamaga ega bo‘ladi. Bunda q-koordinata boshidan tekislikkacha bo’lgan masofa – normal uzunligi E dan berilgan tekislikkacha masofa esa $d=|q-p|$ formuladan topiladi, ya'ni $|q=x_0\cos\alpha+y_0\cos\beta+z_0\cos\gamma-p|$.

Agar tekislik umumuiy Ax+By+Cz+D=0 tenglama bilan berilsa,

$d=\frac{|Ax_0+By_0+Cz_0|}{\sqrt{A^2+B^2+C^2}}$ bo‘lishi ravshan.

Ikki parallel tekislik orasidagi masofani topish uchun, ko‘pincha birinchisidan biror nuqta tanlab, bu nuqtadan ikkinchi tekislikkacha masofa hisoblanadi.

Masala. $A(x_0;y_0;z_0)$ nuqtadan o‘tuvchi $\overrightarrow{a_1}(m_1;n_1;r_1)$, $\overrightarrow{a_2}(m_2;n_2;r_2)$ vektorlarga parallel tekislik tenglamasini yozing.

Tekislik normal vektorini $\vec{N}=\overrightarrow{a_1}x\overrightarrow{a_2}$ deyish mumkin.

$$\vec{N}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ m_1 & n_1 & r_1\\ m_2 & n_2 & r_2\end{vmatrix}=\left(\begin{vmatrix}n_1 & r_1\\ n_2 & r_2\end{vmatrix};\ -\begin{vmatrix}m_1 & r_1\\ m_2 & r_2\end{vmatrix};\ \begin{vmatrix}m_1 & n\\ m_2 & n\end{vmatrix}\right)$$

Nuqtasi va normal vektori berilgan tekislik tenglamasidan

$$\begin{vmatrix}n_1 & r_1\\ n_2 & r_2\end{vmatrix}(x-x_0)-\begin{vmatrix}m_1 & r_1\\ m_2 & r_2\end{vmatrix}(y-y_0)+\begin{vmatrix}m_1 & n\\ m_2 & n\end{vmatrix}(z-z_0)=0$$

kelib chiqadi. Uni $\begin{vmatrix}x-x_0 & (y-y_0) & (z-z_0)\\ m_1 & n_1 & r_1\\ m_2 & n_2 & r_2\end{vmatrix}=0$ ko`rinishida yozish mumkin.

Mavzuga doir misol va masalalar

1. Nuqtasi A(2;-1;3), normal vektori $\vec{N}$=(-1;2;-3) bo‘lgan tekislik tenglamasini yozing.
2. Normal vektori koordinata boshidan A(2;-1;-1) ga yo‘nalgan tekislik tenglamasini yozing.
3. A(3;4;-5) nuqtadan o‘tuvchi $\vec{a}$=(3;1;-1), $\vec{b}$=(1;-2;1) vektorlari parallel tekislik tenglamasini yozing.
4. A(2;-1;3), B(1;3;2) nuqtalardan o‘tuvchi $\vec{a}$=(3;1;-4)ga parallel tekislik tenglamasini yozing.
5. A(1;-1;3), B(2;3;-4), C(0;3;-2) nuqtalardan o‘tuvchi tekislik tenglamasini yozing.
6. Quyidagi tekisliklar orasidagi burchakni toping.

a) $x-y\sqrt{2}+z-1=0$, $x+y\sqrt{2}-z+3=0$, b) 3y-z=0, 2y+z =0

c) 6x+3y-2z=0, x+2y+6z-12=0; d) x+2y+2z-3=0, 16x+12y-15z-1=0;

7. Koordinata boshidan o‘tuvchi, 5x-3y+2z-3=0 tekislikka parallel tekislik tenglamasini yozing.
8. A(1;2;-1) nuqtadan o‘tuvchi x-2y+3z+1=0, 2x–y+z-3-0 tekislikka perpendikulyar tekislik tenglamasini yozing.
9. A(1;1;-3), B(2;3;1) nuqtalardan o‘tuvchi 2x-y+3z-4=0 tekislikka perpendikulyar tekislik tenglamasini yozing.
10. x-2y+z-7=0, 2x+y-z+2=0, x-3y+2z-11=0 umumiy bitta nuqtada kesishadi, shu nuqtani toping.
11. 5x-6y+3z+120=0 tekislik son o‘qlaridan ajratilgan piramida hajmini toping.
12. Tekisliklar normal tenglamasini yozing.

a) x-2y-2z-15=0, b) 2x=y-z-9=0, c) a5y-12z+26=0, d) -4x-4y+2z+1=0

13. Berilgan nuqtalardan berilgan tekislikkacha masofani toping.
a) A(-2;-4;3), 2x-y+2z+3-0; b) (2;-1;-1), 16x-12y+15z-4=0,
c) (1;2;-3), 5x-3y+z+4=0
14. Parallel tekisliklar orasidagi masofani hisoblang.
a) x-2y-2z-12=0, x-2y-2z-6=0, b) 2x-3y+6z-14=0, 4x-6y+12z+21=0
c) 30x-32y+24z-75=0, 15x-16y+12z-25=0
15. Kub ikki yog'i 2x-2y+1+1=0, 2x-2y+z+5=0 tekisliklarda bo'lsa, hajmini toping.
16. Oplikatalar o'qida A(1;-2;0) nuqtalardan va 3x-2y+6z-9=0 tekislikdan bir xil uzoqlikdagi nuqtani toping.

10-mavzu. Fazoda to'g'ri chiziq tenglamalari, asosiy masalalar Ikkinchi tartibli sirtlar

10.1. Fazoda to'g'ri chiziq tenglamalari

To'g'ri chiziqning kanonik tenglamasi

Fazoda biror to'g'ri chiziq berilib, uning E$(x_o;\ y_o\ ;z_0)$ nuqtasi va yo'naltiruvchi vektor deb ataluvchi, to'g'ri chiziqqa parallel $\vec{p} = (m,n,r)$ vektor berilgan bo'lsin.
Agar F(x ; y ; z) to'g'ri chiziqning ixtyoriy nuqtasi bo'lsa,
$\overline{EF}=(x-x_0;y-y_0;z-z_0)$ va $\vec{p} = (m,n,r)$ vektorlar parallelligidan

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{r} \qquad (1)$$

tenglamani hosil qilamiz. Bu tenglama to'g'ri chiziqning kanonik tenglamasi deyiladi.

10.2. To'g'ri chiziqning parametrik tenglamasi

Biror t parametr berilgan bo'lsin. (1) dan $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{r}=t$ deb olib, $x-x_0$=mt, $y-y_0$ =nt, $z-z_0=rt$ tenliklarga ega bo'lamiz.

$$\begin{cases} x = x_0 + mt \\ y = y_0 + \text{nt} \\ z = z_0 + rt \end{cases} \qquad (2)$$

tenglamalar to'g'ri chiziq parametrik tenglamasi deyiladi.

10.3. Ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi

E$(x_o;\ y_o\ ;z_0)$, $F(x_1;y_1;z_1)$ nuqtalardan yagona to'g'ri chiziq o'tishi ma'lum.

Yo'naltiruvchi vektor sifatida $\vec{p} = \overline{EF}=(x-x_0;y-y_0;z-z_0)$, berilgan nuqta sifatida E$(x_o;\ B_2\ ;z_0)$ qaralsa, kanonik tenglama

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0} \qquad (3)$$

ko'rinish oladi. Bu berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenlamasidir.

10.4. To‘g‘ri chiziq - tekisliklar kesishmasi sifatida To‘g‘ri chiziqning umumiy tenglamalari

Fazoda umumiy tenglamasi bilan berilgan ikki tekislik o‘zaro parallel bo‘lmasa, biror to‘g‘ri chiziq bo‘yicha kesishadi.

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

tenglamalar sistemasi **to‘g‘ri chiziqning umumiy tenlamasi** deyiladi.

Umumiy tenglamalarni masalalar yechishda qo‘llash noqulay, shu sababli kanonik yoki parametrik tenglamaga o‘tish kerak bo‘ladi.

Tekisliklar normal vektorlari $\overrightarrow{N_1} = (A_1;B_1;C_1)$, $\overrightarrow{N_2} = (A_2;B_2;C_2)$ vektor ko‘paytmasi $\vec{p} = \overrightarrow{N_1}x\overrightarrow{N_2}$ qaralayotgan to‘g‘ri chiziqqa parallel bo‘ladi, demak, bu vektorni yo‘naltiruvchi vektor sifatida olsa bo‘ladi:

$$\vec{p} = \overrightarrow{N_1}*\overrightarrow{N_2} = \begin{vmatrix} \vec{i} & \vec{k} & \vec{j} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, -\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right)$$

Tog‘ri chiziqda yotuvchi biror nuqta topish kerak. Buning uchun, dastlab, umumiy tenglamadagi z lar o‘rniga biror son qo‘yib yuboramiz: $z=z_0$

$$\begin{cases} A_1x + B_1y = (-C_1z_0 - D_1) \\ A_2x + B_2y = (-C_2z_0 - D_2) \end{cases}$$

Kramer formulasiga ko‘ra: $x_0 = \dfrac{\begin{vmatrix} -C_1z_0-D_1 & B_1 \\ -C_2z_0-D_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$, $y_0 = \dfrac{\begin{vmatrix} A_1 & -C_1z_0-D_1 \\ A_2 & -C_2z_0-D_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$

Endi $E(x_o;\ y_o\ ;z_0)$ nuqta umumiy tenglamalarning ikkalovini ham qanoatlantiradi, ya’ni to‘g‘ri chiziqqa tegishli bo‘ladi. To‘g‘ri chiziqning izlangan kanonik tenlamasi $\dfrac{x-x_0}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \dfrac{y-y_0}{-\begin{vmatrix} A_1 & C \\ A_2 & C_2 \end{vmatrix}} = \dfrac{z-z_0}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$ ko‘rinishida bo‘ladi.

10.5. Fazoda to‘g‘ri chiziq tenglamalariga doir asosiy masalalar Ikki to‘g‘ri chiziq orasidagi burchak

Kanonik tenglamalar bilan berilgan ikki $\frac{x-x_1}{m_1} = \frac{y-y_1}{n_1} = \frac{z-z_1}{r_1}$, $\frac{x-x_2}{m_2} = \frac{y-y_2}{n_2} = \frac{z-z_2}{r_2}$

to‘g‘ri chiziq orasidagi burchak, ularning $\overrightarrow{p_1}=(m_1\ ,n_1,r_1)$, $\overrightarrow{p_2}=(m_2\ ,n_2,\ r_2)$ yo‘naltiruvchi vektorlari orasidagi burchakka teng bo‘ladi. Demak, agar φ o‘sha burchak bo‘lsa

$$\cos\varphi = \frac{\overrightarrow{p_1}\cdot\overrightarrow{p_2}}{|\overrightarrow{p_1}\cdot\overrightarrow{p_2}|} = \frac{m_1m_2+n_1n_2+r_1r_2}{\sqrt{m_1^2+n_1^2+r_1^2}\cdot\sqrt{m_2^2+n_2^2+r_2^2}}$$

Bu to‘g‘ri chiziqlar parallel bo‘lishi uchun, yo‘naltiruvchi vektorlari parallel bo‘lishi kerak, ya’ni $\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{r_1}{r_2}$ tengliklar parallellik shartidir.

Shunga o‘xshash to‘g‘ri chiziqlar perpendikulyar bo‘lishi uchun ham $\overrightarrow{p_1}$ va $\overrightarrow{p_2}$ vektorlar perpendikulyar bo‘lishi kerak, $m_1m_2 + n_1n_2 + r_1r_2 = 0$.

10.6. Nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofa

Fazoda biror $C(x_o;\ y_o\ ;z_0)$ nuqta va $\frac{x-x_1}{m_1} = \frac{y-y_1}{n_1} = \frac{z-z_1}{r_1}$ to‘g‘ri chiziq berilgan bo‘lsin, ular orasidagi eng qisqa d masofani topamiz. $\vec{p}=(m_1\ ,n_1,\ r_1)$ yo‘naltiruvchi vektorni E nuqtadan boshlangan deb hisoblash mumkin. $\overrightarrow{EC}=(x_1 - x_o;\ y_1 - y_0;\ z_1 - z_0)$ va $\overrightarrow{p_1}$ vektorlarga qurilgan parallelogram yuzi

$$S = |\overrightarrow{p_1}x\overrightarrow{EC}| = |\overrightarrow{p_1}|.d$$

ekanligidan $d=\dfrac{|\overrightarrow{p_1}x\overrightarrow{EC}|}{|\overrightarrow{p_1}|} = \dfrac{\left\|\begin{matrix} \vec{i} & \vec{k} & \vec{j} \\ m_1 & n_1 & r_1 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \end{matrix}\right\|}{\sqrt{m_1^2+n_1^2+r_1^2}}$,

yoki $d=\dfrac{\sqrt{\begin{vmatrix} n_1 & r_1 \\ y_1-y_0 & z_1-z_0 \end{vmatrix}^2 + \begin{vmatrix} m_1 & r_1 \\ x_1-x_0 & z_1-z_0 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ x_1-x_0 & y_1-y_0 \end{vmatrix}^2}}{\sqrt{m_1^2+n_1^2+r_1^2}}$

10.7. Ayqash to‘g‘ri chiziqlar orasidagi masofa

Fazoda kesishmaydigan, parallel bo‘lmagan ikki to‘g‘ri chiziq **o‘zaro ayqash** deyiladi.

$\frac{x-x_1}{m_1} = \frac{y-y_1}{n_1} = \frac{z-z_1}{r_1}$ va $\frac{x-x_2}{m_2} = \frac{y-y_2}{n_2} = \frac{z-z_2}{r_2}$ ayqash to‘g‘ri chiziqlar orasidagi eng qisqa masofani topish masalasini ko‘ramiz.

To‘g‘ri chiziqlarning $E_1(x_1;y_1;z_1)$, $E_2(x_2;y_2;z_2)$ nuqtalari va $\overrightarrow{p_1}=(m_1\ ,n_1,\ r_1)$, $\overrightarrow{p_2}=(m_2\ ,n_2,\ r_2)$ vektorlari ma’lum.

$\overrightarrow{E_1E_2}=(x_2 - x_1;\ y_2 - y_1;\ z_2 - z_1)$ vektorni olamiz. $\overrightarrow{p_2}$ning boshini E_1 nuqtaga keltiramiz. $\overrightarrow{p_1},\overrightarrow{p_2},\overrightarrow{E_1E_2}$ vektorga qurilgan parallelepipedni qaraymiz. To‘g‘ri chiziqlar umumiy perpendikulyari uzunligi biz qidirayotgan eng qisqa d masofa bo‘lib, u parallelepiped balandligi hamdir. Demak, $V_{par} = S \cdot d$ yoki $V_{par} = |(\overrightarrow{p_1} x \overrightarrow{p_2}) . \overrightarrow{E_1E_2}| = |\overrightarrow{p_1} x \overrightarrow{p_2}|$ d.

Bundan $d = \frac{|(\overrightarrow{p_1} x \overrightarrow{p_2}) . \overrightarrow{E_1E_2}|}{|\overrightarrow{p_1} x \overrightarrow{p_2}|}$, yoki koordinatalar orqali

$$d = \frac{\left\| \begin{matrix} m_1 & n_1 & r_1 \\ m_2 & n_2 & r_2 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \end{matrix} \right\|}{\sqrt{\begin{vmatrix} n_1 & r_1 \\ n_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & r_1 \\ m_2 & r_1 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2}}.$$

10.8. To‘g‘ri chiziq va tekislik orasidagi burchak

Biror $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$ to‘g‘ri chiziq va Ax+By+Cz+D=0 tekislik berilgan bo‘lsin. Ular orasidagi burchak φ bo‘lsa, yo‘naltiruvchi $\vec{p}(m;n;r)$ va normal $\vec{N}(A;B;C)$ vektorlar orasidagi burchak (90^0-φ) bo‘ladi. Vektorlar orasidagi burchak formulasiga ko‘ra

$\cos(90^0 - \varphi) = \sin\varphi = \frac{\vec{p} \cdot \vec{N}}{|\vec{p}| \cdot |\vec{N}|} = \frac{mA+nB+rC}{\sqrt{m^2+n^2+r^2} \cdot \sqrt{A^2+B^2+C^2}}$

To‘g‘ri chiziq va tekislikning parallellik sharti mA+nB+rC=0 bo‘ladi. Perpendikulyarlik sharti esa, aksincha, $\frac{m}{A} = \frac{n}{B} = \frac{r}{C}$ ko‘rinishidir.

10.9. To‘g‘ri chiziq va tekislik kesishish nuqtasi

$$\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + rt \end{cases}$$

parametrik tenglamali to‘g‘ri chiziq va Ax+By+Cz+D=0 tekislik berilib, ular o‘zaro parallel bo‘lmasin. Unda to‘g‘ri chiziq biror nuqtada tekislikni kesadi. Agar o‘sha kesishish nuqtasi $Q(x_1; y_1; z_1)$ bo‘lsa, uning koordinatalari to‘g‘ri chiziq tenglamasini ham, tekislik tenglamasini ham qanoatlantiradi.

$\begin{cases} x_1 = x_0 + mt \\ y_1 = y_0 + nt \\ z_1 = z_0 + rt \end{cases}$ larni tekislik tenglamasiga qo‘yamiz:

$A(x_0 + mt) + B(y_0 + nt) + C(z_0 + rt) + D = 0$

Hosil bo‘lgan tenglamada faqatgina parametr t noma’lum bo‘lib, uni topish mumkin (faqatgina to‘g‘ri chiziq va tekislik parallel bo‘lgan hol bundan mustasno). Topilgan t ni o‘rniga qo‘yib, $Q(x_1; y_1; z_1)$ nuqta topiladi.

Masalalar. 1) A(2;-4;-1) va $\begin{cases} 3x + 4y + 5z - 26 = 0 \\ 3x - 3y - 2z - 5 = 0 \end{cases}$ to‘g‘ri chiziqning 5x+3y-4z+11=0, 5x+3y-4z-41=0 tekisliklar orasidagi kesmasi o‘rtasidan o‘tuvchi to‘g‘ri chiziq kanonik tenglamasini toping.

Dastlab, $\begin{cases} 3x + 4y + 5z - 26 = 0 \\ 3x - 3y - 2z - 5 = 0 \end{cases}$ to‘g‘ri chiziq parametrik tenglamasini topish kerak.

$z_0 = 2$ desak, $\begin{cases} 3x + 4y = 16 \\ 3x - 3y = 9 \end{cases}$ hosil bo‘ladi. Undan $y_0 = 1$; $x_0 = 4$.

Demak, B(4;1;3) nuqta shu to‘g‘ri chiziqda yotadi.

To‘g‘ri chiziqning yo‘naltiruvchi vektori

$\vec{p} = \overrightarrow{N_1} x \overrightarrow{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 3 & -3 & -2 \end{vmatrix}$=(7;21;-21) yoki $\vec{p}(1;3;-3)$ deyish mumkin.

Bu to‘g‘ri chiziq $\frac{x-4}{1} = \frac{y-1}{3} = \frac{z-2}{-3}$ yoki $\begin{cases} x = 4 + t \\ y = 1 + 3t \\ z = 2 - 3t \end{cases}$ tenglamalarga ega.

Uning parallel tekisliklar bilan kesishgan nuqtalarini topamiz;

a) 5(4+t)+3(1+3t)–4(2-3t)+11=0 dan t=-1 va kesishish nuqtasi $C_1(3;-2;5)$ bo‘ladi.

b) 5(4+t)+3(1+3t)–4(2-3t)-41=0 dan t=1 va kesishish nuqtasi $C_2(5;4;-1)$ bo‘ladi. Bu nuqtalar o‘rtasi C(4;1;2) dir.

A(2;-4;-1), va C(4;1;2), nuqtalardan o‘tuvchi to‘g‘ri chiziq,

$$\frac{x-2}{2} = \frac{y+4}{5} = \frac{z+1}{3}$$

tenglamaga ega.

2) D(1;-4;-3) nuqtadan A(2;1;4), B(1;-2;3),C(0;2;-1) nuqtalardan o‘tuvchi tekislikkacha va $\overrightarrow{AB}$ to‘g‘ri chiziqqacha bo‘lgan masofani toping.

ABC tekislik tenglamasi

$\begin{vmatrix} x - 2 & y - 1 & z - 4 \\ -1 & -3 & -1 \\ -2 & 1 & -5 \end{vmatrix}$=0 yoki 16x-3y-7z-1=0 bo‘ladi.

AB to‘g‘ri chiziq tenglamasi $\frac{x-2}{-1} = \frac{y-1}{-3} = \frac{z-4}{-3}$ dir.

D nuqtadan ABC tekislikkacha masofa $d = \frac{|16 \cdot 1 - 3 \cdot (-4) - 7 \cdot (-3) - 1|}{\sqrt{16^2 + (-3)^2 + (-7)^2}} = \frac{48}{\sqrt{314}}$.

$\overrightarrow{AD}$(-1;-5-7), $\vec{p}$(-1;-3;-1) bo'lgani uchun, D nuqtadan AB to'g'ri chiziqqacha bo'lgan masofa

$$d=\frac{\left|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & -3 & -1\\ -1 & -5 & -7\end{vmatrix}\right|}{|\vec{p}|}=\frac{\sqrt{\begin{vmatrix}-3 & -1\\ -5 & -7\end{vmatrix}^2+\begin{vmatrix}-1 & -1\\ -1 & -7\end{vmatrix}^2+\begin{vmatrix}-1 & -3\\ -1 & -5\end{vmatrix}^2}}{\sqrt{(-1)^2+(-3)^2+(-2)^2}}=\frac{\sqrt{17^2+6^2+2^2}}{\sqrt{1+9+1}}=\sqrt{\frac{329}{11}}.$$

3) $\frac{x+5}{3}=\frac{y+5}{2}=\frac{z-1}{-2}$ va $\begin{cases} x=6t+9\\ y=-2t\\ z=-t+2\end{cases}$ orasidagi eng qisqa masofani toping.

Bu ikki to'g'ri chiziq berilgan nuqtalari $E_1(-5;-5;1)$ va $E_2(9;0;2)$, yo'naltiruvchi vektorlari esa mos ravishda $\overrightarrow{P_1}(3;2;-2)$ va $\overrightarrow{P_2}(6;-2;-1)$ dir.

Demak, $d=\frac{|(P_1P_2)\overrightarrow{E_1E_2}|}{\overrightarrow{P_1P_2}}=\frac{\begin{vmatrix}3 & 2 & -2\\ 6 & -2 & -1\\ 14 & 5 & 1\end{vmatrix}}{\sqrt{\begin{vmatrix}2 & -2\\ -2 & -1\end{vmatrix}^2+\begin{vmatrix}3 & -2\\ 6 & -1\end{vmatrix}^2+\begin{vmatrix}3 & 2\\ 6 & -2\end{vmatrix}^2}}=\frac{147}{21}=7.$

Mavzuga doir masalalar

1. A(1;2;-3) nuqtadan o'tuvchi $\vec{a}$=(2;-3;4)ga parallel to'g'ri chiziq tenglamasini yozing.

2. $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z}{1}$ va 2x-3y+z+1=0 kesishish nuqtasini toping.

3. A(1;-2;3) nuqtadan $\begin{cases}2x+y-z-3=0\\ x+y+z-1=0\end{cases}$ gacha masofani hisoblang.

4. A(2;-3;4) dan B(1;1;0) va C(-2;1;3) lardan o'tuvchi to'g'ri chizqqacha bo'lgan masofani toping.

5. $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{-1}$ va $\begin{cases}2x-y+z-5=0\\ x+y-z+1=0\end{cases}$ orasidagi burchakni va eng qisqa masofani toping.

6. $\{x=1+2t;\ y=-2+3t;\ z=1-6t\}$ va $\begin{cases}2x+y-4z+2=0\\ 4x-y-5z+4=0\end{cases}$ to'g'ri chiziqlar perpendikulyarligini isbotlang.

7. $\frac{x+2}{3}=\frac{y-1}{-2}=\frac{z}{1}$va $\begin{cases}x+y-z=0\\ x-y-5z-8=0\end{cases}$ to'g'ri chiziqlar parallel ekanligini isbotlang.

8. $\begin{cases}5x-3y+2z-5=0\\ 2x-y-z-1=0\end{cases}$ to'g'ri chiziq 4x-3y+7z-7=0 tekislikda yotishini isbotlang.

9. $\begin{cases}2x+2y-z-10=0\\ x-y-z-22=0\end{cases}$ va $\frac{x+7}{3}=\frac{y-5}{-1}=\frac{z-5}{4}$ to'g'ri chiziqlar parallelligini ko'rsating, ular orasidagi masofani hisoblang.

10.10. Ikkinchi tartibli sirtlar

Ikkinchi tartibli sirtlar va ularni kanonik ko'rinishga keltirish

Fazoda $Ax^2+By^2+Cz^2$+2Dxy+2Exz+2Fyz+2Gx+2Hy+2Iz+K=0 (1) tenglama bilan aniqlanadigan nuqtalar to'plami ikkinchi tartibli sirt (ITS)deyiladi.

Teorema: Son o'qlarini parallel ko'chirish, Eyler burchaklari yordamida burish orqali (1) tenglama quyidagi holatlardan biriga keltiriladi:

I. Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, xususan, a=b=c holda $x^2+y^2+z^2=a^2$ sfera tenglamasi hosil bo'ladi.

II. Bir pallali giperboloid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

III. Ikki pallali giperboloid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$.

IV. Konus: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

V. Elliptik paraboloid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$.

VI. Giperbolik paraboloid: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$.

VII. Elliptik silindr: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

VIII. Giperbolik silindr: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

IX. Parabolik silindr: $y^2=2px$

X. Ikki kesishuvchi tekislik: $y^2-k^2x^2=0$

XI. Ikki parallel tekislik: $y^2-k^2=0$

XII Bitta tekislik: $y^2=0$

XIII. To'g'ri chiziq: $x^2+y^2=0$

XIV. Nuqta: $x^2+y^2+z^2=0$

XV. Bo'sh to'plam: $x^2+y^2+z^2=-1$

Isboti: ITCh dagi kabi, koordinata boshini shunday P(a;b;c) nuqtaga

$$\begin{cases}x=x'+a\\ y=y'+b\\ z=z'+c\end{cases}$$

yordamida ko'chiramizki, ya'ni $0'x'y'z'$ sistemada ITS tenlamasida x',y',z' qatnashadigan hadlar bo'lmasin. Buning uchun P(a;b;c) nuqta koordinatalari

$$\begin{cases} aA + bD + cE + G = 0 \\ bB + aD + cF + H = 0 \\ cC + aE + bF + I = 0 \end{cases}$$

sistema yechimlari bo'lishi kifoya. Yangi sistemada ITS $Ax'^2+By'^2+Cz'^2+2Dx'y'+2Ex'z'+2Fy'z'+K'=0$ ko'rinishga keladi.

Endi, Eyler burchaklari yordamida son o'qlarini α,β,γ burchaklarga burib ko'paytmalar qatnashgan hadlar ketma-ket yo'qotiladi.

Masalan, $Ctg2\alpha=\frac{A-B}{2D}$ shartdan α topiladi, ITS tenglamasi yuqoridagi hollardan biriga keladi.

Masalalar. 1) $F_1(0;-5;0)$, $F_2(0;5;0)$ nuqtalargacha masofalari ayirmasi o'zgarmas 6 soniga teng nuqtalar to'plami tenglamasini yozing.

Shartga ko`ra $\sqrt{x^2+(y+5)^2+z^2}=\sqrt{x^2+(y-5)^2+z^2}+6$

Tomonlarni kvadratga ko'tarib, soddalashtirsak,

$$20y-36=12\sqrt{x^2+(y-5)^2+z^2} \text{ yoki}$$

$$5y-9=3\sqrt{x^2+(y-5)^2+z^2}$$

yana kvadratga oshirib, soddalashtiramiz;

$25y^2-90y+81=9(x^2+y^2-10y+25+z^2)$ yoki $9x^2-9y^2+9z^2=-144$

Demak, $\frac{x^2}{16}-\frac{y^2}{9}+\frac{z^2}{16}=-1$ (ikki pallali giperboloid)

2. $x^2+y^2-z^2-2x+4y-4z-4=0$ sirtni kanonik ko'rinishga keltiring.

Chiziqli 2Gx+2Hy+2Iz hadlar yo'qotilishi uchun dastlab

$$\begin{cases} aA + bD + cE + G = 0 \\ bB + aD + cF + H = 0 \\ cC + aE + bF + I = 0 \end{cases} \text{ shartdan } \begin{cases} a-1=0 \\ b+2=0 \\ -2c+2=0 \end{cases} \text{ larni olamiz.}$$

Demak, koordinata boshini P(1;-2;-1) nuqtaga parallel ko'chirish kerak. $\{x=x'+1;\ y=y'-2;\ z=z'-1\}$ almashtirish o'tkazamiz.

$(x'+1)^2+(y'-2)^2-2(z'-1)^2-2(x'+1)+4(y'-2)-4(z'-1)-4=0$

Soddalashtirib, $x'^2+y'^2-2z'^2-7=0$ tenglamaga ega bo'lamiz. Berilgan UTC bir pallali giperboloid ekan.

3. $x^2+yz-5x=1$ sirtni kanonik ko'rinishga keltiriling.

Dastlab, Ox o'qi atrofida y 0 z tekislikni biror γ burchakka buramizki, yangi sistemada yz ko'paytma qatnashmasin.

$Ctg2\gamma=\frac{0-0}{1}=0$ shartdan $2\gamma=90^0$ yoki $\gamma=45^0$. Demak, almashtirish

$$\begin{cases} x = x' \\ y=\frac{\sqrt{2}}{2}(y'-z') \\ z=\frac{\sqrt{2}}{2}(y'+z') \end{cases}$$

bo'ladi. O'rniga qo'yib, $x'^2+\frac{1}{2}(y'^2-z'^2)-5x'=1$.

Bu tenglama $(x'-\frac{5}{2})^2+\frac{1}{2}y'^2-\frac{1}{2}z'^2=\frac{29}{4}$; ko'rinishga keltiriladi. Endi $x=x'-\frac{5}{2}$; $y=y'$, $z=z'$ yordamida parallel ko'chirish o'tkazsak, tenglama $\frac{x^2}{\frac{29}{4}}+\frac{y^2}{\frac{29}{2}}-\frac{z^2}{\frac{29}{2}}=1$ ko'rinishga keladi. Bu UTC ham bir pallali giperboloid ekan.

Mavzuga doir masalalar

1. Quyidagi shartlarda sfera tenglamasini toping.
a) Koordinata boshidan o'tadi, markazi A(4;-4;-2) nuqtada;
b) A(2;-1;-3), nuqtadan o'tadi, markazi B(3;-2;1) nuqtada;
c) A(2;-3;5), B(4;1;-3) nuqtalar biror diametr oxirlaridir;
d) A(1;-2-1), B(-5;10;-1), C(4;1;11), D(-8;-2;2) nuqtalardan o'tadi.
2. Sfera markazi koordinatalari va radiusini toping.
a) $x^2+y^2+z^2+4x-2y+2z-19=0$
b) $x^2+y^2+z^2-6z=0$
3. A(9;-4;-3) nuqtadan $x^2+y^2+z^2+14x-16y-24z+241=0$ sferagacha bo'lgan eng qisqa (uzun) masofani toping.
4. Sirt va to'g'ri chiziq kesishishi nuqtalarini toping.
a) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$ va $\frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4}$,
b) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{4}=1$ va $\frac{x}{4}=\frac{y}{-3}=\frac{z+2}{4}$,
c) $\frac{x^2}{9}-\frac{y^2}{4}=z$ va $\frac{x}{3}=\frac{y-2}{-2}=\frac{z+1}{2}$.
5. ITS larni kanonik ko'rinishga keltiring.
a) $x^2+2y^2-3z^2-4x-8y+12z-10=0$
b) $(x-y)^2+(x+y)^2-z=0$
c) $xy+yz+zx=1$
d) $xy+yx+zx=0$
e) $10x^2-2xy+y^2+z^2+2z=99$
f) $2xy-z^2+x-y=100$
6. Berilgan $F_1(-a;0;0)$, $F_2(a;0;0)$ gacha masofalar kvadratlari yig'indisi $4a^2$ bo'lgan nuqtalar tenglamasini yozing.
7. $F_1(0;-5;0)$, $F_2(0;5;0)$ nuqtalargacha masofalari ayirmasi 6 bo'ladigan nuqtalar tenglamasini yozing.

Fazoda analitik geometriyaga doir joriy nazorat uchun uy vazifasi

I. Fazoda piramida uchlari A(N;-1;-5), B(-N;3;4), C(1;1;-N), D(20-N;10-N;1), bo‘lsa, quyidagilarni toping.

a) AB, AC qirralar uzunligu va ular orasidagi burchak;

b) ABC va ADC tomonlar tenglamalari va ular orasidagi burchak;

c) AD qirra va BCD tomon mumkin bo‘lgan barcha tenglamasi va ular orasidagi burchak;

d) ABCD piramida hajmi va to‘la sirti;

e) D uchidan tushirilgan balandlik tenglamasi va uzunligi, AB tomonga tushirilgan apofema tenglamasi va uzunligi.

f) D uchidan tushirilgan balandlik va ABC tomonning kesishish nuqtasi koordinatalari.

g) AB va CD to‘g‘ri chiziqlar orasidagi eng qisqa masofa.

h) D uchidan o‘tib, AB qirraga parallel (perpendikulyar) to‘g‘ri chiziq tenglamasi.

i) D uchidan o‘tib, ABC tomonga parallel tekislik tenglamasi (perpendikulyar tekisliklardan birortasi tenglamasi).

II. ITS ni kanonik ko‘rinisga keltiring.

a) $Nx^2+(-1)^N y^2+(-1)^{N-1} z^2+(-1)^n 2NX+(-10^{n+1} 4Ny+(-1)^n 4Nz-N=0$

b) $Nxy-(-1)^N y^2+(-1)^{N+1} xz-N=0$

c) $z=Nxy+2Nx+(-1)^N Ny$.

Chiziqli algebra

11-mavzu. Chiziqli algebra (matritsaviy analiz) elementlari

11.1. Vektor fazo. O‘lcham va bazis. Yangi bazisga o‘tish

Tartib bilan yozilgan n ta haqiqiy son sistemasi, yani $\vec{a_i}-(x_1;x_2;x_3...x_n)$ **n-o‘lchamli vektor** deyiladi. Bunda $x_1,x_2,x_3...x_n$ sonlar vektorning koordinatalari deyiladi.

Agar ikki n-o‘lchamli $\vec{x},\vec{y}$ vektorning mos koordinatalari teng; $\vec{x_i}=\vec{y_i}(i=\overline{1,n})$ bo‘lsa, ular teng deyiladi. n-o‘lchamli vektorlar ustida ammallar avvalgidek kiritiladi;

$$\vec{x}\pm\vec{y}=(x_1\pm y_1;x_2\pm y_2;...,x_n\pm y_n),\ \lambda\vec{x}=(\lambda x_1;\lambda x_2;\lambda x_3...\lambda x_n),\ \lambda\epsilon R.$$

Bu chiziqli amallar quyidagi xossalarga ega;

1. $\vec{x}\pm\vec{y}=\vec{y}\pm\vec{x}$ (kommutatuvlik)
2. $(\vec{x}\pm\vec{y})+\vec{z}=\vec{x}+(\vec{y}\pm\vec{z})$ (assotsiativlik)
3. $\alpha(\beta\vec{x})=(\alpha\beta)\vec{x}$
4. $\alpha(\vec{x}+\vec{y})=\alpha\vec{x}+\alpha\vec{y}$
5. $(\alpha+\beta)\vec{x}=\alpha\vec{x}+\alpha\vec{y}$
6. *Shunday* $\vec{0}$=(0;0;...0) vektor mavjudki, $\vec{x}+\vec{o}=\vec{x}$ o‘rinli
7. Ixtiyoriy $\vec{x}$ uchun qarama-qarshi vektor mavjudki, uni $-\vec{x}$ ko‘rinishda belgilaymiz,

   $\vec{x}+(\overline{-\vec{x}})$=0 bo‘ladi.

8. Ixtiyoriy $\vec{x}$ *uchun* $1.\vec{x}=\vec{x}$

Ta’rif: Qo‘shish va songa ko‘paytirish amallari yuqoridagi xossalarga bo‘ysunganda, haqiyqiy koordinatali vektorlar to‘plami **vektor fazo** deyiladi.

Misollar.

1) R, R^2,…, R^n,C vektor fazo bo‘ladi.

2) darajasi n ga teng ko‘phadlar to‘plami $\{P_n(x)\}$ vektor fazo hosil qilmaydi, chunki ikki n-darajali ko‘phad yig‘indisi n-darajali bo‘lmasligi mumkin.

3) n satrli va m ustinli matritsa n, m o‘lchamli vektor sifatida qaralishi mumkin, buning uchun matritsa elementlarini satrma-satr o‘qib chiqish yetarli.

Elementlari funksiyalar yoki sonli ketma-ketliklar bo‘lgan vektor fazolar **funksional fazolar** deyiladi.

Agar $\overrightarrow{a_n}=\lambda_1\overrightarrow{a_1}+\lambda_2\overrightarrow{a_2}+\lambda_{1-1}\overrightarrow{a_{n-1}}$, $\lambda_i \epsilon R$ bo'lsa $\overrightarrow{a_n}$ vektor $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots \overrightarrow{a_{n-1}}$ **vektorlar chiziqli kombinatsiyasi** deyiladi.

Ta'rif: Bir paytda nol bo'lmagan $\lambda_1, \lambda_2, \dots \lambda_{n-1}, \lambda_n$ sonlari mavjud bo'lib, $\lambda_1\overrightarrow{a_1}+\lambda_2\overrightarrow{a_2}+\cdots\lambda_n\overrightarrow{a_n}=0$ o'rinli bo'lsa, $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots \overrightarrow{a_n}$ vektorlar chiziqli bog'liq deyiladi. Aks holda, ya'ni barcha $\lambda_i \epsilon R$lar nol bo'lgandagina $\lambda_1\overrightarrow{a_1}+\lambda_2\overrightarrow{a_2}+\cdots\lambda_n\overrightarrow{a_n}=0$ bo'lsa, bu vektorlar chiziqli erkin deyiladi.

Agar vektorlar chiziqli bog'liq bo'lsalar, ulardan kamida bittasi qolganlarining chiziqli konbinatsiyasi bo'ladi, yoki bir vektor qolganlari chiziqli konbinatsiyasi bo'lsa, ular chiziqli bog'liqdirlar.

1) Agar $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots \overrightarrow{a_n}$ vektorlarning biri nol vektor bo'lsa, ular chiziqli bog'liqdir, chunki masalan; $\overrightarrow{a_i}=0$ bo'lsa, $\lambda_1=\lambda_2=\cdots=\lambda_m=0$ faqatgina, $\lambda_i \neq 0$.

2) Agar $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots \overrightarrow{a_n}$ larning bir qismi chiziqli bog'liq bo'lsa ularning hammasi chiziqli bog'liqdir.

Misol. $\overrightarrow{a_1}=(1;3;1;3)$, $\overrightarrow{a_2}=(2;1;1;2)$, $\overrightarrow{a_3}=(3;-1;1;1)$ chiziqli bog'liqlikka tekshirilsin.

$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$ tenglik qachon bajarilishini tekshiramiz;

$\lambda_1\begin{pmatrix}1\\2\\1\\3\end{pmatrix}+\lambda_2\begin{pmatrix}2\\1\\1\\2\end{pmatrix}+\lambda_3\begin{pmatrix}3\\-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ yoki $\begin{cases}\lambda_1+2\lambda_2+3\lambda_3=0\\3\lambda_1+\lambda_2-\lambda_3=0\\\lambda_1+\lambda_2+\lambda_3=0\\3\lambda_1+2\lambda_2+\lambda_3=0\end{cases}$. Bu sistemani Gaus usilida quyidagi ko'rinishga keltirish mumkin

$\begin{cases}\lambda_1+2\lambda_2+3\lambda_3=0\\\lambda_2+2\lambda_3=0\\0=0\\0=0\end{cases}$. Agar $\lambda_3 = C$ desak, λ_2=-2C va λ_1=C bo'ladi.

(C;-2C;C) ko'rinishdagi nuqtalar to'plamining cheksiz ko'p yechimi mavjud. Ulardan biri, masalan, (1;-2;1) dir. $1\overrightarrow{a_1}-2\overrightarrow{a_2}+\overrightarrow{a_3}=0$ *b*ajarilishi bu uch vektorning chiziqli bog'liqligini bildiradi.

Agar chiziqli fazoda n ta chiziqli erkli vector mavjud bo'lib, ixtiyoriy (n+1) tasi chiziqli bog'liq bo'lsa, bu fazo n o'lchamli deyiladi, boshqacha aytganda, fazo o'lchami undagi chiziqli erkli vektorlar maksimal sonidir.

Biror V fazo o'lchami dim V tarzida belgilanadi. n o'lchamli V fazodagi n ta chiziqli erkli vektorlar to'plami **bazis** deyiladi.

Masalan, R da $\vec{n}$ (1), R^2 da $\overrightarrow{a_1}(1;0)$, $\overrightarrow{a_2}$ (0;1), R^3 da esa (1;0;0), (0;1;0),(0;0;1) lar bazis bo'ladi.

Teorema. V vektor fazo ixtiyoriy elementi yagona ko'rinishida bazis elementlari chiziqli kombinatsiyasi sifatida yoziladi.

Isboti. Agar $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ vektorlar V da bazis bo'lsa, ixtiyoriy $\vec{x} \epsilon V$ uchun $\lambda_1\overrightarrow{l_1}+\lambda_2\overrightarrow{l_2}+\cdots+\lambda_n\overrightarrow{l_n}+\lambda\vec{x}=0$, $\lambda=0$.

Demak, $\vec{x}=-\frac{\lambda_1}{\lambda}\overrightarrow{l_1}-\frac{\lambda_2}{\lambda}\overrightarrow{l_2}-\dots-\frac{\lambda_n}{\lambda}\overrightarrow{l_n}$, yoki $\vec{x}=x_1\overrightarrow{l_1}+x_2\overrightarrow{l_2}+\cdots+x_n\overrightarrow{l_n}=0$.

Oxirgi tenglik $\vec{x}$ vektorning $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ bazis bo'yicha yoyilmasi, $x_1, x_2, \dots, x_n$ esa $\vec{x}$ vektorining shu bazisga nisbatan koordinatalari deyiladi.

Misol. $\overrightarrow{l_1}, \overrightarrow{l_2}, \overrightarrow{l_3}$ bazisda $\overrightarrow{a_1}(1;1;0)$, $\overrightarrow{a_2}(1;-1;1)$, $\overrightarrow{a_3}=(-3;5;-6)$ vektorlar bazis tashkil etilishini isbotlang.

$$\lambda_1\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda_2\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda_3\begin{pmatrix}-3\\5\\-6\end{pmatrix}=0, \begin{cases}\lambda_1+\lambda_2-3\lambda_3=0\\\lambda_1-\lambda_2+5\lambda_3=0\\\lambda_2-6\lambda_3=0\end{cases}$$

Sistemada elementar almashtirishlarni bajarib, $\begin{cases}\lambda_1+\lambda_2-3\lambda_3=0\\-2\lambda_2+8\lambda_3=0\\-8\lambda_3=0\end{cases}$

ko'rinishga keltiramiz. Demak, trivial (0;0;0) yechim bor xolos. Demak, $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}$ vektorlar chiziqli erkli va bazis hosil qiladi.

Endi shu $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}$ bazisdagi $\vec{a}(1;1;4)$, vektor koordinatalarini topamiz, $\lambda_1\overrightarrow{a_1}+\lambda_2\overrightarrow{a_2}+\lambda_3\overrightarrow{a_3}=\vec{a}$,

$$\begin{cases}\lambda_1+\lambda_2-3\lambda_3=1\\\lambda_1-\lambda_2+5\lambda_3=1\\\lambda_2-6\lambda_3=-4\end{cases}$$

Bu sistemani $\begin{cases}\lambda_1+\lambda_2-3\lambda_3=1\\-2\lambda_2+8\lambda_3=0\\-4\lambda_3=-8\end{cases}$ ko'rinishida yozish mumkin.

Demak, $\lambda_3=2$; $\lambda_2=8$; $\lambda_1=-1$, $\vec{a}==-1\overrightarrow{a_1}+8\overrightarrow{a_2}+2\overrightarrow{a_3}$.

V fazoda eski $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ va yangi $\overrightarrow{f_1}, \overrightarrow{f_2}, \dots, \overrightarrow{f_n}$ bazislar berilgan bo'lsa, ularning yangilarini eskilari orqali

$$\begin{cases}\overrightarrow{f_1}=\overrightarrow{a_{11}}\overrightarrow{l_1}+\overrightarrow{a_{12}}\overrightarrow{l_2}+\cdots+\overrightarrow{a_{1n}}\overrightarrow{l_n}\\\overrightarrow{f_2}=\overrightarrow{a_{21}}\overrightarrow{l_1}+\overrightarrow{a_{22}}\overrightarrow{l_2}+\cdots+\overrightarrow{a_{1n}}\overrightarrow{l_n}\\\cdot\quad\cdot\quad\cdot\\\overrightarrow{f_n}=\overrightarrow{a_{n1}}\overrightarrow{l_1}+\overrightarrow{a_{n2}}\overrightarrow{l_2}+\cdots+\overrightarrow{a_{nn}}\overrightarrow{l_n}\end{cases} \quad \text{yoki} \quad \begin{pmatrix}\overrightarrow{f_1}\\\overrightarrow{f_2}\\\vdots\\\overrightarrow{f_n}\end{pmatrix}=\begin{pmatrix}a_{11},a_{12},\dots,a_{1n}\\a_{21},a_{22},\dots,a_{2n}\\\cdots\\a_{n1},a_{n2},\dots,a_{nn}\end{pmatrix}*\begin{pmatrix}\overrightarrow{l_1}\\\overrightarrow{l_2}\\\vdots\\\overrightarrow{l_n}\end{pmatrix}$$

demak, eski $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ bazisdan yangi $\overrightarrow{f_1}, \overrightarrow{f_2}, \dots, \overrightarrow{f_n}$ bazisga o'tish $A=(a_{ij})_{\substack{i=1,n\\j=1,n}}$ o'tish matritsasi orqali amalga oshiriladi. Yangi bazisdan eskisiga o'tish esa A^{-1} teskari matritsa orqali amalga oshiriladi. O'tish matritsasi xos emas, $|A| \neq 0$, shuning uchun yuqoridagilar amalga oshirilishi mumkin.

Berilgan $\vec{x}=(x_1, x_2,\dots,x_n)$ vektor koordinatalarining turli bazislarda o'zaro bog'lanishini ko'ramiz. Yangi koordinatalarda $\vec{x}=(x_1^*,x_2^*,\dots,x_n^*)$ bo'lsin, $\vec{x}= x_1\vec{l_1}+ x_2\vec{l_2}+\dots+ x_n\vec{l_n}=x_1^*\vec{f_1}+x_2^*\vec{f_2}+\dots+x_n^*\vec{f_n}$.

$$x_1\vec{l_1}+ x_2\vec{l_2}+\dots+ x_n\vec{l_n}=x_1^*\begin{pmatrix} a_{11}\\ a_{12}\\ \vdots\\ a_{1n}\end{pmatrix}\vec{l}+x_2^*\begin{pmatrix} a_{21}\\ a_{22}\\ \vdots\\ a_{2n}\end{pmatrix}\vec{l_2}+\dots+x_n^*\begin{pmatrix} a_{n1}\\ a_{n2}\\ \vdots\\ a_{nn}\end{pmatrix}\vec{l_n},$$

demak koordinatalar bog'lanishi

$$\begin{cases} x_1 = a_{11}x_1^* + a_{21}x_2^* + \dots + a_{n1}x_n^* \\ x_2 = a_{12}x_1^* + a_{22}x_2^* + \dots + a_{n2}x_n^* \\ \dots\dots\dots\dots\dots \\ x_n = a_{1n}x_1^* + a_{2n}x_2^* + \dots + a_{nn}x_n^* \end{cases}$$

Matritsaviy ko'rinisda esa, $\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}=A'\begin{pmatrix} x_1^*\\ x_2^*\\ \vdots\\ x_n^*\end{pmatrix}$ yoki $\begin{pmatrix} x_1^*\\ x_2^*\\ \vdots\\ x_n^*\end{pmatrix}=(A^{-1})'\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}$.

Bunda $A',(A^{-1})'$lar A va A^{-1}ning transponirlanganidir.

Misol. $\vec{a}=(1;1;0)$, $\vec{a}=(1;-1;1)$, $\vec{x}=(-3;5;6)$ lar bazis tashkil etishi ko'rsatilgan. Endi $\vec{b}=(4;-4;5)$ vektor $\vec{l_1},\vec{l_2},\vec{l_3}$ basiz bilan berilgan bo'lsa, $\vec{a_1},\vec{a_2},\vec{a_3}$ bazis orqali yozing.

$$\begin{cases} \vec{a_1}=\vec{l_1}+\vec{l_2} \\ \vec{a_2}=\vec{l_1}-\vec{l_2}+\vec{l_3} \\ \vec{a_2}=-3\vec{l_1}+5\vec{l_2}-6\vec{l_3} \end{cases}$$ ko'rinisda bo'ladi. O'tish matritsasi

$A=\begin{pmatrix} 1 & 1 & 0\\ 1 & -1 & 1\\ -3 & 5 & -5\end{pmatrix}$ ko'rinishida bo'lib, $|A|=4$. Algebraik to'ldiruvchilar

$$A_{11}=1 \quad A_{21}=6 \quad A_{31}=1$$
$$A_{12}=3 \quad A_{22}=-6 \quad A_{32}=-1$$
$$A_{13}=2 \quad A_{23}=-8 \quad A_{33}=-2$$

bo'lganligi uchun $A^{-1}=\frac{1}{4}\begin{pmatrix} 1 & 6 & 1\\ 3 & -6 & -1\\ 2 & -8 & -2\end{pmatrix}$. Uning transponirlangani

$A^{-1}=\frac{1}{4}\begin{pmatrix} 1 & 3 & 2\\ 6 & -6 & -6\\ 1 & -1 & -2\end{pmatrix}$. Demak, $\begin{pmatrix} x_1^*\\ x_2^*\\ x_3^*\end{pmatrix}=\begin{pmatrix} 1 & 3 & 2\\ 6 & -6 & -6\\ 1 & -1 & -2\end{pmatrix}*\begin{pmatrix} 4\\ -4\\ 5\end{pmatrix}=\frac{1}{4}$

$\begin{pmatrix} 2\\ 8\\ -2\end{pmatrix}=\begin{pmatrix} 0.5\\ 2\\ -0.5\end{pmatrix}$, ya'ni $\vec{b}$ vektorning $\vec{a_1},\vec{a_2},\vec{a_3}$ bazisdagi koordinatalari (0.5;2;-0.5) bo'ladi. $\vec{b}=0.5\vec{a_1}+2\vec{a_2}-0.5\vec{a_3}$

11.2. Evklid fazolari. Ortoganallashtirish

Chiziqli (vector) fazolarda vektorlarni qo'shish, songa ko'paytirish amallari kiritiladi xolos. Ular yordamida o'lcham, bazis tushunchalari ko'riladi. Endi bu fazoda burchak, uzunlikni hisoblash uchun metrika kiritamiz. Metrika sifatida, masalan, skalyar ko'paytma kiritish mumkin.

Ta'rif: Ikki $\vec{x}=(x_1;x_2;\dots;x_n)$, $\vec{y}=(y_1;y_2;\dots;y_n)\epsilon R^n$ vektorlar **skalyar** ko'paytmasi deb

$\vec{x}.\vec{y}=(\vec{x};\vec{y})=x_1y_1+x_2y_2+\dots+x_ny_n=\sum_{i=1}^{n}x_iy_i$ songa aytiladi.

Skalyar ko'paytmaning iqtisodiy mazmuni: Agar $\vec{x}=(x_1;x_2;\dots;x_n)$ turli tovarlar hajmini ifodalovchi vektor bo'lsa, $\vec{y}=(y_1;y_2;\dots;y_n)$ tovarlar narxini belgilovchi vektor bo'ladi, $(\vec{x};\vec{y})$ esa barcha tovarlar qiymatini bildiradi.

Skalyar ko'paytma quyidagi xossalarga ega:

1^0. $(\vec{x};\vec{y})=(\vec{y},\vec{x})$, 2^0. $(\vec{x};\vec{y}+\vec{z})=(\vec{x};\vec{y})+(\vec{x};\vec{z})$

3^0. $(\alpha\vec{x};\vec{y})=\alpha(\vec{x};\vec{y})$, $\alpha\epsilon R$. 4^0. $(\vec{x};\vec{x})>0$, $(\vec{x};\vec{x})=0\leftrightarrow\vec{x}=\vec{0}$.

Ta'rif: Yuqoridagi 4ta shartga bo'ysinuvchi vektorlar ustida skalyar ko'paytma aniqlangan chiziqli (vektor) fazo **Evklid fazosi** deyiladi.

Skalyar ko'paytma yordamida $\vec{x}$ vektor uzunligi (normasi)

$|\vec{x}|=\sqrt{(\vec{x};\vec{x})}=\sqrt{x_1^2+x_2^2+\dots+x_n^2}$ ko'rinishida aniqlanadi.

$\vec{x}=(x_1;x_2;\dots;x_n)$ vektor uzunligi (normasi) quyidagi xossalarga ega:

1^0. $|\vec{x}|=0 \leftrightarrow \vec{x}=\vec{0}$.

2^0. $|\lambda\vec{x}|=|\lambda|.|\vec{x}|$, $\lambda\epsilon R$.

3^0. $|(\vec{x};\vec{y})|\leq|\vec{x}|.|\vec{y}|$ (Koshi-Bunyakovskiy tengsizligi)

4^0. $|\vec{x}+\vec{y}|\leq|\vec{x}|+|\vec{y}|$ (uchburchak tengsizligi).

Yuqoridagilar yordamida ikki – $\vec{x},\vec{y}$ vektorlar orasidagi φ burchak quyidagicha topiladi: $\cos\varphi=\frac{(\vec{x};\vec{y})}{|\vec{x}|.|\vec{y}|}=\frac{\sum_{i=1}^{n}x_iy_i}{\sqrt{\sum_{i=0}^{n}x_i^2}\cdot\sqrt{\sum_{i=0}^{n}y_i^2}}$, $0\leq\varphi<\pi$

Uzunligi birga teng $\vec{x}$ vektor **normallangan** deyiladi.

Ta'rif: Evklid fazosidagi nol bo'lmagan ikki $\vec{x},\vec{y}$ vektor **ortogonal** deyiladi, agar $(\vec{x};\vec{y})=0$ bo'lsa, n-o'lchamli Evklid fazosidagi $(\vec{l_1},\vec{l_2},\dots,\vec{l_n})$ bazis **ortonormallangan bazis** deyiladi, agar:

$$(\vec{l_i},\vec{l_j})=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}$$ o'rinli bo'lsa.

Masalan, R^n fazoda $\vec{l_1}=(1;0;\dots;0)$, $\vec{l_2}=(0;1;\dots;0)$, $\vec{l_n}=(0;0;\dots;1)$ elementlar ortonormal basiz hosil qiladi.

Endi isbotsiz asosiy teoremani keltiramiz.

Teorema (Gilbert-Shmidtning ortogonallashtirish protsessi): Agar n-o'lchamli Evklid fazosida ixtiyoriy $(\vec{f_1}, \vec{f_2}, \dots, \vec{f_n})$ basiz berilgan bo'lsa, $\vec{l_1} = \vec{f_1}$, $\vec{l_k} = \vec{f_k} - \sum_{i=1}^{k-1} c_i \vec{e_i}$, k=2,3,…,n, bunda $c_i = \frac{(\vec{f_k}, \vec{l_i})}{(\vec{l_i}, \vec{l_i})}$ vektorlar bu fazoda ortogonal bazis tashkil etadi.

Misol. $\vec{f_1}$=(1;2;2;-1), $\vec{f_2} = (1;1;-5;3)$, $\vec{f_3} = (3;2;8;-7)$ lar ortogonallashtirilsin.

$\vec{l_1} = \vec{f_1}$ =(1;2;2;-1)

$$\vec{l_2} = \vec{f_2} - \frac{(\vec{f_2};\vec{l_1})}{(\vec{l_1};\vec{l_1})}\vec{l_1} = \vec{f_2} - \frac{1+2-10-3}{1+4+4+1}\vec{l_1} = \vec{f_2} + \vec{l_1} = (2;3-3;2)$$

$$\vec{l_3} = \vec{f_3} - \frac{(\vec{f_3};\vec{l_1})}{(\vec{l_1};\vec{l_1})}\vec{l_1} - \frac{(\vec{f_3};\vec{l_2})}{(\vec{l_2};\vec{l_2})}\vec{l_2} = \vec{f_3} - \frac{30}{10}\vec{l_1} - \frac{(-26)}{26}$$

$$\vec{l_2} = \vec{f_3} - 3\vec{l_1} + \vec{l_2} = (2;-1;-1-2).$$

Mavzuga doir misol va masalalar

1. Quyidagilar chiziqli vektor fazo bo'ladimi?
a) tekislikdagi vektorlar to'plami;
b) fazodagi vektorlar to'plami;
c) darajasi (n-1) dan kichik ko'phadlar to'plami;
d) fiksirlangan vektorga kolleniar vektorlar to'plami.
2. 1,1+x, 1+x+x^2, 1+x+x^2+x^3 ko'phadlar sistemasi chiziqli erkli ekanligini isbotlang.
3. Chiziqli bog'liqlikka tekshiring.
a) $\vec{a_1}$=(2;-1;3), $\vec{a_2}$=(1;4;-1), $\vec{a_3}$=(0;-9;5)
b) $\vec{a_1}$=(1;2;0), $\vec{a_2}$=(3;-1;1), $\vec{a_3}$=(0;1;1)
4. $\vec{l_1}, \vec{l_2}, \vec{l_3}$ bazisda berilgan $\vec{a}$=(1;2;0), $\vec{b}$=(3;-1;1), $\vec{c}$=(0;1;1) vektorlarning o'zlari ham bazis tashkil etishini isbotlang.
5. $\vec{l_1}, \vec{l_2}, \vec{l_3}$ bazisda $\vec{a} = \vec{l_1} + \vec{l_2} + \vec{l_3}$, $\vec{b} = 2\vec{l_2} + 3\vec{l_3}$, $\vec{c} = \vec{l_2} + 5\vec{l_3}$ vektorlar berilgan. Ular bazis tashkil etishini isbotlang. $\vec{a}, \vec{b}, \vec{c}$ bazisda $\vec{d} = 2\vec{l_1} - \vec{l_2} + \vec{l_3}$ vektor koordinatalarini toping.
6. Agar $\vec{l_1}, \vec{l_2}, \vec{l_3}, \vec{l_3}, \vec{l_5}$ ortonormal bazis bo'lsa, $\vec{x} = \vec{l_1} - 2\vec{l_2} + \vec{l_5}$, $\vec{y} = 3\vec{l_2} + \vec{l_3} - \vec{l_4} + 2\vec{l_5}$ vektorlar skalyar ko'paytmasi va uzunligini toping.
7. $\vec{f_1}$=(2,1,3;-1), $\vec{f_2}$=(7;4;3;-3), $\vec{f_3}$=(1;1;-3;0), $\vec{f_4}$=(5;7;8), vektorlar sistemasi ortogonallashtirilsin.
8. $\vec{l_1}, \vec{l_2}, \vec{l_3}$ bazisdan $\vec{l_3}, \vec{l_2}, \vec{l_1}$ bazisga o'tishi matritsasini yozing.

12-mavzu. Chiziqli operatorlar. Kvadratik formalar

12.1. Chiziqli operatorlar

n va m o'lchamli R^n, R^m fazolarni qaraymiz.

Tarif: Agar har bir $\vec{x}=(x_1; x_2; \dots x_n) \in R^n$ vektorga biror A qonun yoki qoida yordamida yagona $\vec{y}=(y_1; y_2; \dots y_n) \in R^m$ vektor mos qo'yilsa, bu qonun operator (akslantirish, almashtirish) deyiladi va $\vec{y}=A\overrightarrow{(x)}$ tarzida yoziladi.

A: $R^n \to R^m$ operator , A: $R^n \to R$ funksional, A:R $\to R$ funksiya deyiladi.

Operator chiziqli deyiladi, agar $\vec{x}, \vec{y} \in R^n$, $\lambda \in R^n$ uchun

1) $A(\vec{x} + \vec{y}) = A(\vec{x}) + (\vec{y})$ (additivlik)
2) $A(\lambda \vec{x}) = A\,\vec{x}$, (bir jinslilik)

$\vec{y}=A\overrightarrow{(x)}$ vektor $\vec{x}$ vektor obrazi (tasviri), $\vec{x}$ vektor esa $\vec{y}$ ning proobrazi (asli) deyiladi.

Agar R^n, R^m fazolar ustma-ust tushsa, A operator R^nni o'zini o'ziga akslantiradi. Biz aynan shunday operatorlarni qaraymiz.

R^n fazoda $\vec{l_1}, \vec{l_2} \dots \vec{l_n}$ bazis berilsa, ixtiyoriy $\vec{x} \in R^n$ uchun

$\vec{x} = x_1\vec{l_1} + x_2\vec{l_2} + \dots + x_n\vec{l_n}$.

A operator chiziqliligidan: $A(\vec{x}) = x_1A(\vec{l_1}) + x_2A(\vec{l_2}) + \dots + x_nA(\vec{l_n})$

Lekin $A(\vec{l_i})$ $(i=\overline{1,n}) \in R^n$, ularni ham $\vec{l_1}, \vec{l_2} \dots \vec{l_n}$ bazis bo'yicha yoyish mumkin

$A(\vec{l_i}) = a_{1i}(\vec{l_1}) + a_{2i}(\vec{l_2}) + \dots + a_{ni}(\vec{l_n})$ $(i=\overline{1,n})$. U holda

$A(\vec{x}) = x_1(a_{11}\vec{l_1} + a_{21}\vec{l_2} + \dots + a_{n1}\vec{l_n}) + x_2(a_{12}\vec{l_1} + a_{22}\vec{l_2} + \dots + a_{n2}\vec{l_n}) + \dots$
$x_n(a_{1n}\vec{l_1} + a_{2n}\vec{l_2} + \dots + a_{nn}\vec{l_n}) = (a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n)\vec{l_1} + (a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n)\vec{l_2} + + (a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n)\vec{l_n}$

Boshqa tomondan, $\vec{y}=A\overrightarrow{(x)}$ vektorning $\vec{l_1}, \vec{l_2}, \dots, \vec{l_n}$ bazisdagi y_1, y_2, … y_n koordinatalari $A(\vec{x}) = y_1\vec{l_1} + y_2\vec{l_2} + \dots + y_n\vec{l_n}$; ko'rinishida yoziladi. $\vec{y}$ vektorni $\vec{l_1}, \vec{l_2}, \dots, \vec{l_n}$ bazis bo'yicha yoyilmasi yagona ekanligidan:

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ y_n = a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n; \end{cases}$$

Bundagi $\Delta=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{3n} \\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}$ matritsa A operator matritsasi, Δ ning rangi esa A operator rangi deyiladi.

Umuman, har bir n-tartibli matritsaga n-o'lchamli fazodagi bitta chiziqli operator mos keladi va aksincha.

$\vec{x}=(x_1; x_2; \dots x_n)$ va $\vec{y}=A(\vec{x})=(y_1; y_2; \dots y_n)$ orasidagi bog'liqlik matritsaviy ko'rinishda bo'ladi: $\begin{pmatrix}y_1\\y_2\\y_n\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{3n} \\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_n\end{pmatrix}$.

Misol. R^3 da A operator l_1, l_2, l_3 bazisda

$$\Delta=\begin{pmatrix}3&2&4\\-1&5&6\\1&8&2\end{pmatrix}$$

matritsa bilan berilgan. $\vec{x}=4\vec{l_1}-3\vec{l_2}+\vec{l_3}$ vektor obrazi $\vec{y}=A(\vec{x})$ni toping.

$$\begin{pmatrix}y_1\\y_2\\y_n\end{pmatrix}=\begin{pmatrix}3&2&4\\-1&5&6\\1&8&2\end{pmatrix}\begin{pmatrix}4\\-3\\1\end{pmatrix}=\begin{pmatrix}10\\-13\\-18\end{pmatrix}$$

Demak, $y=10\vec{l_1}-3\vec{l_2}-18\vec{l_3}$.

Chiziqli operatorlar ustida amallar quyidagicha kiritiladi:

1) $(A+B)(\vec{x})=A(\vec{x})+B(\vec{y})$, 2) $(\lambda A)(\vec{x})=\lambda A(\vec{x})$, 3) $(AB)(\vec{x})=A(B(\vec{x}))$

Natijaviy operatorlar ham additiv, bir jinsli, ya'ni chiziqli bo'ladi. Ixtiyoriy $\vec{x}=(x_1,x_2,\dots x_n)\in R^n$ uchun $O(\vec{x})=\vec{0}$ operatori nol operator, $E(\vec{x})=\vec{x}$ esa ayniy (birlik) operatori deyiladi.

Turli bazislarda operator matritsalari orasidagi bog'lanish quyidagi teoremada ifodalangan.

Teorema. A chiziqli operator $l_1, l_2,\dots l_n$ va $f_1, f_2,\dots f_n$ bazislardagi matritsalari mos ravishda Δ va Δ^* bo'lsa, $\Delta^*=C^{-1}\Delta C$, bunda C eski bazisdan yangisiga o'tish matritsasi.

Isbot: $y=\Delta x$, $y^*=\Delta^* x$ matritsaviy tengliklar o'rinli. Agar C o'tish matritsasi bo'lsa $x=Cx^*$, $y=Cy^*$. Birinchi tenglikni chapdan Δ ga ko'paytiramiz $\Delta x=\Delta Cx^*$, ya'ni $y=\Delta Cx^*$, yoki $Cy^*=\Delta Cx^*$, bundan esa $y^*=C^{-1}\Delta Cx^*$, ya'ni $\Delta^*=C^{-1}\Delta C$ kelib chiqadi.

Misol. $\vec{l_1},\vec{l_2}$ bazisda A operator $\Delta=\begin{pmatrix}17&6\\6&8\end{pmatrix}$ matritsaga ega. A operatorning $f_1=\vec{l_1}+2\vec{l_2}$, $f_2=-2\vec{l_1}+\vec{l_2}$ bazisdagi matritsasini toping.

$C=\begin{pmatrix}1&2\\-2&1\end{pmatrix}$, unga teskari matritsa, $C^{-1}=\frac{1}{5}\begin{pmatrix}1&-2\\2&1\end{pmatrix}$dir.

Demak, $\Delta^*=C^{-1}\Delta C=\frac{1}{5}\begin{pmatrix}1&-2\\2&1\end{pmatrix}\begin{pmatrix}17&6\\6&8\end{pmatrix}\begin{pmatrix}1&2\\-2&1\end{pmatrix}=\begin{pmatrix}5&0\\0&20\end{pmatrix}$.

Ta'rif: Nol bo'lmagan $\vec{x}\neq\vec{0}$ A chiziqli operatorning xos vektori deyiladi, agar shunday λ son topilib, $A(\vec{x})=\lambda\vec{x}$ o'rinli bo'lsa.

Bunda λ soni A operatorning (Δ matritsaning) xos soni deyiladi ($\vec{x}$ vektorga mos).

Demak, xos vektor operator ta'sirida o'ziga kolleniar vektorga o'tadi, o'zi songa ko'payadi, xolos.

Ta'rif: matritsaviy yozuvda $\Delta x=\lambda x$ yoki

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=\lambda x_1\\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=\lambda x_2\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=\lambda x_n\end{cases}$$

Soddalashtirsak,

$$\begin{cases} (a_{11}-\lambda)x_1+a_{12}x_2+\dots+a_{1n}x_n=\lambda x_1\\ a_{21}x_1+(a_{22}-\lambda)x_2+\dots+a_{2n}x_n=\lambda x_2\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}x_1+a_{n2}x_2+\dots+(a_{nn}-\lambda)x_n=\lambda x_n\end{cases}$$

Bu bir jinsli sistema trivial $x=\vec{0}=(0;0;\dots;0)$ yechimga doimo ega. Noldan farqli, notrivial yechim mavjud bo'lishi uchun sistema asosiy determinanti nolga teng bo'lishi kerak.

$$|\Delta-\lambda E|=\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{1n}\\ a_{21} & a_{22}-\lambda & a_{2n}\\ \dots & \dots & \dots\\ a_{n1} & a_{n2} & a_{nn}-\lambda\end{vmatrix}=0$$

Bu determinant λ ga nisbatan n-darajali ko'phad bo'lib, uni A operatop (yoki Δ matritsa) xarakteristik ko'phadi deyiladi, $|\Delta-\lambda E|=0$ tenglama A operator xarakteristik tenglamasi deyiladi.

Chiziqli operator xarakteristik ko'phadi bazis tanlanishiga bog'liq emas.

Misol. $\Delta=\begin{pmatrix}4&12\\3&4\end{pmatrix}$ matritsa bilan berilgan chiziqli operator xos sonlari va xos vektorlarini toping.

Xarakteristik tenglama tuzamiz: $|\Delta-\lambda E|=\begin{vmatrix}4-\lambda & 12\\ 3 & 4-\lambda\end{vmatrix}=0$, yoki $(4-\lambda)^2-6^2=0$.

Bundan chiziqli operator xos sonlari $\lambda_1=-2$, $\lambda_2=10$.

Dastlab, $\lambda_1=-2$ ga mos $\vec{x}^{(1)}$ xos vektorni qidiramiz. Buning uchun $(\Delta-\lambda_1 E)\vec{x}^{(1)}=0$ yoki $\begin{pmatrix}6&12\\3&6\end{pmatrix}\begin{pmatrix}\vec{x_1}^{(1)}\\ \vec{x_2}^{(1)}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ matritsaviy tenglamani yechamiz. $\overrightarrow{x_1}^{(1)} = -2\overrightarrow{x_2}^{(1)}$ munosabatga egamiz. Agar $\overrightarrow{x_2}^{(1)} = C$ desak, $\overrightarrow{x_1}^{(1)} = -2C$ bo'ladi.

Demak, λ_1=-2 ga mos xos vektorlar $\vec{x}^{(1)}$ =(-2C;C) ko'rinishida bo'ladi. C$\neq 0$

$\lambda_2 = 10$ da $\overrightarrow{x_2}^{(2)}$xos vektor uchun $\begin{pmatrix} -6 & 12 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} \overrightarrow{x_2}^{(1)} \\ \overrightarrow{x_2}^{(2)} \end{pmatrix} = 0$ dan $\overrightarrow{x_2}^{(1)}$=$2\overrightarrow{x_2}^{(2)}$ munosabatni olamiz. $\overrightarrow{x_2}^{(2)} = C$ desak, $\overrightarrow{x_2}^{(1)} = 2C$ bo'ladi. Demak, $\lambda_2 = 10$ ga mos xos vektorlar C$\neq$ 0da $x^{(2)}$=(2C;C) ko'rinishida bo'ladi.

Agar A chiziqli operator n ta chiziqli erkli $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ xos vektorlarga va $\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}, \dots, \overrightarrow{\lambda_n}$ xos sonlarga ega bo'lsa, $\overrightarrow{l_1}, \overrightarrow{l_2}, \dots, \overrightarrow{l_n}$ vektorlar bazis deb olinsa, A($\overrightarrow{l_i}$)=$a_{1i}\overrightarrow{l_1}$+$a_{2i}\overrightarrow{l_2} + \dots + a_{ii}\overrightarrow{l_i} + \dots + a_{ni}\overrightarrow{l_n} = \lambda_i\overrightarrow{l_i}$

Undan i$\neq$ jda a_{ij}=0, i=j da esa $a_{ij} = \lambda_i$.

Shunday qilib, xos vektorlardan iborat bazisda A operatop matritsasi diagonal ko'rinishdadir. $\nabla = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & \lambda_n \end{pmatrix}$ va aksincha agar A operator matritsasi diagonal ko'rinishda bo'lsa, bu bazis barcha vektorlari xosdir.

12.2. Kvadratik formalar

Ta'rif: Quyidagi L(x_1,x_2,...x_n)=$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$ yig'indi n o'zgaruvchi kvadratik formasi deyiladi, unda har bir qo'shiluvchi biror o'zgaruvchi kvadrati yoki ikki o'zgaruvchi ko'paytmasidan iborat.

$a_{ij} \in R$, a_{ij}=a_{ji} deb faraz qilamiz. Unda Δ= (a_{ij}) matritsa simmetrikdir, uni kvadratik forma matritsasi deyiladi.

Kvadratik formani matritsaviy yozuvda L=$x'\Delta x$, aslida esa

$$L=(x_1,x_2,\dots x_n)\begin{pmatrix} a_{11} + a_{12} + \dots + a_{1n} \\ a_{21} + a_{22} + \dots + a_{2n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1} + a_{n2} + \dots + a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=(x_1,x_2,\dots x_n)\begin{pmatrix} \sum a_{1j}x_j \\ \sum a_{2j}x_j \\ \vdots \\ \sum a_{nj}x_j \end{pmatrix} =$$

$\sum_{j=1}^{n} a_{1j}x_1x_j$+$\sum_{j=1}^{n} a_{2j}x_2x_j$+...+$\sum_{j=1}^{n} a_{nj}x_nx_j$=$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ yoziladi.

Misol. L(x_1;x_2;x_3)=4x_1^2-12x$_1$x$_2$-10x$_1$x$_3$+x_2^2-3x_3^2 ni matritsaviy ko'rinishda yozing. $L(x_1;x_2;x_3)\begin{pmatrix} 4 & -6 & -5 \\ -6 & 1 & 0 \\ -5 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Agar X=$(x_1,x_2,\dots x_n)'$, Y=$(y_1,y_2,\dots y_n)'$, C=(c_{ij}) lar uchun X=CY bo'lsa, kvadratik forma L=$x'\Delta x$ = $(CY)'\Delta(CY)$=$(Y'C')\Delta(CY)$=$Y'(C'\Delta C)Y$.

Demak, bunday xosmas chiziqli almashtirishda kvadratik forma matritsasi Δ^*=$C'\Delta C$ ko'rinish oladi.

Misol. L(x_1,x_2)=2x_1^2 + 4x_1x_2-3x_2^2 kvadratik forma berilgan.

Unda $x_1 = 2y_1 - 3y_2$, x_2=y_1+y_2 chiziqli almashtirish natijasida olinadigan L(y_1, y_2) kvadratik formani toping.

$\Delta = \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$, C=$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ bo'lgani uchun Δ^*= $C'\Delta C$ dan

$\Delta^* = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}\cdot\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -17 \\ -17 & 3 \end{pmatrix}$

Demak, L(y_1; y_2)=13y_1^2 − 34y_1y_2 + 3y_2^2.

Ta'rif. $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ kvadratik forma kanonik ko'rinishiga keltirilgan deyiladi, agar $i \neq j$ da $a_{ij} = 0$ bo'lsa, ya'ni L=$a_{11}x_1^2 + a_{22}x_2^2 + \dots + a_{nn}x_n^2$=$\sum_{i=1}^{n} a_{ii}x_i^2$

Kanonik ko'rinishdagi kvadratik forma matritsasi diagonal ko'rinishida bo'ladi.

Teorema. Ixtiyoriy kvadratik forma xosmas chiziqli almashtirish yordamida kanonik ko'rinishga keltiriladi.

Misol. L(x_1, x_2, x_3)= $x_1^2 - 10x_1x_2 + 6x_1x_3 + 4x_2x_3 + 26x_2^2$+$x_3^2$ o'zgaruvchilarni xosmas chiziqli almashtirish yordamida kanonik ko'rinishga keltiring.

$$\left[x_1^2 - 2x_1\left(\frac{1}{2}(10x_2 - 6x_3)\right) + (5x_2 - 3x_3)^2\right] - (5x_2 - 3x_3)^2 + +4x_2x_3 + 26x_2^2 + x_3^2 =$$

L(x_1, x_2, x_3)=$(x_1 - 5x_2 + 3x_3)^2 - 25x_2^2 + 30x_2x_3 - 9x_3^2$ + $4x_2x_3 + 26x_2^2 + x_3^2 = (x_1 - 5x_2 + 3x_3)^2 +$ $x_2^2 + 34x_2x_3 - 8x_3^2 = (x_1 - 5x_2 + 3x_3)^2 +$

$+(x_2+17x_3)^2-(17x_3)^2 - 8x_3^2 = (x_1 - 5x_2 + 3x_3)^2+(x_2+17x_3)^2-27x_3^2$.

Demak, xosmas chiziqli almashtirish:

$y_1 = x_1 - 5x_2 + 3x_3$, $y_1 = x_2 + 17x_3$, $y_3 = x_3$ desak, kvadratik forma $L_1(y_1, y_2, y_3)$=$y_1^2 + y_2^2 - 297y_3^2$ kanonik ko'rinishiga keladi.

Teorema. (kvadratik forma inersiya qonuni). Ixtiyoriy xosmas chiziqli almashtirishlarda olingan kanonik tenglamada musbat (manfiy) koeffitsiyentli qo'shiluvchilar soni o'zgarmaydi.

Agar L=$(x_1,x_2,\dots x_n)$>0 $[L = (x_1,x_2,\dots x_n) < 0]$ bo'lsa, kvadratik forma musbat (manfiy) aniqlangan deyiladi.

Masalan, $L_1=(3x_1^2+9x_2^2+4_3^2)$ musbat aniqlangan, $L_1=-x_1^2+2x_2x_3-x_2^2$ manfiy aniqlangan.

Teorema. $L=x'\Delta x$ kvadratik forma musbat (manfiy) aniqlangan bo'lishi uchun Δ matritsaning barcha xos sonlari λ_i lar musbat (manfiy) bo'lishi zarur va yetarlidir.

Ishora aniqlash uchun ko'p hollarda quyidagi Silvestr alomatidan foydalaniladi.

Teorema (Silvestr). Kvadratik forma musbat (manfiy) aniqlangan bo'lishi uchun kvadratik forma matritsasining bosh minorlari musbat, ya'ni

$$\Delta_1>0,\ \Delta_2>0,\ \dots,\ \Delta_n>0,\ \text{bunda}\ \Delta_n=\begin{vmatrix} a_{11}+a_{12}+\dots+a_{1n} \\ a_{21}+a_{22}+\dots+a_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}+a_{n2}+\dots+a_{nn} \end{vmatrix}$$

bo'lishi zarur va yetarlidir.

Agar kvadratik forma manfiy aniqlangan bo'lsa, bosh minoralar ishoralari musbatdan boshlab almashinadi.

Misol. $L=13x_1^2-6x_1x_2+5x_2^2$ kvadratik forma ishorasini tekshiring.

1-usul. Kvadratik forma matritsasi $\Delta=\begin{vmatrix} 13 & -3 \\ -3 & 5 \end{vmatrix}$ bo'lgani uchun $|\Delta-\lambda E|=\begin{vmatrix} 13-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix}=0$ yoki $\lambda^2-18\lambda+56=0$. $\lambda_1=14$, $\lambda_2=4$ musbat xos sonlar bo'lgani uchun berilgan kvadratik forma musbat aniqlangan.

2-usul. $|a_{11}|=13$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=\begin{vmatrix} 13 & -3 \\ -3 & 5 \end{vmatrix}=56$ ekanligi Silvestr alomatiga ko'ra L ning musbat aniqlanganligini bildiradi.

12.3. Almashishning chiziqli modeli

Matritsaning xos son va xos vektori tushunchalariga olib keluvchi almashishning chiziqli modeli (xalqaro bozor modeli) ni ko'rib chiqamiz.

Milliy foydasi $x_1, x_2, \dots, x_n$ bo'lgan $S_1, S_2, \dots, S_n$ davlatlar berilgan bo'lsin. S_j davlatning S_i davlatdan tovarlar sotib olishga sarflaydigan milliy foydaning qismini a_{ij} deb belgilaymiz. Agar milliy foydaning hammasi yoki shu davlat ichida yoki chet davlatdan tovar sotib olishiga sarflanadi deb hisoblansa, $\sum_{j=0}^{n} a_{ij}=1$ (j=1,2,. . ., n) bo'ladi.

Quyidagi $\Delta=\begin{pmatrix} a_{11} & a_{12}\dots a_{1n} \\ a_{21} & a_{22}\dots a_{2n} \\ \cdot & \cdot\ \ \cdot \\ a_{n1} & a_{n2}\dots a_{nn} \end{pmatrix}$ matritsa savdoning **strukturali matritsasi** deyiladi. Unda ixtiyoriy ustun elementlari yig'indisi 1 ga teng.

Ichki va tashqi savdodan ixtiyoriy $S_i(i=\overline{1,n})$ davlat daromadi $P_i=a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n$ bo'ladi.

Har bir davlat $P_i\geq x_i$ $(i=\overline{1,n})$ bo'lishga harakat qiladi, albatta. U holda

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n\geq x_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n\geq x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n\geq x_n \end{cases}$$

Barcha tengsizliklarni qo'shib, guruhlab:

$$x_1(a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n)+x_2(a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n)+\dots$$
$$+x_n(a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n)>x_1+x_2+\dots+x_n.$$

Lekin tengsizlik chap tomoni ham $x_1+x_2+\dots+x_n$ ga teng, qavs ichidagilar yig'ndisi 1. Ziddiyat kelib chiqdi, ya'ni:

$$x_1+x_2+\dots+x_n>x_1+x_2+\dots+x_n.$$

Shunday qilib, $P_i\geq x_i$ $(i=\overline{1,n})$ qilib olish mumkin emas. U holda $P_i=x_i$ $(i=\overline{1,n})$ shart qoladi xolos.

Iqtisodiy mazmuni quyidagicha: Barcha davlatlar bir paytda foyda olmaydi.

$\vec{x}=(x_1,x_2,\dots,x_n)'$ vektor kiritsak, $\Delta\vec{x}=\vec{x}$ tenglamaga ega bo'lamiz, ya'ni qaralgan masala Δ matritsaning $\lambda=1$ xos songa mos xos vektorini izlashga keladi.

Misol. Uchta S_1, S_2, S_3 davlat strukturali matritsasi $\Delta=\begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & 0 \end{pmatrix}$ bo'lsa, balanslangan savdodagi har bir davlat milliy foydasini toping.

$(\Delta-E)\vec{x}=\vec{0}$ tenglamani yechib, $\lambda=1$ ga mos $\vec{x}$ xos vektorni qidiramiz.

$$\begin{pmatrix} \frac{1}{3}-1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2}-1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & 0-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ sistemani yechib, $x=(\frac{3}{2}c; 2c; c)$ xos vektorni topamiz. Demak, bu uch davlat orasidagi sodda balanslangan bo'lishi uchun ularning milliy foydasi $\frac{3}{2}; 2; 1$ nisbatda bo'lishi kerak.

12.4. Ko‘p tarmoqli iqtisodda Leontyev modeli (balans analiz)

N ta ishlab chiqarish korxonasini o‘rganaylik. Ularning har biri o‘z mahsulotini ishlab chiqaradi. Bu mahsulotlar bir qismi korxonaning o‘ziga, qolganlari boshqa korxonalarga taqsimlanadi. Barcha ehtiyojlarni qondirish uchun har bir korxona ishlab chiqarish hajmi qanday bo‘lishi kerak? Bu savolga 1936 yilda amerikalik V.Leontyev tomonidan yaratilgan matematik model javob beradi. Bu model ishlab chiqarish korxonalari sohalararo balansini ifodalovchi jadvalni analiz qiladi.

Biror muddatda ichlab chiqarish protsessini tekshiraylik. i-ishlab chiqarish tarmog‘i ishlab chiqqan umumiy tovar hajmi x_i bo‘lsin. $(i=\overline{1,n})$.

Ishlab chiqarishda j-tarmoq i-tarmoq mahsulotini iste’mol etishi x_{ij} $(i,j=\overline{1,n})$ bo‘lsin. i-tarmoq mahsulotini y_i qismi boshqa zaruriyatlarga ishlatilsin.

U holda i-tarmoq umumiy ishlab chiqarish hajmi $x_i=\sum_{j=0}^{n}x_{ij}+y_i$ $(i=\overline{1,n})$ bo‘ladi. Bu tenglama **balans munosabatlari** deyiladi.

Quyidagi $a_{ij}=\frac{x_{ij}}{x_j}$ $(i,j=\overline{1,n})$ koeffitsiyent **to‘g‘ridan-to‘g‘ri harajatlar koeffitsiyenti** deyiladi. U j-tarmoq birlik mahsulotini chiqarishga sarf bo‘ladigan i-tarmoq mahsuloti sarfini ko‘rsatadi.

Qaralayotgan muddatda a_{ij} koeffitsiyentlar o‘zgarmas bo‘lsa, $x_{ij}=a_{ij}x_j$, $(i,j=\overline{1,n})$ chiziqli bog‘liqlik o‘rinli bo‘ladi. U holda balans munosabatlari: $x_i=\sum_{j=0}^{n}a_{ij}x_j+y_i$ $(i=\overline{1,n})$ bo‘ladi.

Agar $X=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$, $A=\begin{pmatrix}a_{11}a_{12}\dots a_{1n}\\a_{21}a_{22}\dots a_{2n}\\\dots\\a_{n1}a_{n2}\dots a_{nn}\end{pmatrix}$, $Y=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$ belgilashlar kiritsak, bunda X – umumiy ishlab chiqarish vektori, Y – oxirgi mahsulot vektori, A – xarajatlar matritsasi, yuqoridagi tenglik X=AX+Y ko‘rinish oladi.

Tarmoqlararo balansning asosiy vazifasi shunday X ni topishki, X=AX+Y o‘rinli bo‘lsin.

Tenglamani (E-A)X=Y ko‘rinishida yozamiz. Agar (E-A) matritsa xosmas bo‘lsa, ya’ni $|E-A|\neq 0$, u holda $X=(E-A)^{-1}*Y$. $(E-A)^{-1}$ matritsa **to‘la sarf-xarajatlar matritsasi** deyiladi.

Masalan. Tekshirilgan muddatda ikki korxona balansi quyidagicha edi:

Soha		Sarf-harajat		Oxirgi mahsulot	Umumiy i/ch
		1-soha	2-soha		
I/ch	1-soha	7	12	72	100
	2-soha	12	15	63	100

1-sohaning oxirgi sarf-harajati 2 marta oshirilsa, 2-soha avvalgiday ishlasa, umumiy ishlab chiqarish hajmi zaruriy hajmini hisoblang.

$x_1=100$, $x_2=100$, $x_{11}=7$, $x_{12}=21$, $x_{21}=12$, $x_{22}=15$, $y_1=72$, $y_1=63$.

Formuladan $a_{11}=0.07$, $a_{12}=0.21$, $a_{21}=0.12$, $a_{22}=0.15$

Demak, $A=\begin{pmatrix}0.07 & 0.21\\0.12 & 0.15\end{pmatrix}$ $X=(E-A)^{-1}Y$ dagi $E-A=\begin{pmatrix}0.93 & -0.21\\-0.12 & 0.85\end{pmatrix}$

$|E-A|=0.7653\neq 0$.

Teskari matritsa $(E-A)^{-1}=\frac{1}{0.7653}\begin{pmatrix}0.85 & 0.12\\0.21 & 0.93\end{pmatrix}$

U holda $X=\frac{1}{0.7653}\begin{pmatrix}0.85 & 0.12\\0.21 & 0.93\end{pmatrix}\begin{pmatrix}169.8\\116.1\end{pmatrix}$.

Demak, 1-sohani 169.8, 2-sohani 116.1 gacha kuchaytirish kerak.

Mavzuga doir misol va masalalar

1. $\vec{l_1},\vec{l_2}$ bazisda chiziqli operator matritsasi $\begin{pmatrix}3 & 2\\-1 & 5\end{pmatrix}$ bo‘lsa $\vec{x}=4\vec{l_1}$-3$\vec{l_2}$ vektor obrazini toping.

2. $\vec{l_1},\vec{l_2},\vec{l_3}$ bazisda chiziqli operator matritsasi $\begin{pmatrix}-1 & 0 & 2\\2 & 1 & 1\\3 & 0 & 1\end{pmatrix}$ bo‘lsa, $\vec{x}=2\vec{l_1}+4\vec{l_2}-\vec{l_3}$ vektor obrazini toping.

3. $\vec{l_1},\vec{l_2}$ bazisda operator matritsasi $\begin{pmatrix}2 & 4\\-3 & 3\end{pmatrix}$ bo‘lsa, operatorning $\vec{l_1}^{*}=\vec{l_2}$-$2\vec{l_1}$, $\vec{l_2}^{*}=2\vec{l_1}$- $4\vec{l_2}$ bazisdagi matritsani yozing.

4. Matritsalari berilgan chiziqli operatorlar xos son va xos vektorlarini toping.

1) $\begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$ 3) $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$ 4) $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$

5) $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$ 6) $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$ 7) $\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$

5. $L=2x_1^2+3x_2^2-x_3^2+4x_1x_2-6x_1x_3+10x_2x_3$ kvadratik formaning matritsasini yozing.

6. $L(x_1,x_2)=3x_1^2-x_2^2+4x_1x_2$ kvadratik forma $x_1=2y_1-y_2$, $x_2=y_1+y_2$ chiziqli almashtirishda qanday o‘zgaradi?

7. Kvadratik formalarni kanonik ko‘rinishga keltiring.

1) $x_1^2+5x_2^2-4x_3^2+2x_1x_2-4x_1x_3$, 2) $x_1x_2+x_2x_3+x_1x_3$

3) $4x_1^2+x_2^2+x_3^2-4x_1x_2+4x_1x_3-3x_2x_3$,

4) $x_1^2+x_2^2+5x_3^2-6x_1x_2+\quad +6x_1x_3-6x_2x_3$

8. Kvadratik forma ishorasini aniqlang.

1) $x_1^2+4x_2^2+3x_3^2+2x_1x_2$, 2) $2x_2^2-x_1^2-x_1x_3++x_2x_3-2x_3^2$.

Chiziqli algebra elementlariga doir joriy nazorat uchun uy vazifasi
(N-talabaning jurnaldagi tartib nomeri)

I. Biror bazisda $\vec{a}(1;2;N)$, $\vec{b}(-2;0;3)$, $\vec{c}(3;N;1)$, vektorlari berilgan. Bu vektotlar bazis tashkil etishini isbotlang va shu bazisda $\vec{d}(N;10;-N;1)$ vektor koordinatalarini toping.

II. Ikkita chiziqli almashtirish berilgan:

$$\begin{cases} y_1 = Nx_1+(20-N)x_2+x_3 \\ y_2 = 2x_1-Nx_2-4x_3 \\ y_3 = x_1+2x_2-(N-10)x_3 \end{cases} \qquad \begin{cases} z_1 = y_1-Ny_2+(20-N)y_3 \\ z_2 = Ny_1+y_2-4y_3 \\ z_3 = (10-N)y_1+2y_2-y_3 \end{cases}$$

Matritsaviy usulda z_1,z_2,z_3 ni x_1,x_2,x_3 orqali ifodalovchi almashtirishni toping.

III. $\begin{pmatrix} 1 & (10-N) & -3 \\ N & N-10 & 10-N \\ 4 & 20-N & 2 \end{pmatrix}$ matritsasi chiziqli almashtirish xos son va xos vektorlarini toping.

IV. $f_1(1;-2;3;N)$, $f_1(N;0;-3;5)$, $f_3(-1;N;4;2)$ va $f_4(-3;1;10-N;2)$ vektorlar ortogonallashtirilsin.

V. Kvadratik formani kanonik ko‘rinishga keltiring va ishorasini aniqlang.

$L=x_1^2+(20-N)x_2^2+Nx_3^2-6x_1x_2+Nx_1x_3-(30-N)x_2x_3$.

13-mavzu. To‘plam. Funksiya

13.1. “To‘plam” tushunchasi

“To‘plam” tushunchasi boshlang‘ich tushunchalardan bo‘lib, uni boshqa soddaroq tushunchalar bilan ta’riflab bo‘lmaydi.

Umuman, to‘plam deganda biror xususiyatiga ko‘ra qaralayotgan narsalar, obyektlar tushuniladi. To‘plam hosil qilayotgan obyektlar to‘plam elementlari yoki nuqtalari deyiladi. To‘plamlar bosh harflarda, uning elementlari kichik harflarda belgilanadi. a element A to‘plam elementi ekanligi $a\in A$, b elementi A to‘plamga tegishli emasligi $A\not\ni b$ tarzida yoziladi.

Birorta ham elementi bo‘lmagan to‘plam **bo‘sh to‘plam** deyiladi va $\emptyset$ ko‘rinishida belgilanadi. Masalan, $x^2+1=0$ tenglama haqiqiy ildizlari to‘plami bo‘shdir.

Agar B to‘plam elementlari A to‘plamga ham element bo‘lsa, B to‘plam A to‘plam qismi (qism to‘plami) deyiladi va $B\subset A$ ko‘rinishida yoziladi.

Bir xil elementlardan tuzilgan ikki to‘plam **teng** deyiladi.

Ikki A va B to‘plam yig‘indisi deb, ularning kamida bittasiga tegishli barcha elementlardan tuzilgan C to‘plamga aytiladi va $C=A\cup B$ tarzida yoziladi.

Ikki A va B to‘plam kesishmasi deb, ulaming umumiy elementlaridan tuzilgan D to‘plamga aytiladi va $D=A\cap B$ ko‘rinishida yoziladi.

A to‘plamdan B to‘plamning ayirmasi deb, faqatgina A ga xos-tegishli elementlar to‘plami tushuniladi. U masalan, $E=A\setminus B$ tarzida yoziladi.

Agar $A\subset B$ bo‘lsa, A to‘plamning B to‘plamgacha to‘ldiruvchisi deb, B ning A ga tegishli bo‘lmagan nuqtalari to‘plami tushuniladi va C_BA tarzida yoziladi.

$$C_BA=B\setminus A$$

Misol. $A=\{1;2;3;4\}$, $B=\{3;4;5\}$, $D=\{1;2;3\}$, bo‘lsa, $A\cup B=\{1;2;3;4;5\}$, $A\cap B=\{3;4\}$, $A\setminus B=\{1;2\}$, $B\setminus A=\{5\}$, $C_AD=\{4\}$.

A va B to‘plam xos elementlaridan tuzilgan to‘plam **simmetrik ayirma** deyiladi va $A\Delta B$ tarzida yoziladi.

$$A\Delta B=(A\setminus B)\cup(B\setminus A)=(A\cup B)\setminus(A\cap B)$$

A va B to'plamlar barcha elementlaridan tuzilgan (a;b) ko'rinishidagi juftliklar to'plami **Dekart ko'paytmasi** deyiladi va AXB deb yoziladi.

$$AXB=\{(a;b): \forall a \in A, \forall b \in B\}.$$

Misol. A={1; 2}, B={2; 3}, bo'lsa, $A\Delta B=\{1\}\cup\{3\}=\{1;3\}$.
$AXB=\{(1;2),(1;3),(2;2),(2;3)\}$

Shunga o'xshash, $AXBXC=\{(a;b;c): \forall a \in A, \forall b \in B, \forall c \in C\}$,...

Elementlari haqiqiy sonlar bo'lgan to'plamlar **sonli to'plam** deyiladi. Elementar matematikadan ma'lumki, R-haqiqiy sonlar, Q-ratsional sonlar, I-irratsional sonlar, Z-butun sonlar, N-natural sonlar to'plamlari edi.

$N\subset Z\subset Q\subset R\subset C$, $R=Q\cup I$, $RXR=R^2$, $RXRXR=R^3$ kabi munosabatlar tushunarli.

13.2. Funksiyaning xossalari va turlari

Faraz qilaylik, $X,Y\subset R$ to'plamlar berilgan bo'lsin.

Ta'rif. Agar har bir $x\epsilon X$ son uchun biror f qoidaga ko'ra yagona $y\epsilon Y$ son mos qo'yiigan bo'lsa, X to'plamda y=f(x) funksiya berilgan deyiladi.

Tekislikning $\{x; f(x)\}=\{(x; f(x);\ x\epsilon X, f(x)\epsilon Y\}$ ko'rinishdagi nuqtalari to'plami berilgan **funksiya grafigi** deyiladi.

X to'plam funksiyaning aniqlanish sohasi, Y to'plam esa o'zgarish sohasi deyiladi va mos ravishda D(y), E(y) ko'rinishida belgilanadi.

y=f(x) yozuvda x erkli o'zgaruvchi (argument), y esa erksiz o'zgaruvchidir.

Funksiya, asosan, 3 xil: analitik, jadval, grafik usulda beriladi.

Analitik usulda funksiya y=f(x) formula yordamida beriladi, jadval usulida erkli o'zgaruvchili x ning qiymatlariga mos keladigan y ning qiymatlari beriladi.

Grafik usulda funksiya tenglamasini qanoatlantiradigan $(x;y)\ \epsilon R^2$ nuqtalar to'plami beriladi.

Funksiyani o'rganish uning aniqlanish sohasini topishdan boshlanadi:

Misol. $y=\dfrac{\sqrt{\delta^2-16}}{\log_2(x^2+3x-10)}$ funksiya aniqlanish sohasini toping.

Ma'lumki, bu funksiya:

$$\begin{cases} x^2-16\ge 0 \\ x^2+3x-10>0,\ \log_2(x^2+3x-10)\ne 0 \end{cases}$$ shartlar o'rinli bo'lgandagina aniqlanadi.

$$\begin{cases} (x-4)(x+4)\ge 0 \\ (x-2)(x+5)>0,\ x^2+3x-11\ne 0 \end{cases}$$

Bu tengsizliklar yechimlari mos ravishda:

$(-\infty;-4\,]\cup[4;+\infty\,),(-\infty;-5\,]\cup[2;+\infty\,),\ (-\infty;-\frac{3+\sqrt{53}}{2})\cup(-\frac{3+\sqrt{53}}{2};\frac{-3+\sqrt{53}}{2})\cup(\frac{-3+\sqrt{53}}{2};+\infty)$

bo'lganligi uchun, berilgan funksiyalarning barchasi o'rinli bo'lgan

$(-\infty;-\frac{3+\sqrt{53}}{2})\cup(-\frac{3+\sqrt{53}}{2};-5)\cup[4;+\infty)$ sohada aniqlanadi, xolos.

Funksiyaning asosiy xossalari bilan tanishamiz.

13.3. Juft-toqlik

Funksiya aniqlanish sohasi koordinatalar boshiga nisbatan simmetrik bo'lsin, ya'ni agar funksiya biror $x\epsilon R_+$ da aniqlansa, $(-x)\,\epsilon R_-$ da ham aniqlanishi shart.

Ta'rif: Agar $f(-x)=f(x)$ $[f(-x)=-f(x)]$ tenglik o'rinli bo'lsa, funksiya qaralayotgan sohada **juft (toq)** deyiladi.

Masalan, $y=x^2$, y=cosx funksiyalari juft, $y=x^3$, y=sinx funksiyalari toq.

Yuqoridagi ikkala shartga ham bo'ysunmaydigan funksiya juft ham, toq ham emas, **umumiy holdagi funksiya** deyiladi.

Masalan, $y=x^2-x$, $y=1-x+x^2-x^3$ funksiyalar shular jumlasidan.

Ta'rifdan, juft funksiya grafigi ordinata o'qiga nisbatan, toq funksiya grafigi koordinata boshiga nisbatan simmetrik joylashishi kelib chiqadi.

Juft funksiya yig'indisi, ayirmasi, ko'paytmasi, bo'linmasi (maxrajdagi funksiya noldan farqli bo'lsa) yana juft funksiya bo'ladi.

Toq funksiyalar yig'indisi, ayirmasi toq funksiya, lekin ko'paytmasi, bo'linmasi juft funksiya bo'ladi.

Misol. 1) $y=2^x+2^{-x}$ juft-toqlikka tekshirilsin.

$y(-x)=2^{-x}+2^{-(-x)}=2^{-x}+2^{x}=y(x)$ o'rinliligidan juftdir.

2) $y=\ln(x+\sqrt{1+x^2})$ juft-toqlikka tekshirilsin.

$y(-x)=\ln(\sqrt{1+x^2}+(-x))=\ln(\sqrt{1+x^2}-x)\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}=\ln\frac{1}{\sqrt{1+x^2}+x}=$
$=\ln(x+\sqrt{1+x^2})^{-1}=-\ln(x+\sqrt{1+x^2})=-y(x)$. Demak, bu funksiya toqdir.

13.4. Chegaralanganlik

Ta'rif. X to'plamda aniqlangan f(x) funksiya yuqoridan (quyidan) chegaralangan deyiladi, agar har bir $x \epsilon X$ uchun shunday M (m) soni topilib, $f(x) \leq M$ ($f(x) \geq m$) tengsizlik o'rinli bo'lsa, X to'plamda ham yuqoridan, ham quyidan chegaralangan funksiyalar chegaralangan deyiladi.

Masalan. $y=x^2$ funksiya quyidan 0 bilan, $y=1-x^2$ funksiya yuqoridan 1 bilan chegaralangan.

y=sinx funksiya esa chegaralangan, chunki $-1 \leq sinx \leq 1$

Agar X to'plamda $f_1(x)$, $f_2(x)$ funksiyalar chegaralangan bo'lsa,

$f_1 \pm f_2$, $f_1 f_2$, Cf_1, $f_2 \neq 0$ da $\frac{f_1}{f_2}$ funksiyalar ham chegaralangan bo'ladi.

Funksiyaning X to'plamda chegaralangan ekanligi: $|f(x)| \leq C$ tengsizlik o'rinli bo'ladigan C sonning ko'rsatilishi demakdir.

Misol. $y=3^{sin^2x}+3cos4x$ funksiyaning chegaralanganligini ko'rsating. $|3^{sin^2x}+3cos4x| \leq 3^{sin^2x}+3|cos4x| \leq 3+3=6$ va $3^{sin^2x} \geq 1$ bo'lishini e'tiborga olsak, $3^{sin^2x}+3cos4x \geq 1-3=-2$ $-2 \leq 3^{sin^2x}+3cos4x \leq 6$, berilgan funksiya chegaralangan.

13.5. Davriylik

X to'plamda y=f(x) funksiyani aniqlang.

Ta'rif: Agar shunday $T \neq 0$ son mavjud bo'lsaki, ixtiyoriy $x \epsilon X$ da

1) $x-T, x+T \epsilon X$,
2) $f(x+T)=f(x)$

bo'lsa, f(x) funksiya davriy deyiladi. Bunday T sonlarning eng kichik musbati **funksiyaning davri** deyiladi.

Masalan, y=sinx, y=cosx funksiya davri 2π, y=tgx, y=ctgx funksiyalar davri esa π dir.

Agar f_1, f_2, funksiyalar davri T bo'lsa, $f_1 \neq f_2$, $f_1 x f_2$, $\frac{f_1}{f_2}$ $(f_2 \neq 0)$ funksiyalar ham davriy va davri T dir. Agar f_1, f_2, funksiyalar davri $T_1 va T_2$ bo'lsa $f_1 \pm f_2$, funksiya davri $EKUK(T_1, T)_2$ bo'ladi. Masalan,

y=sin2x+cos3x funksiyada birinchi qo'shiluvchi davri π, ikkinchisiniki $\frac{2\pi}{3}$, funksiyaning osi esa EKUK $(\pi; \frac{2\pi}{3})=2\pi$ davrlidir.

13.6. Monotonlik

X to'plamda y=f(x) funksiya berilgan bo'lsin.

Ta'rif. Agar ixtiyoriy $x_1, x_2 \epsilon X$ qiymatlari uchun $x_1<x_2$ bo'lishidan $f(x_1)<f(x_2)$ ($f(x_1)>f(x_2)$) kelib chiqilsa, f(x) funksiya X to'plamda o'suvchi (kamayuvchi) deyiladi. O'suvchi va kamayuvchi funksiyalar monoton funksiyalar deyiladi.

Agar f_1, f_2 funksiyalar X to'plamda o'suvchi (kamayuvchi) bo'lsa, f_1+c, f_1+f_2, c>0 da c f_1 funksiyalar o'suvchi (kamayuvchi), c<0 da esa cf_1 funksiya kamayuvchi (o'suvchi) bo'ladi.

Misol. 1) $y=x^2$ funksiya $(-\infty; 0]$ *da* kamayuvchi, $[0;+\infty)$ da o'suvchi. Haqiqatan, $x_1, x_2 \in [0;+\infty)$, $x_1<x_2$ bo'lsin. Unda $f(x_1)-f(x_2)=x_1^2-x_2^2=(x_1+x_2)(x_1-x_2)<0$, chunki $x_1-x_2<0$. $x_1<x_2$ dan $f(x_1)<f(x_2)$ kelib chiqdi, demak, $y=x^2$ funksiya $[0;+\infty)$ da o'suvchi.

Endi ba'zi elementar funksiyalarni sanab o'tamiz.

1. $y=|x|=\begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, bu funkslya modul deyiladi.

2. $y=sgnx=\begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x, 0 \end{cases}$, bu funkslya x ning ishorasi deyiladi.

3. $y=[x]$ ko'rinishda x ning butun qismi belgilanadi. Berilgan sonning butun qismi – o'ziga teng yoki undan kichik eng katta butun sondir, masalan, $[1,5]=1, [-1,5]=-2, [-1]=-1$

4. $y=\{x\}$ ko'rinishda x ning kasr qismi belgilanadi. $x=[x]+\{x\}$ bo'lishi tushunarlidir.

5. Darajali funksiya: $y=x^n$

6. Ko'rsatkichli funksiya: $y=a^x$ $(a>0, a\neq 1)$

7. Logorifmik funksiya: $y=\log_a x$ $(a>o, a\neq 1)$

8. Trigonometrik funksiyalar: y=sinx,y=cosx,y=tgx,y=ctgx

9. Teskari trigonometrik funksiyalar: y=arcsinx, y=arccosx, y=arctgx, y=arcctgx

10. Giperbolik funksiyalar:

a) sinus giperbolik funksiya: $y=\frac{e^{+x}-e^{-x}}{2}=shx$;

b) kosinus giperbolik funksiya: $y=\frac{e^{+x}+e^{-x}}{2}=chx$;

c) tangens giperbolik funksiya: $y=\frac{e^{+x}-e^{-x}}{e^{x}+e^{-x}}=thx$;

d) kotangens giperbolik funksiya: $y=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=cthx$.

Giperbolik funksiyalar quyidagi xossalarga ega:

$Sh0=0, ch0=1, thx=\frac{shx}{chx}$, $cthx=\frac{chx}{shx}$, $ch^2x-sh^2x=1$, $ch^2x+sh^2x=ch2x$, $2shxchx=sh2x$ $Sh(x+y)=shxchy+chxshy$, $ch(x+y)=chxchy+shxshy$, $th(x+y)=\frac{thx+thy}{1+thxthy}$.

Bu funksiyalar aslida ko'rsatkichli funksiyalar yordamida quriladi, lekin xossalari trigonometrik funksiyalar xossalariga o'xshashligidan shunday nomlanishiga sabab bo'lgan.

Elementar funksiyalardan arifmetik amallar yordamida olinadigan barcha funksiyalar elementar funksiyalardir.

Funksiyalar quyidagicha berilishi mumkin:

1. **Oshkor funksiya.** Bunda funksiya shunday tenglama bilan eriladiki, uning o'ng tomoni faqat erkli o'zgaruvchiga bog'liq bo'ladi, masalan, $y=x^2-4x+3$.

2. **Oshkormas funksiya.** Funksiya y ga nisbatan yechilmagan F(x;y)=0 tenglama bilan beriladi, masalan, $\frac{x^2-y^2}{a^2+b^2}=1$

3. **Teskari funksiya.** y=f(x) funksiya bitta x ga yagona y ni mos qo'yadi. Endi shu y∈Y ga yagona x∈X ni mos qo'yuvchi $x=\varphi(y)$ funksiyalarni qarash mumkin. Bu funksiya y=f(x) ga teskari deyiladi. Uni qayta belgilab (x ni y,y ni x deb), $y=f^{-1}(x)$ tarzida yoziladi. Masalan, $y=a^x$ ga $y=\log_a x$ funksiya teskaridir. Ixtiyoriy qat'iy monoton funksiya teskarisi doimo mavjud ekanligini isbotlash mumkin.

4. **Murakkab funksiya.** Agar y=f(u) funksiya argumenti ham $u=\varphi(x)$ funksiya bo'lsa, ular superpozitsiyasi $y=f(\varphi(x))$ murakkab funksiya deyiladi. Masalan, y=lnsinx, $y=(2x-1)^3$,……

5. **Parametrik funksiya.** Agar funksiya ham, argumenti ham biror t parametr yordamida aniqlansa, ya'ni $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ funksiya parametrik usulda berilgan deyiladi. Masalan, $x^2+y^2=R^2$ aylanani $\begin{cases} x=R\cos t \\ y=R\sin t \end{cases}$ ko'rinishda berish mumkin.

Funksiyalar quydagicha klassifikatsiyalashtiriladi

1. Butun ratsional funksiyalar: $y=P_n(x)=a^n x^n+a_{n-2}x^{n-1}+\ldots+a_2+a_0$.

2. Kasr ratsional funksiyalar: $y=\frac{Q_m(x)}{P_n(z)}=\frac{Q_m x^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0}{a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}$

3. Irratsional funksiyalar; masalan. $y=\sqrt{x}$, $y=x+\sqrt{x}$,…

4. Transsendent funksiyalar, ratsional yoki irratsional bo'lmagan funksiyalar **transsendent funksiyalar** deyiladi, masalan, y=sinx, y=arcsinx, $y=\cos x+x$;…

Funksiya grafiklari ustida arifmetik amallarni bajarish uchun har bir x∈X da berilgan amal bajariladi, natijasi tekislikda belgilanadi. Olingan nuqtalar birlashtirilsa, natijaviy funksiya hosil bo'ladi.

Mavzuga doir misol va masalalar

1. Quyidagilarni isbotlang.

1) $A\cup(A\cap C)=(A\cup B)\cap(A\cup C)$ 2) $(B\backslash C)(B\backslash A)\subset A\backslash C$

3) $(A\backslash B)\backslash C=(A\backslash C)(B\backslash C)$ 4) $A\Delta B=(A\cap CB)\cup(B\cap CA)$

5) $(A\cap B)\cup(A\cap CB)\cup(CA\cap B)=A\cup B$

6) $A\times(B\times C)=(A\times B)\cup(A\times c)$

2. Aniqlanish sohasini toping.

1) $y=\frac{1}{4-x^2}$ 2) $y=\sqrt{4x-x^3}$ 3) $y=\sqrt{\frac{1+x}{1-x}}$ 4) $y=\sqrt{\frac{x^2-9x+14}{x^2-9}}$

5) $y=\sqrt{9-x^2}+\sqrt{x^2-4}$ 6) y=lg(5-x) 7) $y=\log_2\log_3\log_4 x$

8) $y=\sqrt{\sin x-\frac{1}{2}}$ 9) $y=\sqrt{1-tg^2 4x}$ 10) $y=\arcsin\frac{x-1}{2}$

3. Juft-toqligini tekshiring.

1) $y=x^2-4x^4$ 2) $y=x^3-5x^5$ 3) y=xsinx 4) y=xcosx

5) $y=\frac{1}{1-x}+\frac{1}{1+x}$ 6) y=thx 7) $y=\ln\frac{1-x}{1+x}$ 8) $y=\ln(\sqrt{1+x^2}-x)$

4. Chegaralanganligini tekshiring.

1) $y=\frac{x^2+x+1}{x^2+1}$ 2) y=2sinx+cosx 3) $y=\sqrt{9-x^2}$ 4) $y=2^{\sqrt{1-x}+\sqrt{x}}$

5) $y=\frac{x^2+x+2}{x^2+x+1}$ 6) $y=2^{\cos 2x}+5\sin x$ 7) $y=2^{1-\sin x}+2\cos x$ 8) y=2sin3x+5cos3x

5. Davrini aniqlang.

1) y=cosx+sinx 2) $y=\frac{\cos x}{4+\sin^2 x}$ 3) y=cosx+cos5x 4) $y=\cos 3\pi x+\sin 2\pi x$

5) $y=|\cos x|$ 6) y=ln (sinx) 7) $y=\ln\sin x$ 8) $y=\sin\sqrt{3}x+\cos\frac{\sqrt{3}}{2}x+tg7\sqrt{3}x$.

6. Monotonlikka tekshiring.

1) $y=x^3$ 2) $y=\log_2 x$ 3) $y=(\frac{1}{2})^x$ 4) $y=\frac{1}{x-1}$ 5) $y=\frac{2x+3}{x+1}$

7. Grafigini chizing.

1) y=x^2, y=$(x-1)^2$, y = $(x+1)^2$, y=$2x^2$, y=$\frac{1}{2}x^2$, y=$-2(x+1)^2$, y=x^2+1, y=$2(x+1)^2+1$, y=8x-$2x^2$, y=x^2-3x+2.
2) y=x^3, y=$-2x^3$, y=$(x-1)^3$, y=$2(x-1)^3+1$
3) y=$\frac{1}{x}$, y=$\frac{1}{x-1}$, y=$\frac{1}{x+1}$, y=$\frac{1-x}{1+x}$. 4) y=sin^2x, y=sin^3x, y=sin^2x+cos^2x
5) 1+x+e^x, y=x+sinx, y=xsinx, y=x sgn(sinx)
8. Tenglamani grafik usulda yeching. 1) $x^3-4x-1=0$ 2) lgx=0.1x
3) lgx=x.

14-mavzu. Funksiya va ketma-ketlik limiti

14.1. Ketma-ketlik va uning limiti

Ta'rif: Agar y=f(x) funksiya aniqlanish sohasi natural sonlar to'plami N bo'lsa, bu funksiya **ketma-ketlik** deyiladi, u a_n=f(n) o'rniga $\{a_n\}$ ko'rinishida belgilanadi.

Boshqacha aytganda, biror qonun bo'yicha har bir natural songa biror a_n son mos qo'yilgan bo'lsa, $\{a_n\}$ ketma-ketlik berilgan deyiladi.

$a_1, a_2, \ldots, a_n$ sonlar ketma-ketlik hadlari, a_n – umumiy hadi deyiladi.

Ketma-ketlik hadlari son o'qida tasvirlanadi.

Ketma-ketliklar ustida arifmetik amallar quyidagicha kiritiladi.

m$\{a_n\}=\{ma_n\}=\{ma_1, ma_2, ma_3, \ldots, ma_n, \ldots\}$

$\{a_n\} \pm \{b_n\}=\{a_n \pm b_n\}=\{a_1 \pm b_1; a_2 \pm b_2; a_3 \pm b_3; \ldots; a_n \pm b_n; \ldots\}$

$\{a_n\}\{b_n\}=\{a_n b_n\}=\{a_1, b_1; a_2 b_2; a_3 b_3; \ldots; a_n b_n; \ldots\}$

$\frac{\{a_n\}}{\{b_n\}}=\left\{\frac{a_n}{b_n}\right\}=\left\{\frac{a_1}{b_1}; \frac{a_2}{b_2}; \frac{a_3}{b_3}; \ldots \frac{a_n}{b_n}; \ldots\right\}$.

Ta'rif: $\{a_n\}$ ketma-ketlik uchun shunday m, M sonlar mavjud bo'lib, ixtiyoriy had uchun m$\le a_n \le$M o'rinli bo'lsa, ketma-ketlik chegaralangan deyiladi.

Agar C=max$\{|m|, |M|\}$ bo'lsa, chegaralanganlik shartini $|a_n| \le C$ ko'rinishida yozish mumkin.

$\{a_n\}$ ketma-ketlik chegaralanmagan deyiladi, agar har bir musbat C soni uchun $|a_n| > C$ shatrni qanoatlantiruvchi element topilsa.

Agar ixtiyoriy musbat C son uchun shunday N nomer topilsaki, n>N bo'lganda $|a_n| > C$ bajarilsa, bu $\{a_n\}$ ketma-ketlik cheksiz katta deyiladi.

Cheksiz katta ketma-ketlik chegaralanmagan, lekin chegaralanmagan ketma-ketlik cheksiz katta bo'lishi shart emas. Masalan, $\{1; 2; 1; 3; 1; 4; \ldots; 1; n; 1; n+1; \ldots\}$ ketma-ketlik chegaralanmagan, lekin cheksiz katta emas, chunki $|a_n| > C$ barcha toq nomerli hadlar uchun bajarilmaydi.

$\{a_n\}$ ketma-ketlik cheksiz kichik deyiladi, agar ∀>0 ∃N, n>N larda $|a_n| \le \varepsilon$ o'rinli bo'lsa.

Demak, $\{a_n\}$ cheksiz katta bo'lsa $\left\{\frac{1}{a_n}\right\}$ cheksiz kichik bo'ladi va aksincha.

$\{n\}$ cheksiz katta ekanligi ma`lum, demak, $\left\{\frac{1}{n}\right\}$ cheksiz kichikdir.

Ta'rif: a soni $\{a_n\}$ **ketma-ketlik limiti** deyiladi, agar ixtiyoriy musbat ε soni uchun shunday N nomer topilsaki, n>N larda $|a_n - a| < \varepsilon$ o'rinli bo'lsa.

Limitga ega ketma-ketlik yaqinlashuvchi, aks holda uzoqlashuvchi deyiladi.

Simvolik tarzda limit $\lim_{n\to\infty} a_n = a$ yoki $a_n \xrightarrow[n\to\infty]{} a$ tarzida yoziladi. "Limit" so'zi lotincha "limes" so'zidan olingan.

$\lim_{n\to\infty} \frac{n}{n+3} = 1$ ekanligini isbotlang.

$\forall\varepsilon>0$ olamiz. $|a_n - 1| = \left|\frac{n}{n+3} - 1\right| = \frac{3}{n+3} < \varepsilon$ dan $n+3 > \frac{3}{\varepsilon}$. Demak, $N=[\frac{3}{\varepsilon} - 3]$ deyilsa, n>N larda

$|a_n - 1| < \varepsilon$ bajariladi, demak, $\lim_{n\to\infty} \frac{n}{n+3} = 1$.

Cheksiz katta ketma-ketlik limitga ega emas. Lekin, uni cheksiz limitga ega deb, $\lim_{n\to\infty} a_n = \infty$ ko'rinishida yozish mumkin. Cheksiz kichiklarning yaqinlashuvchi va limiti nol ekanligi tushunarli.

Yaqinlashuvchu ketma-ketliklar quyidagi xossalarga ega:

Agar $\{a_n\}, \{b_n\}$ yaqinlashuvchi ketma-ketliklar bo'lsa, $Ca_n, a_n \pm b_n$, $a_n b_n$, $\frac{a_n}{b_n}$ ($b \neq 0$) ketma-ketliklar ham yaqinlashuvchi va

$\lim_{n\to\infty} Ca_n = C\lim_{n\to\infty} a_n;$ $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n;$

$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Ta'rif: Agar ixtiyoriy $n\epsilon N$ va $a_n < a_{n+1}$ bo'lsa, $\{a_n\}$ ketma-ketlik o'suvchi deyiladi, $a_n \le a_{n+1}$ bo'lsa, kamaymaydigan, $a_n > a_{n+1}$ bo'lsa, kamayuvchi, $a_n \ge a_{n+1}$ bo'lsa o'smaydigan deyiladi.

Bunday ketma-ketliklar umumiy nom bilan **monoton ketma-ketliklar** deyiladi. Ular hech bo'lamagnda bir tomondan chegaralangan bo'ladi.

Teorema. Monoton chegaralangan ketma-ketlik yaqinlashuvchidir.

Agar a soni o'suvchi $\{a_n\}$ elementlarini, masalan, yuqoridan chegaralasa, $a_n \le a$, u holda $\lim_{n\to\infty} a_n = a$ ekanligini isbotlash mumkin.

Teorema. Ichma-ich joylashgan $[a;b] \supset [a_1; b_1] \supset [a_2; b_2] \supset \ldots \supset [a_n; b_n] \ldots$ kesmalar uchun, ularning barchasiga tegishli yagona nuqta mavjud.

Misol. $a_n = (1 + \frac{1}{n})^n$ ketma-ketlik yaqinlashishini isbotlaymiz.

Buning uchun uning o'suvchi va yuqoridan chegaralanganligini ko'rsatamiz.

Nyuton binomi formulasiga ko'ra,

$a_n = (1 + \frac{1}{n})^n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}.\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}.\frac{1}{n^3} + \cdots + \frac{n(n-1)\ldots 1}{n!}.\frac{1}{n^n}$.

Bu ifodani quyidagicha yozish mumkin.

$a_n = 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{2}{n}) + \ldots + \frac{1}{n!}(1-\frac{1}{n})\ldots(1-\frac{n-1}{n})$.

U holda $a_{n+1} = 2 + \frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \ldots \frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+2})\ldots(1-\frac{n}{n+1})$

Bu hadlarda $1-\frac{k}{n} < 1-\frac{k}{n+1}$ bo'lganligi uchun $a_n < a_{n+1}$, $\{a_n\}$ ketma-ketlik o'suvchi ekan.

$$a_{n+1} = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} = 1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3$$

Demak, $\{a_n\}$ ketma-ketlik yuqoridan chegaralangan, limitga ega. Bu limit e harfi bilan belgilanadi, $2<e<3$. Aslida, e=2.7182818284590….

Misollar. 1) $\lim_{n\to\infty} \frac{1000n}{n^2+1} = \lim_{n\to\infty} \frac{1000}{n+\frac{1}{n}} = 0$. Chunki $\frac{1}{n} \to 0, \frac{1000}{n} \to 0$.

2) $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

3) $\lim_{n\to\infty}(\sqrt{2}.\sqrt[4]{2}.\sqrt[8]{2}\ldots..\sqrt[2^n]{2}) = \lim_{n\to\infty} 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}} = \lim_{n\to\infty} 2^{1-\frac{1}{2^n}} = 2$.

4) $\lim_{n\to\infty}(\frac{n+2}{n})^n = \lim_{n\to\infty}(1+\frac{2}{n})^{\frac{n}{2}\cdot 2} = [\lim_{n\to\infty}(1+\frac{2}{n})^{\frac{n}{2}}]^2 = e^2$.

14.2. Funksiya limiti

Biror a nuqta va unga yaqinlashuvchi $x_1, x_2, \ldots, x_n, \ldots$ a ϵX, ketma-ketlik berilgan bo'lsin. X to'plamda aniqlangan f(x) funksiya berilsa, $f(x_1)$, $f(x_2)$, …, $f(x_n)$, … ham sonli ketma-ketlik bo'ladi. Uning limiti mavjudligi masalasini ko'ramiz.

Ta'rif: Agar $x_n \to a$ da $f(x_n) \to A$ bo'lsa, A soni f(x) funksiyaning $x = a$ nuqtadagi limiti deyiladi va $\lim_{x\to a} f(x) = A$ ko'rinishida yoziladi. $\{f(x_n)\}$ ketma-ketlik yagona limitga egaligidan A son ham yagona bo'ladi.

Ta'rif: Agar ixtiyoriy $\forall\varepsilon > 0$ soni uchun shunday $\delta>0$ soni mavjud bo'lsaki, barcha $x\epsilon X$ lar uchun $|x - a| < \delta$ ekanligidan $|f(x) - A| < \varepsilon$ tengsizlik kelib chiqsa, A soni $f(x)$ ning $x = a$ nuqtadagi limiti deyiladi.

Birinchi ta'rifni sonli ketma-ketliklar tilidagi, ikkinchisini "$\varepsilon - \delta$ tilidagi" **limit ta'riflari** deyiladi va ular o'zaro ekvivalentdir.

Ta'rif: A soni $f(x)$ funksiyaning $x = a$ nuqtadagi chap (o'ng) limiti deyiladi, agar a ga $\{x_n\}$ elementlari chapdan (o'ngdan) yaqinlashganda $f(x_n)$ ketma-ketlik A ga yaqinlashsa.

Bu limitlar **bir tomonli limitlar** deyiladi va $\lim\limits_{x\to a-} f(x)=A$ ($\lim\limits_{x\to a+} f(x)=A$) ko'rinishida yoziladi.

Misol. $f(x)=sgnx$ funksiyaning chap limiti ($x = 0$ da) $\lim\limits_{x\to 0-} sgnx=-1$; $\lim\limits_{x\to 0+} sgnx=1$ bundan tashqari: $sgn0 = 0$.

Teorema. $f(x)$ funksiyaning $x = a$ nuqtada limiti mavjud bo'lishi uchun, bu nuqtada chap va o'ng limitlar mavjud va teng bo'lishi zarur va yetarlidir. Bu holda funksiya limiti ham bir tomonli limitlarga tengdir.

Ta'rif: A soni $f(x)$ ning $x\to\infty$ dagi limiti deyiladi, agar argumentning cheksiz katta qiymatli ketma-ketliklarida $\{f(x_n)\}$ ketma-ketlik A soniga yaqinlashsa. Uni $\lim\limits_{n\to\infty} f(x)=A$ ko'rinishida yoziladi.

Misol. 1) $\lim\limits_{x\to\infty}\frac{1}{x}=0$, 2) $\lim\limits_{x\to\infty}\frac{2x^2-5}{x^2+x}=\lim\limits_{x\to\infty}\frac{2-\frac{5}{x^2}}{1+\frac{1}{x}}=\frac{2}{1}=2$.

Agar biror tovar narxi x, unga talab y ekanligi berilib, ular bog'liqligi y=200:(x+2) bo'lsa, tovar narxi oshganda talab nolga intilishi kelib chiqadi.

Agar $\lim\limits_{x\to\infty} f(x)=A$, $\lim\limits_{x\to\infty} g(x)=B$ bo'lsa,

1) $\lim\limits_{x\to a}[f(x)\pm g(x)]=A\pm B$, 2) $\lim\limits_{x\to a}[f(x).g(x)]=AB$, 3) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{A}{B}$,
4) $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} h(x)=A$ bo'lib, qaralayotgan sohada $f(x)\le g(x)\le h(x)$ bo'lsa, $\lim\limits_{x\to\infty} g(x)=A$ ham o'rinlidir.

Misollar. 1) $\lim\limits_{x\to\infty}\frac{(3x+1)^{10}(4x-2)^{20}}{(2x-1)^{30}}=\lim\limits_{x\to\infty}\left(\frac{3x+1}{2x-1}\right)^{10}\left(\frac{4x-2}{2x-1}\right)^{20}=$
$\lim\limits_{x\to\infty}\left(\frac{3+\frac{1}{x}}{2-\frac{1}{x}}\right)^{10}\left(\frac{4-\frac{2}{x}}{2-\frac{1}{x}}\right)^{20}=\left(\frac{3}{2}\right)^{10}\left(\frac{4}{2}\right)^{20}=\frac{3^{20}}{2^{10}}.2^{20}=3^{10}.2^{10}=6^{10}$.

2. $\lim\limits_{x\to 2}\frac{x^3+x^2-8x+12}{x^3-5x^2+8x-4}=\lim\limits_{x\to 2}\frac{(x-2)^2(x+3)}{(x-2)^2(x-1)}=\lim\limits_{x\to 2}\frac{x+3}{x-1}=\frac{5}{1}=5$.

3. $\lim\limits_{x\to 2}\frac{\sqrt{7+x}-3}{\sqrt[3]{6+x}-2}=\lim\limits_{x\to 2}\frac{\sqrt{7+x}-3}{\sqrt[3]{6+x}-2}\cdot\frac{\sqrt{7+x}+3}{\sqrt{7+x}+3}\cdot\frac{\sqrt[3]{(6+x)^2}+2\sqrt[3]{6+x}+4}{\sqrt[3]{(6+x)^2}+2\sqrt[3]{6+x}+4}=$

$=\lim\limits_{x\to 2}\frac{(x-2)[\sqrt[3]{(6+x)^2}+2\sqrt[3]{6+x}+4]}{(x-2)[\sqrt{7+x}+3]}=\lim\limits_{x\to 2}\frac{\sqrt[3]{(6+x)^2}+2\sqrt[3]{6+x}+4}{\sqrt{7+x}+3}=\frac{4+4+4}{3+3}=\frac{12}{6}=2$.

4. $\lim\limits_{x\to\infty}[\sqrt{(x+a)(x+b)}-x]=\lim\limits_{x\to\infty}[\sqrt{(x+a)(x+b)}-x].\frac{[\sqrt{(x+a)(x+b)}+x]}{[\sqrt{(x+a)(x+b)}+x]}=$
$\lim\limits_{x\to\infty}\frac{(x+a)(x+b)-x^2}{\sqrt{(x+a)(x+b)}+x}=\lim\limits_{x\to\infty}=\frac{(a+b)x+ab}{x\left(\sqrt{\left(1+\frac{a}{x}\right)\left(1+\frac{b}{x}\right)}+1\right)}=\lim\limits_{x\to\infty}\frac{(a+b)+\frac{ab}{x}}{\sqrt{\left(1+\frac{a}{x}\right)\left(1+\frac{b}{x}\right)}+1}=\frac{a+b}{2}$.

1. Birinchi ajoyib limit. $\lim\limits_{x\to 0}\frac{sinx}{x}=1$ ekanligini isbotlaymiz.

Isbotlash uchun markazi koordinata boshida bo'lgan, radiusi bir doirani qaraymiz. Radian o'lchovi $x(0<x<\frac{\pi}{2})$ bo'lgan markaziy burchakni qaraymiz. $S_{\Delta AOB}<S_{\Delta AOB\ sektor}<S_{\Delta AOC}$ ekanligidan $\frac{1.sinx}{2}<\frac{x.1^2}{2}<\frac{1.tgx}{2}$ yoki $sinx<x<tgx$.

Tengsizlik tomonlarini $sinx$ ga bo'lsak, $1<\frac{x}{sinx}<\frac{1}{cosx}$ $\frac{x}{sinx}, cosx$ funksiyalari juftligi uchun olingan tengsizlik; $-\frac{\pi}{2}<x<0$ da ham o'rinli bo'ladi. $\lim\limits_{x\to 0}1=\lim\limits_{x\to 0}cosx=1$ va 4^0 xossa yordamida $\lim\limits_{x\to 0}\frac{x}{sinx}=1$ ekanligini olamiz. Demak, $\lim\limits_{x\to 0}\frac{x}{sinx}=1$, $\lim\limits_{x\to 0}\frac{sinx}{x}=1$.

Misollar. 1) $\lim\limits_{x\to 0}\frac{tgx}{x}=\lim\limits_{x\to 0}\frac{sinx}{x}.\frac{1}{cosx}=1$

2) $\lim\limits_{x\to 0}\frac{1-cos4x}{x^2}=\lim\limits_{x\to 0}\frac{2sin^2 2x}{x^2}=\lim\limits_{x\to 0}8\left(\frac{sin2x}{2x}\right)^2=8$.

2. Ikkinchi ajoyib limit. $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Isbotlash uchun, avvaldan ma'lum $\lim\limits_{x\to 0}(1+\frac{1}{n})^n=e$ dan foydalanamiz. So'ngra $x=\frac{1}{n}$ almashtirish yordamida ikkinchisi isbotlanadi.

Misollar. 1) $\lim\limits_{x\to 0}(1+5x)^{\frac{1}{x}}=\lim\limits_{x\to 0}(1+5x)^{\frac{1}{5x}\cdot 5}=e^5$.

2. $\lim\limits_{x\to\infty}(1+\frac{5}{x})^{3x}=\lim\limits_{x\to\infty}[\left(1+\frac{5}{x}\right)^{\frac{x}{5}}]^{\frac{5}{x}\cdot 3x}=\lim\limits_{x\to\infty}[\left(1+\frac{5}{x}\right)^{\frac{x}{5}}]^{15}=e^{15}$.

Misol. P% yillik foyda beradigan bankka Q_0 miqdorda omonat qo'yildi. t yildan so'ng qo'yilgan omonat Q_t qancha bo'lishini toping.

Har yilda qo'yilgan omonat $(1+\frac{p}{100})$ marta oshadi. $Q_1=Q_0(1+\frac{p}{100})$, $Q_2=Q_0(1+\frac{p}{100})^2,\dots,$ $Q_t=Q_0(1+\frac{p}{100})^t$. uzluksiz berilsa, t yildan so'ng omonat, $Q_t=\lim\limits_{n\to\infty}[Q_0(1+\frac{p}{100n})^{nt}]=$ $Q_0\lim\limits_{n\to\infty}[(1+\frac{p}{100n})^{\frac{100n}{p}}]^{\frac{pt}{100}}=Q_0e^{\frac{pt}{100}}$. ko'rinishida bo'ladi.

Ta'rif: Agar $\lim\limits_{x \to x_0(\infty)} \alpha(x) = 0$ bo'lsa, $\alpha(x)$ funksiya cheksiz kichik miqdor deyiladi. Agar $\lim\limits_{x \to x_0(\infty)} f(x) = A$ bo'lsa, $f(x) = A + \alpha(x)$ bo'ladi.

Cheksiz kichik miqdorlar yig'indisi (ayirmasi), ko'paytmasi yana cheksiz kichik miqdor bo'lishi ravshan.

Ta'rif: $f(x)$ funksiya $x \to x_0$ da cheksiz katta miqdor deyiladi, agar yetarli katta M>0 uchun shunday δ>0 mavjud bo'lsaki, $|x - x_0| < \delta$ shartga bo'ysinuvchi x lar uchun $|f(x)| > M$ bo'lsa,va quyidagicha yoziladi: $\lim\limits_{x \to x_0} f(x) = \infty$

Misol. 1) $x \to \frac{\pi}{2}$ da tgx, $x \to \infty$ da $\sqrt{5x-7}$ lar cheksiz kattadir.

Agar $\lim\limits_{x \to x_0 \infty} \alpha(x) = 0$ cheksiz kichik bo'lsa, $f(x) = \frac{1}{\alpha(x)}$ funksiya $x \to x_0(\infty)$ da cheksiz katta bo'ladi.

Iqtisodiyotda tovarlar ikki xilga ajratiladi: zaruriy (masalan, non) va to'kinlikni bildiruvchi (masalan, mashina). Ularni mos ravishda y(x), va z(x) desak, $y(x) = \frac{b_1(x-a_1)}{x-c_1}, x \geq a_1$, $z(x) = \frac{b_2 x(x-a_2)}{x-c_2}, x \geq a_1, a_2 \geq a_1$ bo'ladi. Tovarlar soni cheksiz kattalashtirilsa, birinchisining limiti o'zgarmas songa, ikkinchisiniki esa cheksizlikka tenglashadi.

15-mavzu. Funksiyaning uzluksizligi

f(x) funksiya x_0 nuqtaning biror atrofida aniqlangan bo'lsin.

Ta'rif: f(x) funksiya $x = x_0$ nuqtada uzluksiz deyiladi, agar bu nuqtada funksiyaning limiti va qiymati teng bo'lsa, ya'ni $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

$\lim\limits_{x \to x_0} x = x_0$ ekanligidan $\lim\limits_{x \to x_0} f(x) = f\left(\lim\limits_{x \to x_0} x\right)$ kelib chiqadi, ya'ni uzluksiz funksiyalarda limit va funksiya belgisi o'rinlarini almashtirish mumkin.

Ketma-ketliklar tilida funksiya uzluksizligi quyidagichadir: $f(x)$ funksiya x_0 nuqtada uzluksiz deyiladi, agar x_0 ga yaqinlashuvchi $x_1, x_2, x_3, \ldots, x_n, \ldots$ ketma-ketlik uchun mos $f(x_1), f(x_2), f(x_3), \ldots, f(x_n), \ldots$ ketma-ketlik $f(x_0)$ ga yaqinlashsa.

"$\varepsilon - \delta$ tilida" bu ta'rif quyidagicha bo'ladi.

$f(x)$ funksiya x_0 nuqtada uzluksiz deyiladi, agar ixtiyoriy $\varepsilon > 0$ uchun shunday δ>0 mavjud bo'lsaki, $|x - x_0| < \delta$ shartga bo'ysunuvchi x lar uchun $|f(x) - f(x_0)| < \varepsilon$ tengsizlik o'rinli bo'lsa.

Agar $\lim\limits_{x \to x_0+} f(x) = f(x_0)(\lim\limits_{x \to x_0-} f(x) = f(x_0))$ bo'lsa, $f(x)$ funksiya x_0 **nuqtada o'ngdan (chapdan) uzluksiz** deyiladi.

$\Delta x = x - x_0$, $\Delta y = f(x_0 + \Delta x) - f(x_0)$ kattaliklar mos ravishda argument va funksiyaning x_0 nuqtadagi orttirmasi deyiladi.

Uzluksizlik bu tilda $\lim\limits_{\Delta x \to 0} \Delta y = 0$ ko'rinishida yoziladi.

Teorema: Agar x_0 nuqtada $f(x), g(x)$ funksiyalar uzluksiz bo'lsa, $f \pm g$, $f.g$, $\frac{f}{g}$ $(g \neq 0)$ funksiyalar ham bu nuqtada uzluksiz bo'ladi.

Algebraik ko'phadlar, $sinx, cosx, |x|$ kabi funksiyalar ixtiyoriy nuqtada uzluksizdir. [x],{x}, *sgnx, tgx, ctgx* kabi funksiyalar uzluksiz bo'lmaydigan nuqtalarni ko'rsatish mumkin.

Ta'rif: Agar x_0 nuqtada $f(x)$ funksiya uzluksiz bo'lmasa, u holda x_0 nuqta $f(x)$ funksiya uchun **uzilish nuqtasi** deyiladi.

x_0 uzilish nuqtasi I tur deyiladi, agar $\lim\limits_{x \to x_0-} f(x) \neq \lim\limits_{x \to x_0+} f(x)$ o'rinli bo'lsa.

Masalan, $f(x) = sgnx$ uchun $x = 0$ nuqta I tur uzilish nuqtasidir, chunki $\lim\limits_{x \to x_0-} sgnx = -1$, $\lim\limits_{x \to x_0+} sgnx = 1$

x_0 uzilish nuqtasi II tur deyiladi, agar x_0 nuqtada hech bo'lmaganda bitta bir tomonlama limit mavjud emas yoki cheksiz bo'lsa.

Masalan, $f(x) = \frac{1}{x}$ uchun $x = 0$ nuqta II tur uzilish nuqtasidir, chunki

$\lim\limits_{x \to x_0 -} \frac{1}{x} = -\infty, \lim\limits_{x \to x_0 +} \frac{1}{x} = +\infty.$

Chekli nuqtada I tur uzilishga ega, qolgan nuqtalarda uzluksiz funksiyalar qaralayotgan sohada bo'lakli uzluksiz deyiladi.

Masalan, $f(x) = [x]$, $f(x) = \{x\}$ funksiyalari bo'lakli uzluksizdir.

Agar x_0 nuqtada $f(x)$ funksiya uchun $\lim\limits_{x \to x_0 -} f(x) = \lim\limits_{x \to x_0 -} f(x) \neq f(x_0)$ munosabatlar o'rinli bo'lsa, x_0 nuqta yo'nalishining **mumkin bo'lgan uzilish nuqtasi** deyiladi.

Uzluksiz funksiyalarning asosiy xossalarini keltiramiz

Teorema (Bolsano-Koshi birinchi teoremasi): $f(x)$ funksiya [a;b] kesmada uzluksiz va $f(a).f(b) < 0$ bo'lsin. U holda shunday c∈(a,b) mavjudki, $f(c) = 0$ bo'ladi.

Isboti: [a;b] kesmani teng ikkiga bo'lamiz. Agar o'rta nuqtada $f(x) = 0$ bo'lsa teorema isbotlanadi, aks holda bo'laklardan chegaralardagi ishoralari turlichasini $[a_1; b_1]$ deb olib, uni ham teng ikkiga bo'lamiz. Natijada, ichma-ich joylashgan $[a;b] \supset [a_1; b_1] \supset [a_2; b_2] \supset \ldots [a_n; b_n] \supset \ldots$ oraliqlar paydo bo'lib, ularning umumiy nuqtasi c da $f(c) = 0$ dir.

Teorema (Bolsano-Koshining ikkinchi teoremasi): $f(x)$ funksiya [a;b] kesmada uzluksiz, $f(a) = A$, $f(b) = B$ bo'lsin. Agar $A < C < B$ bo'lsa, shunday c∈(a;b) mavjudki, $f(c) = C$ bo'ladi.

Teorema (Veyershtrassning birinchi teoremasi): Agar, $f(x)$ funksiya [a;b] kesmada aniqlangan, uzluksiz bo'lsa, bu kesmada chegaralangan hamdir.

Teorema (Veyershtrassning ikkinchi teoremasi): Agar, $f(x)$ funksiya [a;b] kesmada uzluksiz bo'lsa, u holda funksiya bu oraliqda aniq quyi (yuqori) chegarasiga erishadi, ya'ni shunday $x_1, x_2 \epsilon [a; b]$ mavjudki, $f(x_1) = M = \sup\limits_{[a;b]} f(x)$, $f(x_2) = m = \inf\limits_{[a;b]} f(x)$

Isboti elementar funksiyalar uzluksizligiga asoslangan muhim limitlarni qarab chiqamiz:

I. $\lim\limits_{x \to 0} \frac{(1+x)^p - 1}{x} = p$.

Isboti. Nyuton binomi formulasiga ko'ra:

$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + x^p.$

Demak,

$$\lim_{x \to 0} \frac{(1+x)^p - 1}{x} = \lim_{x \to 0} \frac{1 + px + \frac{p(p-1)}{2!} x^2 + \cdots + x^p - 1}{x} = \lim_{x \to 0} \frac{x[p + \frac{p(p-1)}{2!} x + \cdots + x^{p-1}]}{x} = p.$$

II. $\lim\limits_{x \to 0} \frac{\log_a(1+x)}{x} = \lim\limits_{x \to 0} \frac{1}{x} \log_a(1 + x) = \lim\limits_{x \to 0} \log_a(1 + x)^{\frac{1}{x}} =$ $\log_a \left(\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} \right) = \log_a e$

III. $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$.

Isboti. $a^x - 1 = t$ almashtirish o'tkazamiz. Undan x=$\log_a(1 + t)$ kelib chiqadi. x→ 0 da $t \to 0$ ekanligini hisobga olib,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{t \to 0} \frac{t}{\log_a(1+t)} = \frac{1}{\log_a e} = \ln a.$$

Misollar. 1) $\lim\limits_{x \to 0} \frac{(1+\sin^2 3x)^{10} - 1}{x^2} = \lim\limits_{x \to 0} \frac{(1+\sin^2 3x)^{10} - 1}{\sin^2 3x} . (\frac{\sin 3x}{3x})^2 .9 = 90$

2) $\lim\limits_{x \to 0} \frac{\ln(1+tg2x)}{x} = \lim\limits_{x \to 0} \frac{\ln(1+tg2x)}{tgx} . \frac{tg2x}{2x} .2 = 2$

3) $\lim\limits_{x \to 0} \frac{8^x + 4^x - 2}{4^x - 2^x} = \lim\limits_{x \to 0} \frac{(8^x - 1) + (4^x - 1)}{(4^x - 1) - (2^x - 1)} = \lim\limits_{x \to 0} \frac{\frac{8^x - 1}{x} + \frac{4^x - 1}{x}}{\frac{4^x - 1}{x} - \frac{2^x - 1}{x}} = \frac{\ln 8 + \ln 4}{\ln 4 - \ln 2} = \frac{\ln 32}{\ln 2} = 5$

$$\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{\ln(1+\sin 4x)}} = \lim_{x \to 0} \left[1 + \frac{(a^x - 1) + (b^x - 1)}{2} \right]^{\frac{2}{(a^x-1)+(b^x-1)} \cdot \frac{(a^x-1)+(b^x-1)}{2\ln(1+\sin 4x)}} = e^{\frac{1}{2} \lim\limits_{x \to 0} \frac{\frac{a^x-1}{x} + \frac{b^x-1}{x}}{\frac{\ln(1+\sin 4x)}{\sin 4x} \cdot \frac{\sin 4x}{4x} \cdot 4}} =$$

4) $e^{\frac{1}{8}[\ln a + \ln b]} = e^{\ln(ab)^{\frac{1}{8}}}$ $= \sqrt[8]{ab}.$

Mavzuga doir misol va masalalar

1. Tengliklarni isbotlang.

1) $\lim_{n \to \infty} \frac{n}{2^n} = 0$, 2) $\lim_{n \to \infty} \sqrt[n]{a} = 1$, a>0, 3) $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

2. Quyidagilarni toping.

1) $\lim\limits_{n \to \infty} \frac{4n^2 - 5n^2 + 1}{1 + n + n^2}$; 2) $\lim\limits_{n \to \infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$; 3) $\lim\limits_{n \to \infty} \frac{1 + 2 + \cdots + (n-1)}{n^2}$;

4) $\lim\limits_{n \to \infty} \frac{1 + a + a^2 + \cdots + a^n}{1 + b + b^2 + \cdots + b^n}$; (IaI<1; IbI<1), 5) $\lim\limits_{n \to \infty} \frac{1}{n^3} [1^2 + 2^2 \ldots + (n-1)]^2$;

6) $\lim\limits_{n \to \infty} [\frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{n(n+1)}]$; 7) $\lim\limits_{n \to \infty} \frac{1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2}{[1 + 4 + 7 + \cdots + (3n-2)]^2}$.

3. Funksiya limiti ta'rifidan foydalanib isbotlang

1) $\lim_{n\to2}(2x-1)=3$; 2) $\lim\limits_{n\to1} x^3=1$; 3) $\lim\limits_{n\to x_0}\frac{1}{x}=\frac{1}{x_0}$; 4) $\lim\limits_{n\to x_0}\sin x=\sin x_0$

4. Limitlarni toping.

1) $\lim\limits_{n\to1}\frac{x^3+1-2x^2-2}{x^2-4}$; 2) $\lim\limits_{n\to1}\frac{x^2+1-2x^2-2}{x^2-4}$; 3) $\lim\limits_{n\to2}\frac{(x^2-7x+10)^{20}}{(x^3-5x^2+2+8)^{10}}$;

4) $\lim\limits_{n\to1}\frac{x+x^2+\cdots x^n-n}{x-1}$; 5) $\lim\limits_{n\to1}\frac{x^m-1}{x^n-1}$; 6) $\lim\limits_{n\to1}(\frac{m}{1-x^m}-\frac{n}{1-x^n})$;

5. Limitlarni toping.

1) $\lim\limits_{n\to2}\frac{\sqrt{7+x}-3}{2-\sqrt{2+x}}$; 2) $\lim\limits_{n\to2}\frac{2-\sqrt[3]{6+x}}{\sqrt{14+x-4}}$; 3) $\lim\limits_{n\to16}\frac{\sqrt[4]{x^3}-8}{\sqrt{x}-4}$; 4) $\lim\limits_{n\to1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$;

5) $\lim\limits_{n\to+\infty}(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x})$; 6) $\lim\limits_{n\to+\infty}(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x})$;

6. Ajoyib limitlar yordamida toping.

1) $\lim\limits_{n\to0}\frac{\sin mx^2}{nx^2}$; 2) $\lim\limits_{n\to0}\frac{\sin x}{\sin 6x-\sin 7x}$; 3) $\lim\limits_{n\to0}\frac{\sin mx}{\sin nx}$;

4) $\lim\limits_{n\to0}\frac{\tan x-\sin x}{\sin^3 x}$; 5) $\lim\limits_{n\to0}\frac{\arcsin 5x}{x}$; 6) $\lim\limits_{n\to1}\frac{\sin 7\pi x}{\sin 2\pi x}$;

7) $\lim\limits_{n\to0}\frac{\sqrt{1+tgx}-\sqrt{1+\sin x}}{x^3}$; 8) $\lim\limits_{n\to0}(1+2x)^{\frac{1}{x}}$; 9) $\lim\limits_{n\to0}(1+3tg^2x)^{ctg^2x}$;

10) $\lim\limits_{n\to0}\frac{(1+4x)^{10}-1}{\sin x}$; 11) $\lim\limits_{n\to0}\frac{(1+tg^2 4x)^{10}-1}{x^2}$; 12) $\lim\limits_{n\to0}\frac{2^{tg^2x}-1}{x^2}$;

13) $\lim\limits_{n\to0}\frac{e^{ax}-e^{bx}}{\sin ax-\sin bx}$; 14) $\lim\limits_{n\to0}\frac{e^{x^2}-\cos x}{\sin^2 x}$; 15) $\lim\limits_{n\to0}(\frac{1+x2^x}{1+x3^x})^{\frac{1}{x^2}}$;

16) $\lim\limits_{n\to0}(\frac{a^x+b^x+c^x}{3})^{\frac{1}{x}}$; (a>0, b>0, c>0); 17) $\lim\limits_{n\to\frac{\pi}{2}}(\sin x)^{tg^2x}$;

7. Uzliksizlika tekshiring, uzilish nuqtasi tipini aniqlang .

1) $y=\frac{x^2-1}{x-1}$; 2) $y=\{\frac{x^2-1}{x-1};\ x\neq1$; 3) $y=\frac{1}{x-1}$; 4) $y=\frac{1}{1+2^{\frac{1}{x-1}}}$;

5) $y=[x^2]$; 6) y=x[x]; 7) $y=\lim\limits_{n\to0}\cos^{2n}x$.

Mavzuga doir joriy nazorat uchun uy vazifasi
(N-talabaning jurnaldagi nomeri)

1. Quyidagi funksiyalar aniqlanish sohasini toping.

1) $y=\sqrt{\frac{Nx-x^2}{x^2-Nx-2N^2}}$; 2) $y=\log_{Nx}\frac{N-x}{Nx+1}$; 3) $y=\arcsin(\lg\frac{x}{N})$;

4) $y=\sqrt{A-\sin Nx+(-1)^N\cos Nx}$;

5) bunda A={$\frac{1}{2}$, agar N=3k-2, $\frac{\sqrt{2}}{2}$, agar N=3k-1, $\frac{\sqrt{3}}{2}$, agar N=3k.

2. Quyidagi funksiyalar juft toqligini tekshiring.

1) $Y=x^N-Nx^{N+2}$; 2) $y=\frac{1}{N-x}+\frac{1}{N+x}+x^N$; 3) $y=\ln\frac{N-x}{N+x}$.

3. Quyidagi funksiyalar chegaralanganligini tekshiring.

1) $y=\frac{x^2+Nx+N}{x^2+N}$; 2) y=NsinNx-2NcosNx.

4. Quyidagi funksiyalar monotonligini tekshiring.

1) $Y=x^N$; 2) $y=\log_{N+1}x$; 3) $y=\frac{Nx+2}{x+1}$;

5. Quyidagi funksiyalar davrini aniqlang.

1) y=sinNx+cos(N+1)x; 2) $y=\sin\frac{N\pi}{2}x-\cos\frac{N\pi}{2}x$;

6. Quyidagi funksiyalar grafigini chizing.

1) $Y=|2x^2-6Nx+4N^2|$; 2) $y=-2\sin2(x-\frac{\pi}{N})+1$;

3) $y=|x-\frac{N}{2}|+|x+\frac{N}{2}|-|x-N|$

7. Limitlarni toping.

1) $\lim\limits_{n\to\infty}\frac{N\cdot n^{N+1}-n+1}{1+n-n^{N+1}}$ 2) $\lim\limits_{x\to N}\frac{\sqrt{N^2-N+}-N}{\sqrt[3]{x-N+8}-2}$ 3) $\lim\limits_{x\to0}\frac{1-\cos Nx}{Nx^2}$ 4) $\lim\limits_{x\to0}(1+tg^2Nx)$

5) $\lim\limits_{x\to0}\frac{(1+\sin^2Nx)^{N+5}-1}{\sin^2(N+)x}$ 6) $\lim\limits_{x\to0}\frac{\log_{N+1}(1+tgNx)}{x}$ 7) $\lim\limits_{x\to0}\frac{(N+4)^x-(N+2)^x}{(N+3)^x+(N+1)^x-2}$.

$$8)\ f(x)=\begin{cases}-N-x, & x\le -N\\ (x+N)^2, & -N<x\le0\\ N-2x, & 0<x<\frac{N}{2}\\ N, & x\ge\frac{N}{2}\end{cases}$$

Funksiya uzilish nuqtalarini toping, sxematik tasvirlang.

16-mavzu. Hosila mazmuni, hisoblash qoidalari

16.1. Hosila ta'rifi, mazmuni

X oraliqda aniqlangan f(x) funksiyani ko'rib chiqamiz. Biror $x \epsilon X$ga Δx orttirma beramiz, $x+\Delta x$ ham X ga tegishli bo'lsin. Funksiya:

$\Delta y = f(x+\Delta x)-f(x)$ orttirma oladi.

Ta'rif. y=f(x) funksiyaning x nuqtadagi **hosilasi** deb, funksiya orttirmasining argument orttirmasiga nisbatini, argument orttirmasi nolga intilgandagi limiti $\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}$ ga aytiladi va $y'(x)$, $f'(x)$, $\frac{df}{dx}$ ko'rinishida belgilanadi. $f'(x)=\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Agar $\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\pm\infty$ bo'lsa, funksiya bu nuqtada cheksiz hosilaga ega deyiladi.

Ta'rifni $x_1, x_2 \epsilon X$ nuqtalar uchun $f'(x)=\lim_{x_2 \to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$ ko'rinishida kiritish mumkin.

Misollar. 1) $y=x^p$ $(p\neq-1)$ funksiyaning ixtiyoriy $x\epsilon R$ nuqtadagi hosilasini hisoblaymiz:

$$y'(x)=\lim_{\Delta x \to 0}\frac{(x-\Delta x)^p-x^p}{\Delta x}=\lim_{\Delta x \to 0}x^p\frac{(1+\frac{\Delta x}{\Delta x})^p-1}{x\frac{\Delta x}{x}}=px^{p-1};$$

$$2)\ y=\sin x,\ y'=\lim_{\Delta x \to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\lim_{\Delta x \to 0}\frac{2\sin\frac{\Delta x}{2}\cos\frac{2x+\Delta x}{2}}{2\frac{\Delta x}{x}}=\cos x$$

$$3)\ y=\cos x,\ y'=\lim_{\Delta x \to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}=\lim_{\Delta x \to 0}\frac{-2\sin\frac{\Delta x}{2}\sin\frac{2x+\Delta x}{2}}{2\frac{\Delta x}{x}}=-\sin x$$

$$4)\ y=a^x\ (a>0,\ a\neq 1),\ y'=\lim_{\Delta x \to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}=a^x\lim_{\Delta x \to 0}\frac{a^{\Delta x}-1}{\Delta x}=a^x\ln a$$

Xususan, $(e^x)'=e^x$.

Hosilaning geometrik ma'nosi

y=f(x) funksiya uchun f(x)=A, $f(x+\Delta x)$=B bo'lsin. A va B dan o'tuvchi to'g'ri chiziq f(x)ga kesuvchi bo'ladi, uning Ox musbat yo'nalishi bilan hosil qilgan burchagi φ bo'lsin.

Agar $\Delta x \to 0$ bo'lsa, B nuqta A nuqtaga yaqinlashadi, kesuvchi to'g'ri chiziq A(x;f(x)) nuqtadan o'tuvchi urinmaga aylanadi, $y'=\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\mathrm{tg}\alpha$ hosil bo'ladi. Bu hosilaning x nuqtadagi qiymati urinmaning ox o'q musbat yo'nalishi bilan hosil qilgan burchagi tangensiga teng ekanligini bildiradi .

Masalan $E(x_0;f(x_0))$ nuqtadan o'tadigan nuqta urinma tenglamasini $y-y_0=k(x-x_0)$ ko'rinishini o'zgartirib, $y-f(x_0)=f'(x_0)(x-x_0)$ tarzida yozish mumkin bo'ladi.

Hosilaning fizik ma'nosi

Moddiy nuqta S=S(t) qonuniyati bilan harakatlanayotgan bo'lsin. Unda t_1 vaqtgacha $S(t_1)$, t_2 vaqtgacha $S_1(t_2)$ yo'l bosiladi.

$S'(t_1)=\lim_{t_2 \to t_1}\frac{S(t_2)-S(t_1)}{t_2-t_1}=v(t_1), v'(t_1)=\lim_{t_2 \to t_1}\frac{v(t_2)-v(t_1)}{t_2-t_1}=a(t_1)$,

munosabatlar bosib o'tilgan yo'l hosilasi tezlik, tezlik hosilasi esa tezlanish ekanligini bildiradi.

Hosilaning iqtisodiy ma'nosi

Biror t vaqt ishlab chiqarilgan mahsulotni U=U(t) funksiya ifodalasin. t dan $t+\Delta t$ vaqtgacha ishlab chiqarilgan mahsulot soni U=U(t) dan $U+\Delta U = U(t+\Delta t)$ gacha o'zgaradi. Unda o'rtacha mehnat unumdorligi $Z_{o'rt}=\frac{\Delta U}{\Delta t}$.

Demak, t vaqtdagi unumdorligi $Z=\lim_{\Delta t \to 0} Z_{o'rt}=\lim_{\Delta t \to 0+}\frac{\Delta U}{\Delta t}=U'(t)$ ko'rinishda topiladi, ya'ni ishlab chiqarilgan mahsulot hajmi hosilasi t vaqtdagi mehnat unumdorligi ekan.

Ta'rif: $f'_+(x)=\lim_{\Delta x \to 0+}\frac{\Delta y}{\Delta x}$ $(f'_-(x)=\lim_{\Delta x \to 0-}\frac{\Delta y}{\Delta x})$ limitlar **o'ng (chap) hosilalar** deyiladi.

Agar f(x) funksiya x nuqtada hosilaga ega bo'lsa, bu nuqtada bir tomonli hosilalar mavjud va teng bo'lishi zarur.

$f(x)=|x|$ uchun x=0 nuqtada $f'_+(0)=\lim_{\Delta x \to 0+}\frac{\Delta y}{\Delta x}=1$, $f'_-(0)=\lim_{\Delta x \to 0-}\frac{\Delta y}{\Delta x}=-1$ bo'lganligi uchun berilgan funksiyaning x=0 nuqtada hosilasi mavjud emas.

Dastlab, o'zgarmas son hosilasi nolga, y=x funksiya hosilasi esa birga tengligini aytib o'tamiz, chunki, $(C)'=\lim_{\Delta x \to 0}\frac{c-c}{\Delta x}=0$, $(x)'=\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x \to 0}\frac{\Delta x}{\Delta y}=1$.

16.2. Hosila hisoblash qoidalari

Endi hosilasi mavjud U(x), V(x) funksiyalar berilgan deb hisoblab, hosilani hisoblash qoidalarini keltirib chiqaramiz.

1. Agar U=U(x), V=V(x) funksiyalar x nuqtada hosilalarga ega bo'lsa, ularning yig'indisi, ayirmasi, ko'paytmasi, bo'linmasi ($v'(x) \neq 0$), songa ko'paytmasi ham hosilaga ega bo'lib, quyidagi qoidalar o'rinli:

$$(C_1U \pm C_2V)' = C_1U' \pm C_2V',\ (UV)' = U'V + UV',\ \left(\frac{U}{V}\right)' = \frac{U'V - UV'}{V^2}.$$

Isboti. $(C_1U \pm C_2V) = \lim\limits_{\Delta x \to 0} \frac{[(C_1U(x+\Delta x) \pm C_2V(x+\Delta x)] - [(C_1U(x) \pm C_2V(x)]}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \left[\frac{C_1U(x+\Delta x) - C_1V(x)}{\Delta x} \pm \frac{C_2V(x+\Delta x) - C_2V(x)}{\Delta x}\right] = C_1U \pm C_2V.$

$(UV) = \lim\limits_{\Delta x \to 0} \frac{U(x+\Delta x)V(x+\Delta x) - U(x)V(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{[U(x)+\Delta U][V(x)+\Delta x] - UV}{\Delta x} =$

$\lim\limits_{\Delta x \to 0} \frac{U(x)V(x) + \Delta UV(x) + U(x)\Delta V + \Delta U\Delta V - UV}{\Delta x} = UV + UV.$

$\left(\frac{U}{V}\right)' = \lim\limits_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{U(x+\Delta x)}{V(x+\Delta x)} - \frac{U(x)}{V(x)}\right] = lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x+\Delta x)}{\Delta x\, v(x+x)v(x)} =$

$lim_{\Delta x \to 0} \frac{[u(x)+\Delta u]v(x) - u(x)[v(x)+\Delta v]}{\Delta x\, v(x+\Delta x)v(x)} = lim_{\Delta x \to 0} \frac{uv + \Delta uv - uv - u\Delta v}{\Delta x\, v(v+\Delta v)} = \frac{u'v - uv'}{v^2}$

Bu qoidalardan

$(c_1u_1 \pm c_2u_2 \pm \cdots \pm c_nu_n) = c_1u_1' \pm c_2u_2' \pm \cdots \pm c_nu_n'[(\sum_{k=1}^{n} c_ku_k) = \sum_{k=1}^{n} c_ku_k']$

$(u_1u_2 \ldots u_n)' = u_1'u_2 \ldots u_n + u_1u_2' \ldots u_n + \ldots + u_1u_2 \ldots u_n'[(\prod_{k=1}^{n} u_k)' = \sum_{k=1}^{n} u_1u_2 \ldots u_k' \ldots u_n]$,

umumiy qoidalarni ham keltirib chiqarishi mumkin.

Misollar.

1) $(\mathrm{tg}x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos)'}{\cos^2 x} = \frac{(\sin x)'\cos x - \sin x(\cos)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

2) Shunga o'hshash $(\mathrm{ctg}x)' = -\frac{1}{\cos^2 x}.$

2. Teskari funksiya hosilasi

Teorema. Agar y=f(x) funksiya biror x nuqtada $f'(x) \neq 0$ hosilaga ega, $x=\varphi(y)$ uning teskari funksiyasi bo'lsa, $\varphi'(y) = \frac{1}{f'(x)}$ tenglik o'rinlidir.

Isbot. $y_x' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \to 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{x_y'}$

Bu teorema sodda geometrik ma'noga ega. y=f(x) ga x nuqtada o'tkazilgan urinma ox o'qi bilan α burchak hosil qilsa, oy o'qi bilan β burchak hosil qiladi va $\alpha + \beta = \frac{\pi}{2}$, $f'(y) = \mathrm{tg}\beta$.

Isbot esa $\varphi'(y) = \mathrm{tg}\beta = \frac{1}{\mathrm{ctg}\beta} = \frac{1}{\mathrm{ctg}(\frac{\pi}{2} - \alpha)} = \frac{1}{\mathrm{tg}\alpha} = \frac{1}{f'(x)}$ tengliklardan kelib chiqadi.

Misol. 1) y=arcsinx funksiya va x=siny funksiyalar o'zaro teskari funksiyalar ekanligidan $(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ kelib chiqadi.

2) $(\arccos x)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}};$

3) $(\mathrm{arctg}x)' = \frac{1}{(\mathrm{tg}y)'} = \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{\frac{1}{1+\mathrm{tg}^2 y}} = \frac{1}{1+x^2};$

4) $(\mathrm{arcctg}x)' = \frac{1}{(\mathrm{ctg}y)'} = \frac{1}{-\frac{1}{\sin^2 y}} = -\frac{1}{1+\mathrm{ctg}^2 y} = -\frac{1}{1+x^2};$

5) $(\log_a x)' = \frac{1}{(a^y)'} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$, xususan, $(\ln x)' = \frac{1}{x}$

3. Murrakab funksiya hosilasi

Teorema. Agar $x=\varphi(t)$ funksiya $t=t_0$ nuqtada hosilaga ega bo'lsa, y=f(x) funksiya esa mos $x_0=\varphi(t_0)$ nuqtada hosilaga ega bo'lsa, u holda $y=f(\varphi(t))$ murrakab funksiya ham t_0 nuqtada hosilaga ega va $y'=[f(\varphi(t))]'=f'(x_0)\varphi'(t_0)$ o'rinlidir.

Isbot. $y'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t} = y_x' \cdot x_t'$

Umuman, $y=f_1(f_2(\ldots f_n(x)))$ berilsa, $y' = f_1' \cdot f_2' \ldots f_n'$ formula o'rinli bo'ladi.

Misollar

1) $(\ln^2(\sin^3 x))' = 2\ln(\sin^3 x)\frac{1}{\sin^3 x}3\sin^2 x\cos x = 6\mathrm{ctg}x\ln(\sin^3 x);$

2) $(\mathrm{sh}x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \mathrm{ch}x;$

3) $(\mathrm{ch}x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \mathrm{sh}x;$

Oxirgi misollar va bo'linma hosilasi formulasidan foydalanib, $(\mathrm{th}x)' = \frac{1}{\mathrm{ch}^2 x}$, $(\mathrm{cth}x)' = -\frac{1}{\mathrm{sh}^2 x}$ formulani keltirib chiqarish mumkin.

4. *Daraja ko'rsatkichli funksiya hosilasi*

$[lnf(x)]'=\frac{f'(x)}{f(x)}$ logorifmik hosila deb ataladi, uning yordamida $y=U(x)^{V(x)}$ daraja ko'rsatkichli funksiya hosilasi uchun formula keltirib chiqaramiz:

$\ln y=v(x)\ln U(x)$ ekanligidan $\frac{y'}{y}=V'(x)\ln U(x)+V(x)\cdot\frac{U(x)}{U'(x)}$ va

$y'=U(x)^{V(x)}[V'(x)\ln U(x)+V(x)\frac{U'(x)}{U(x)}]$ formula hosil bo'ladi .

Misollar.

1) $(x^x)'=x^x[1\ln x+x\frac{1}{x}]=x^x(\ln x+\ln e)=x^x\ln(ex)$;

2) $(sinx^{cosx})'=sinx^{cosx}[-\sin x\ln\sin x+\frac{cos^2x}{sinx}]$.

5. *Parametrik funksiya hosilasi*

Agar funksiya $x=x(t)$, $y=y(t)$ parametrik ko'rinishda berilib, bu funksiyalar $t=t_0$ nuqtada hosilaga ega bo'lsa, $y=f(x)$ funksiya hosilasi ham mavjud va $y'_x=\frac{y'_t}{x'_t}$.

Isboti. $y'_x=\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta t\to 0}\frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}=\frac{y'_t}{x'_t}$

Misol

1) $\begin{cases}X=R\,cost\\ y=R\,sint\end{cases}$, $(0\le t\le\pi)$ bo'lsa, $y'_x=\frac{R\,cost}{-R\,sint}=-\frac{\frac{x}{R}}{\sqrt{1-\frac{x^2}{R^2}}}=-\frac{x}{\sqrt{R^2-x^2}}$ $(x\neq\pm R)$

Aslida, $x^2+y^2=R^2$ da, $y=\sqrt{R^2-x^2}$ deb olinsa $y'_x=\frac{1}{2\sqrt{R^2-x^2}}(-2x)=-\frac{x}{\sqrt{R^2-x^2}}$ hosil bo'ladi .

6. *Oshkormas funksiya hosilasi*

Agar funksiya y ga nisbatan yechilmagan $F(x;y)=0$ tenglama bilan berilsa, undan murakkab funksiya kabi hosila olish, so'ngra y'_x ni topish mumkin.

Umuman, $y'_x=-\frac{F'_x(x;y)}{F'_y(x;y)}$ formula o'rinli bo'ladi, lekin, F'_x , F'_y larni topishda ikkinchi o'zgaruvchi o'zgarmas hisoblanadi.

Misol. 1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

Agar bu funksiyadan murakkab funksiya kabi hosila olsak, $\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$ yoki $y'=-\frac{b^2x}{a^2y}$ kelib chiqadi. Bu natija yozilgan formula bo'yicha ham kelib chiqadi.

Yuqorida elementar funksiyalar hosilalari uchun topilgan natijalardan foydalanib, quyidagi **hosilalar jadvalini** hosil qilamiz.

1) $(C)'=0$
2) $(x^p)'=px^{p-1}$, xususan, $(\frac{1}{x})'=-\frac{1}{x^2}$, $(\sqrt{x})'=\frac{1}{2\sqrt{x}}$
3) $(a^x)'=a^x\ln a$, xususan, $(e^x)'=e^x$
4) $(\log_a x)'=\frac{1}{x\ln a}$, xususan, $(\ln x)'=\frac{1}{x}$
5) $(\sin x)'=cosx$;
6) $(\cos x)'=-sinx$;
7) $(tgx)'=\frac{1}{cos^2x}$;
8) $(ctg)'=-\frac{1}{sin^2x}$;
9) $(\arccos x)'=\frac{1}{\sqrt{1-x^2}}$;
10) $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$;
11) $(arctgx)'=\frac{1}{1+x^2}$;
12) $(arcctgx)'=-\frac{1}{1+x^2}$;
13) $(shx)'=chx$;
14) $(chx)'=shx$;
15) $(thx)'=\frac{1}{ch^2x}$;
16) $(cthx)'=-\frac{1}{sh^2x}$;

17-mavzu. Differensial. Yuqori tartibli hosila va differensiallar

17.1. Yuqori tartibli hosilalar

Funksiyaning $f'(x)$ hosilasi **birinchi tartibli hosila** deb ataladi. $f'(x)$ ham funksiya bo‘lganligi uchun undan yana bir marta hosila olish mumkin.

$(f'(x))'$;

Bu hosila **ikkinchi tartibli hosila** deyiladi, umuman funksiyaning (n-1)-tartibli hosilasidan olingan hosila n-tartibli hosila deyiladi.

Bu yuqori tartibli hosilalar, $y^{(n)}$ ko‘rinishida belgilanadi.

Demak, $y^{(n)}=(y^{(n-1)})'$.

Misollar.

1) $y=a^x$, (a>0, a≠1), $y'=a^x\ln a$, $y''=a^x\ln^2 a$, $y'''=a^x\ln^3 a$, $y^n=a^x\ln^n a$, xususan, $y=e^x$ bo‘lsa, $(e^x)^{(n)}=e^x$

2) y=sinx uchun $y'=\cos x=\sin(\frac{\pi}{2}+x)$; $y''=-\sin x=\sin(2\frac{\pi}{2}+x)$;
$y'''=-\cos x=\sin(3\frac{\pi}{2}+x)$; $y^{IV}=\sin x=\sin(4\frac{\pi}{2}+x)$; tengliklardan
$y^n=(\sin x)^{(n)}=\sin(n\frac{\pi}{2}+x)$; kelib chiqadi.

3) $(\cos x)^{(n)}=\cos\left(n\frac{\pi}{2}+x\right)$ formula o‘rinliligi yuqoridagidek tekshiriladi.

Endi (UV) ko‘paytmaning yuqori tartibli hosilasini olish masalasini ko‘rib chiqamiz.

$(UV)'=U'V+UV'$,

$(UV)''=(U'V+UV')'=U''V+U'V'+U'V'+UV''=U''V+2U'V'+UV''$;

$(UV)'''=(U''V+2U'V'+UV'')'=U'''V+U''V'+2U''V'+2U'V''+U'V''+UV'''=U^{(3)}V+3U^{(2)}V^{(1)}+3U^{(1)}V^{(2)}+UV^{(3)}$;

$(UV)^{(n)}=U^{(n)}V+\frac{n}{1!}U^{(n)}V'+\frac{n(n-1)}{2!}U^{(n-2)}V''+\frac{n(n-1)(n-2)}{3!}U^{(n-3)}V'''+\ldots UV^{(n)}$

Oxirgi tenglik Leybnis formulasi deb ataladi.

Misol. 1) $y=x^2e^x$ funksiyaning 50-hosilasini toping. $U=e^x$, $V=x^2$ desak, $V'=2x$, $V''=2$, $V'''=0$, ekanligidan, $(x^2e^x)^{(50)}=x^2e^x+\frac{50}{1!}e^x2x+\frac{5049}{1!}e^x2$, chunki qolgan qo‘shiluvchilar nollardan iborat bo‘ladi.

1) y=xcosx ning 10-hosilasini toping.

Agar u=cosx, V=x desak, $V'=1$, $V''=0$, ekanligidan, $(x\cos x)^{(50)}=\cos(10\frac{\pi}{2}+x)x+\frac{10}{1!}\cos(9\frac{\pi}{2}+x)1=x\cos x(5\pi+x)+10\cos x(4{,}5\pi+x)$ $=-x\cos x-10\sin x$.

Yuqori tartibli hosilalar olishda ayniyatlardan foydalanish mumkin.

1) $Y=\sin^4 x$ ning n-tartibli hosilasini toping.

$\sin^4 x=(\frac{1-\cos 2x}{2})^2=\frac{1}{4}[1-2\cos x2x+\cos^2 2x]=\frac{1}{4}[1-2\cos x2x+\frac{1-\cos 4x}{2}]=$

$=\frac{1}{8}[3-4\cos 2x+\cos 4x]$ ekanligini e’tiborga olib,

$y^{(n)}=-\frac{1}{2}.2^n\cos(n.\frac{\pi}{2}+2x)+\frac{1}{8}.4^n\cos(n.\frac{\pi}{2}+4x)$

2) $y=\frac{1}{x^2-9x+20}=\frac{1}{(x-4)(x-5)}=\frac{1}{(x-5)}-\frac{1}{(x-4)}=(x-5)^{-1}-(x-4)^{-1}$

bo’lganligi uchun $y^{(n)}=(-1)^n n![(x-5)^{-n-1}-(x-4)^{-n-1}]$;

$F(x,y)=0$ tenglama bilan berilgan oshkormas funksiyaning ikkinchi hosilasini olish uchun, tenglamaning tomonlaridan qiymati qo‘yilib, y'' topiladi va hokazo.

17.2. Differensial ma’nosi, hisoblash qoidalari

Ta’rif. Agar y=f(x) funksiya orttirmasini $\Delta y=A\Delta x+\alpha(\Delta x)\Delta x$, bunda A-son, $\alpha(\Delta x)$-cheksiz kichik, ko‘rinishda yozish mumkin bo‘lsa, u **differensiallanuvchi** deyiladi.

Funksiya differensiallanuvchi bo‘lishi uchun chekli hosila mavjud bo‘lishi zaruriy va yetarli shart hisoblanadi, chunki $f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=$ $=\lim_{\Delta x\to 0}(A+\alpha(\Delta x))=A$;

Funksiyani differensiallanuvchi bo‘lishi uning uzluksizligini ham keltirib chiqaradi, chunki,

$\lim_{\Delta x\to 0}\Delta y=A\lim_{\Delta x\to 0}\Delta x+\lim_{\Delta x\to 0}\alpha(\Delta x)\lim_{\Delta x\to 0}\Delta x=0$;

ya’ni argument va funksiya orttirmalari bir paytda nolga intiladi, bu esa funksiya uzluksizligini bildiradi.

Funksya orttirmasining $\Delta y=A\Delta x+\alpha(\Delta x)\Delta x$ ko‘rinishida, $A\Delta x$ orttirmaning chiziqli **bosh qismi**, $\alpha(\Delta x)\Delta x$ esa **qoldiq qismi** deyiladi.

Ta’rif. Funksiya orttirmasining chiziqli bosh qismi uning differensiali deyiladi va $dy=A\,\Delta x$ tarzida yoziladi.

$y'(x)=A$ ekanligini hisobga olsak, $dy=y'(x)\Delta x$, agar y=x deyilsa, $dx=1\Delta x$ bo‘ladi va differensial uchun $dy=y'(x)dx$ formula hosil qilamiz.

Differensial uchun topilgan dy=$y'(x)dx$ formula yordamida quyidagi, diferensial hisoblash qoidalarini topish mumkin.

1) $d(C_1U \pm C_2V)=(C_1U \pm C_2V)'dx=(C_1U' \pm C_2V')dx=C_1dU \pm C_2dV$

2) $d(UV)=(UV)'dx=(U'V+UV')dx=VdU+UdV$

3) $d=(\frac{U}{V})=(\frac{U}{V})'dx=\frac{VU'-UV'}{V^2}dx=\frac{VdU-UdV}{V^2}$.

Agar y=f(x), x=φ(t) funksiyalar yordamida tuzilgan y=f(φ(t)) murakkab funksiya qaralsa, differensial $dy=y'_x t'_x dt = y'_x dx$ ko'rinishida yoziladi, o'z holatini saqlaydi. Differensial o'z ko'rinishini o'zgartirmaslik xususiyati uning invariantligi deyiladi.

y=f(x) funksiya biror nuqtadagi birinchi differensialidan shu nuqtada olingan differensial uning ikkinchi differensiali deyiladi, $d^2y=d(dy)$ ko'rinishida yoziladi. Shunga o'xshash, $d^3y=d(d^2y)$, $d^ny=d(d^{n-1}y)$ lar ham ko'riladi.

Yuqori tartibli hosila, differensiallarini hisoblashda dx ixtiyoriy va x ga bog'liqmas son ekanini, uni o'zgarmas ko'paytiruvchi sifatida qarash lozimligini unutmaslik lozim.

$d^2y=d(dy)=d(y'dx)=d(y')dx=(y''dx)dx=y''dx^2$,

$d^3y=d(d^2y)=d(y''dx^2)=d(y'')dx^2=(y'''dx)dx^2=y'''dx^3$,

Umuman, $d^ny=y^{(n)}dx^n$,

Agar $y=x^n$, funksiyaning yuqori tartibli differensiali hisoblansa, $d(x^n)$, $d^2(x^n)$, ko'rinishida yoziladi.

Yuqori tartibli differensiallarda invariantlik xossasi o'rinli bo'lmaydi, chunki, y=f(φ(t)) funksiya uchun

$d^2y=d(y'_x dx)=d(y'_x)dx+y'_x d(dx)=y''_{x^2}dx^2+y'_{x^2}d^2x$ hosil bo'ladi.

Biror $x=x_0$ nuqtada $dy \approx \Delta y$ ekanligidan taqribiy hisoblashlarda unumli foydalaniladi.

$\Delta y=f(x_0+\Delta x)-f(x_0) \approx f'(x_0)\Delta x$ dan $f(x_0+\Delta x) \approx f(x_0)+f'(x_0)\Delta x$ taqribiy hisoblash formulasi kelib chiqadi.

Misol. $\sqrt{26}$ taqribiy qiymatini toping.

$f(x)=\sqrt{x}$, $x_0=25$ deb, $\sqrt{25+1} \approx \sqrt{25}+\frac{1}{2\sqrt{25}}1=5+\frac{1}{10}=5{,}1$ ekanligini topamiz.

17.3. Hosilani iqtisodiy masalalarga tatbiq etish

Ishlab chiqarilgan mahsulot hajmi hosilasi mehnat unumdorligi ekanligini o'rgandik.

Agar ishlab chiqarilayotgan mahsulot birligi x ni biror ishlab chiqarish qoldig'i y ning argumenti deb qaralsa, $\frac{\Delta y}{\Delta x}$ har bir ishlab chiqarilayotgan mahsulotga mos o'rtaga qoldiq bo'ladi. $\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}$ hosila esa, ishlab chiqarish limit qoldig'ini bildiradi va har bir ishlab chiqarilayotgan mahsulotga qo'shimcha sarflanishi zarur xarajatni taqribiy ifodalaydi.

Shu usulda limit foyda, limit mahsulot, limit foydalilik kabi kattaliklar kiritilishi va ishlab chiqarish xususiyatlarini ochib berishi mumkin. Ular iqtisodiy obyekt o'zgarishlari jarayonini ochib beradi.

Iqtisodiy jarayonlarni tekshirish, tatbiqiy masalalarni yechishda "funksiya elastikligi" tushunchasidan foydalaniladi.

Ta'rif. y=f(x) funksiya elastikligi $E_x(y)$ deb, funksiya va argument nisbiy orttirmallari nisbatining $\Delta x \to 0$ dagi limitiga aytiladi.

$E_x(y)=\lim_{\Delta x \to 0}(\frac{\Delta y}{y}:\frac{\Delta x}{x})=\frac{x}{y}\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\frac{x}{y}y'$;

Elastiklik funksiyasi, funksiya argumenti x 1% ga o'zgarganda y=f(x) funksiya qancha foizga o'zgarishini ko'rsatadi.

$T_y=(\ln y)'=\frac{y'}{y}$ logorifimik hosila iqtisodda **funksiya o'zgarishi tempi** deyiladi.

Elastiklik funksiyasi quyidagi xossalarga ega.

1. Elastiklik funksiyasi argument va o'zgarish tempi funksiyalari ko'paytmasiga teng; $E_x(y)=xT_y$

2. $E_x(UV)=E_x(U)+E_x(V)$, $E_x(\frac{U}{V})=E_x(U)-E_x(V)$, chunki

$E_x(UV)=x\frac{(UV)'}{UV}=x\frac{U'V+UV'}{UV}=x(\frac{U'}{U}+\frac{V'}{V})=xT_u+xT_V=E_x(U)+E_x(V)$,

$E_x(\frac{U}{V})=x\frac{(U)'}{V}\frac{V}{U}=x\frac{U'V-UV'}{UV^2}V=x(\frac{U'}{U}-\frac{V'}{V})=xT_u-xT_V=E_x(U)-E_x(V)$,

Agar tovar narxi x 1% o'zgarsa, talab elastikligi y ning qancha o'zgarishini ko'rsatadi.

Agar talab elastikligi $E_x(y)>1$ bo'lsa talab elastik, $E_x(y)=1$ bo'lsa talab neytral, $E_x(y)<1$ da esa talab noelastik hisoblanadi.

Masalalar. 1. Agar mahsulot ishlab chiqarishda mahsulot hajmi x va ishlab chiqarish qoldiqlari y orasidagi bog'lanish $y=50x-0{,}05x^3$ (so'm) funksiya bilan berilsa, 10 birlik mahsulot tayyorlashdagi o'rta va limit qoldiqlarini hisoblang.

Birlik mahsulot o'rtacha qoldig'i funksiyasi $y=\frac{y}{x}=50x-0{,}05x^2$ bo'ladi, 10 birlik mahsulot uchun $y_1(10)=50-0{,}05.\ 10^2=45$(so'm).

Limit qoldig'i esa $y'=50-0{,}15x^2$ hosila yordamida aniqlanib, 10 birlik mahsulot uchun $y'(10)=35$(so'm).

2. Mahsulot ishlab chiqarish x(mlrd so'm) va bitta mahsulot tannarxi y(ming so'm) bog'lanishi y=-0,5x+80 funksiya bilan berilgan. 60 mlrd.so'm mahsulot ishlab chiqarilgandagi mahsulot tannarxi elastikligini toping.

$E_x(y)=\frac{x}{y}y'$ formulaga ko'ra, $E_x(y)=\frac{-0.5}{-0.5+80}=\frac{x}{x-160}$.

x=60 da $E_{60}(y)=-0{,}6$, ya'ni 60 mlrd.so'mlik mahsulot ishlab chiqarishda, uni 1% ga oshirish tannarxi 0,6% ga pasayishini bildiradi.

Mavzuga doir misollar va masalalar

1. Ta'rif yordamida hosilalarni toping.

1) $y=x^2$; 2) $y=\frac{1}{x}$; 3) $y=\sqrt{x}$; 4) y=tgx; 5) y=argsinx;

6) $y=x(x-1)^2(x-2)^3\ldots(x-10)^{11}$ bo'lsa $f'(0)$, $f'(1)$ ni toping.

2. Hosilalar jadvali va qoidalari yordamida hisoblang.

1) $y=\frac{x^3}{3}+\frac{x^2}{2}-2x$; 2) $y=\frac{ln5}{x}+\pi^2+e$; 3) y=(x-a)(x-b); 4) $y=(x-1)(x-2)^2(x-3)^3$;

5) $y=\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}$; 6) $y=\frac{2x}{1-x^2}$; 7) $y=x+\sqrt{x}+\sqrt[3]{x}$; 8) $y=\frac{1}{x}+\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}$; 9) $y=x\sqrt{1+x^2}$;

10) $y=\frac{x}{\sqrt{a^2-x^2}}$; 11) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$; 12) y=sin4x-4cos2x; 13) $y=\frac{cosx}{2sin^2x}$;

14) $y=\frac{sin^2x}{sinx^2}$; 15) $y=tgx-\frac{1}{3}tg^3x+\frac{1}{5}tg^5x$; 16) $Y=2^{tg\frac{1}{x}}$; 17) $y=e^x+e^{e^x}$;

18) $y=lg^3x^2$; 19) y=ln(ln(lnx)); 20) $y=\ln(x+\sqrt{a^2+x^2})$; 21) $y=\ln tg\frac{x}{2}$;

22) $y=\ln\sqrt{\frac{1-sinx}{1+sinx}}$; 23) $y=arctg\frac{x^2}{a}$; 24) y=arcsin(sinx-cosx); 25) $y=\arcsin\frac{1-x^2}{1+x^2}$;

26) $y=\sqrt[x]{x}$; 27) $y=x^{sinx}$; 28) y=sh(tgx); 29) $y=x^2+2xy-y^2-2x=0$;

30) y^2-2px; 31) $\sqrt{x}+\sqrt{y}=\sqrt{a}$;

3. Berilgan funksiyaga berilgan nuqtadagi urinma tenglamasini yozing.

1) $y=2+x-x^2$, $x_0=1$; 2) $y=\sqrt{5-x^2}$, $x_0=1$; 3) $y=\frac{8}{4+x^2}$; $x_0=2$;

4. Berilgan funksiyalar qanday burchak ostida kesishishini toping.

1) $y_1=x^2$ va $y_2=\sqrt{x}$; 2) y_1=sinx va y_2=cosx; 3) $y_1=\frac{1}{x}$ va $y_2=\sqrt{x}$;

5. Taqribiy qiymatlarini toping.

1) $\sqrt[3]{65}$; 2) sin 29^0; 3) lntg47^0; 4) lg 11;

6. n-tartibli hosilalarini toping.

1) $y=e^{-\frac{x}{a}}$; 2) y=lnx; 3) $y=cos^2x$; 4) $y=\frac{1}{x^2-6x+8}$; 5) $y=\frac{ax+b}{cx+d}$;

6) $y=\frac{x}{\sqrt{1+x}}$; 7) $y=x^2-sin2x$; 8) $y=\frac{1}{1-x^2}$; 9) y=sinax cosbx;

10) $y=sin^4x+cos^4x$; 11) $y=e^{ax}cosbx$; 12) $y=x\ln\frac{5+x}{5-x}$; 13) $y=\frac{lnx}{x}$;

Masalalarni yeching

1) Ish kunida sexning mahsulot ishlab chiqarish hajmi u vaqt t bilan o'zaro $u=-t^3-5t^2+75t+425$ funksiya yordamida bog'langan. Ish boshlangandan 2 soat keyin mehnat unumdorligini toping.

2) Ishlab chiqarish qoldiqlari u (so'm) va mahsulot hajmi x (dona) o'zaro $y=10x-o.04x^3$ formula bilan bog'langan. O'rta va limit qoldiqlarni 5 dona mahsulot uchun toping.

3) Talab q va taklif s funksiyalari p narx bilan quyidagiga berilgan: q=7-p; s=p+1. Quyidagilar topilsin:

a) Turg'un narx;

b) Talab va taklif elastikligi;

c) Narx 5% oshirilganda foyda necha foiz ortadi?

Mavzuga doyir joriy nazorat uchun uy vazifasi.

(N-talabalarning ro'yxatdagi nomeri)

1. Ta'rif yordamida berilgan funksiyalar hosilasini toping.

1) $y=x^{100-N}+Nx-N$; 2) y=sinNx+NcosNx; 3) $y=x(x-N)^2(x-2N)^3$, $y'(0)$=? $y'(N)$=?

2. Jadval va qoidalar yordamida berilgan funksiya hosilalarini toping.

1) $Y=x^{N+1}+\frac{1}{x^{N+1}}+\sqrt[N+1]{x+N}-\frac{N}{\sqrt{x^2-Nx}}$; 2) $y=sin^{N+1}[cos^{100-N}Nx]$;

3) $Y=\frac{sin^{10+N}Nx+N}{N^2+x}$; 4) $y=x^{N+10}(N-x^{N+1})(1+N^{100-N})$;

5) y=,3; 6) $y=[\sin(N+1)x]^{\frac{N}{x}}$;

7) $x^{N+1}-Nxy+Nx-(N+5)y-y^{100-N}=0$; 8) $\begin{cases} x=Nt-\sin Nt+e^{2Nt} \\ y=N+\cos(20+N)t-e^{-N} \end{cases}$;

3. Berilgan funksiyalar n-tartibli hosilasini toping.

1) $Y=\frac{2}{2x^2-3N+N^2}$; 2) $y=\frac{x+N}{Nx-1}$; 3) $y=\frac{x}{\sqrt[N+1]{x+N}}$; 4) y=sin(N+1)xcos(100-N)x;

5) $y=x^3cosNx$; 6) $y=sin^6Nx+cos^6Nx$; 7) $y=\frac{x^2}{N-x}$;

4. Talabalarning o'tilgan darslarni qabul qilishi U vaqtga bog'liqligi

$U(t)=-t^3-Nt^2+(100-N)t,\ 1\le t\le 6$; tenglama bilan berilgan. Darslar boshlangandan 1 soat keyin va darslar tugashiga 1 soat qolgandagi dars qabul qilish samaradorligini, o'zgarishi tempi va tezligini toping.

5. Iqtisodiyot yo'nalishi bitiruvchilari ishga taklif etilishi narxi P bo'lsa, kuzatuvlar natijasida bakalavrlarga talablar $q=\frac{P+N}{P+(100-N)}$; takliflar esa $S=P+\frac{1}{N}$; bo'lsa, quyidagilarni toping.

a) talab va taklif teng bo'ladigan talaba turg'un narxini;
b) topilgan narx talab va taklif elastikligini.
c) talab narxi N% oshirilsa, olinadigan foyda.

18-mavzu. Differensial hisob asosiy teoremalari va ularni qo'llash

18.1. Differensial hisob asosiy teoremalari

Teorema (P.Ferma): Agar differensiallanuvchi f(x) funksiya X oraliq ichki $c\epsilon X$ nuqtasida eng katta (kichik) qiymatiga erishsa, $f'(c)=0$ bo'ladi.

Isbot. Aniqlik uchun f(x) funksiya c nuqtada eng katta qiymatga erishadi deylik, $f(x)\le f(c)$.

$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ mavjudligidan, bir tomonli

$0\le \lim_{x\to c-0}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c+0}\frac{f(x)-f(c)}{x-c}\le 0$ limitlar teng bo'lishi kerak. Bu $f'(c)=0$ da bajariladi xolos. Boshqacha aytganda, funksiyaning eng katta (kichik) qiymatlarida grafikka o'tkazilgan urinma abssissalar o'qiga parallel bo'ladi.

Teorema (M.Poll): f(x) funksiya quyidagi shartlarni qanoatlantirsin.

1) [a;b] kesmada uzluksiz;
2) (a;b) intervalda differensiallanuvchi;
3) f(a)=f(b)

U holda kamida bitta $c\in(a;b)$ nuqta topiladiki, $f'(c)=0$ bo'ladi.

Isbot. f(x) funksiya Veyershtrass teoremasiga ko'ra eng katta M, eng kichik m qiymatlarga erishadi.

Ikki holat bo'lishi mumkin.

1) m=M. Bu holda f(x) funksiya [a;b] oraliqda o'zgarmas bo'ladi, hamma ichki $c\in(a;b)$ nuqtada eng katta qiymatga erishishi kelib chiqadi va bu nuqtalarda $f'(c)=0$;

2) M va m turlicha. f(a)=f(b) shartdan biror ichki $c\in(a;b)$ nuqtada eng katta M, eng kichik m qiymatlarga erishishi kelib chiqadi va Ferma teoremasiga ko'ra: $f'(c)=0$.

Teorema (Lagranj): f(x) funksiya quyidagi shartlarni qanoatlantirsin.

1) [a;b] kesmada uzluksiz;
2) (a;b) oraliqda differensiallanuvchi.

U holda shunday kamida bitta $c\in(a;b)$ nuqta mavjud, $\frac{f(b)-f(a)}{b-a}=f'(c)$ tenglik o'rinli bo'ladi.

Isbot. Yordamchi $F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$ funksiyani qaraymiz. Bu funksiya uchun Roll teoremasi shartlari o'rinli, shunday c∈(a;b) nuqta mavjudki, $F'(c)=0$ bo'ladi, ya'ni:

$0=F'(c)=f'(c)-\frac{f(b)-f(a)}{b-c}$. Bundan **Lagranj tengligi** kelib chiqadi.

Teorema (Koshi): f(x) va g(x) funksiyalar quyidagi shartlarni qanoatlantirsin;

1) f(x), g(x) funksiyalar [a;b] kesmada uzluksiz;

2) f(x), g(x) lar (a;b) da chekli hosilalarga ega, $g'(x)\neq 0$

U holda kamida bitta c∈(a;b) nuqta topiladiki, quyidagi Koshi tengligi bajariladi: $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$.

Isbot. Avvalo, $g(b)\neq g(a)$, aks holda, Poll teoremasiga ko'ra, $g'(c)=0$ bo'lib qolishi mumkin.

Endi yordamchi $F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$ funksiyani qaraymiz. Bu funksiya uchun ham Poll teoremasi shartlari o'rinli, ya'ni shunday c∈(a;b) mavjudki, $0=F'(c)=f'(c)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(c)$ bo'ladi, bundan esa **Koshi tengligi** kelib chiqadi.

Misollar. 1) [1;4] kesmada $f(x)=x^2$ funksiya uchun **Lagranj tengligi** o'rinli bo'ladigan nuqtani toping .

$\frac{4^2-1^2}{4-1}=2C$ dan $C=\frac{5}{2}$;

2) $[0;\frac{\pi}{2}]$ kesmada f(x)=sinx, g(x)=cosx funksiyalar uchun Koshi tengligi o'rinli bo'ladigan nuqtani aniqlang.

$\frac{\sin\frac{\pi}{2}-\sin 0}{\cos\frac{\pi}{2}-\cos 0}=\frac{\cos c}{-\sin c}$ tenglikdan ctgC=1, ya'ni $C=\frac{\pi}{4}$.

18.2. Differensial hisob asosiy teoremalarini qo'llash Noaniqliklarni ochish, Lopital qoidalari

Teorema (Lopital). Biror a∈x nuqta atrofida f(x), g(x) aniqlanib, hosilalar mavjud bo'lsin. $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$, $g'(x)\neq 0$. U holda, agar $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ mavjud bo'lsa, $\lim_{x\to a}\frac{f(x)}{g(x)}$ ham mavjud va $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

Isbot. f(a)=g(a)=0 deb qabul qilinsa, masalan, [a;x] oraliqda f,g funksiyalar uchun Koshi teoremasi shartlari o'rinli bo'ladi, shunday c∈(a;x) mavjudki, $\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}$ o'rinli, bundan $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ kelib chiqadi, chunki $x\to a$ *da* $c\to a$ bo'lishi tabiiy.

Agar f,g funksiyalar hosilalari ham yuqoridagi shartlarga bo'ysunsa, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$.

Ya'ni noaniqlik yo'qolguncha Lopital qodasini qo'llash mumkin.

Misol: 1) $\lim_{x\to 0}\frac{x-\sin x}{x^2}\triangleq\lim_{x\to 0}\frac{1-\cos x}{3x^2}\triangleq\lim_{x\to 0}\frac{\sin x}{6x}\triangleq\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$.

Qarab chiqilgan noaniqlik $\frac{0}{0}$ tipidagi deyiladi.

Agar $x\to\pm\infty$ bo'lsa ham, yuqoridagi teorema o'rinlidir, chunki $x=\frac{1}{t}$ almashtirish o'tkazilsa, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f(\frac{1}{t})}{g(\frac{1}{t})}\triangleq\lim_{x\to 0}\frac{f'(\frac{1}{t})(-\frac{1}{t^2})}{g'(\frac{1}{t})(-\frac{1}{t^2})}\triangleq\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$.

Misol. 1) $\lim_{x\to\infty}\frac{\ln x}{x}\triangleq\lim_{x\to\infty}\frac{1}{x}=0$.

Agar $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\infty$ bo'lsa ham, Lopital qoidasi $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ o'rinli bo'ladi.

Misol. 1) $\lim_{x\to 0}\frac{\ln(\sin ax)}{\ln(\sin bx)}\triangleq\lim_{x\to 0}\frac{a\sin bx\cos ax}{b\sin ax\cos bx}=\frac{a}{b}\lim_{x\to 0}\frac{\mathrm{tg}bx}{\mathrm{tg}ax}=\frac{a}{b}\lim_{x\to 0}\frac{b\cos^2 ax}{a\cos^2 bx}=1$;

$0.\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0 ko'rinishidagi noaniqliklar ham $\frac{0}{0}$, $\frac{\infty}{\infty}$ ko'rinishidagi noaniqliklarga keltiriladi.

Misollar. 1) $\lim_{x\to 0+}x\ln x=\lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0+}(-x)=0$

2) $\lim_{x\to\frac{\pi}{2}}(\frac{1}{\cos x}-tgx)=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}\triangleq\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=0$

3) $\lim_{x\to 0+}x^x=\lim_{x\to 0+}e^{x\ln x}=e^{\lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}}}=e^0=1$

18.3.Teylor formulasi

Teorema (Teylor Bruk): f(x) funksiya c nuqta va uning atrofida (n+1)-tartibli hosilaga ega bo‘lsin. c va x orasida shunday ξ nuqta mavjudki, quyidagi formula o‘rinli.

$$f(x)=f(c)+\frac{f'(c)}{1!}(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1};$$

Agar Teylor formulasida C=0 bo‘lsa Makloren K. formulasi $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{n}(0)}{n!}x^n+R_{n+1}(x)$ hosil bo‘ladi.

Makloren formulasi bo‘yicha quyidagi yoyilmalarni olish mumkin.

1. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+0(x^n),$
2. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+0(2^{2n}),$
3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+0(2^{2n+1}),$
4. $(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\cdots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+0(x^n),$
5. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+0(x^n).$

Misollar. 1) $f(x)=\sqrt{x}$ funksiyani x-1 darajalari bo‘yicha yoyilmasi uchta hadini toping.

Teylor formulasi bo‘yicha

$$f(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2, f'(x)=\frac{1}{2\sqrt{x}}, f''(x)=-\frac{1}{4\sqrt{x^3}}$$

Demak, $\sqrt{x}=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+0((x-1)^2).$

2) $e^{i\varphi}=\cos\varphi+i\sin\varphi$ Eyler ayniyatini isbotlang.

Funksiyalarning Makloren formulasi bo‘yicha yoyilmalaridan foydalanamiz.

$$e^{i\varphi}=1+i\varphi-\frac{\varphi^2}{2!}-\frac{i\varphi^3}{3!}+\frac{\varphi^4}{4!}+\frac{i\varphi^5}{5!}-\frac{\varphi^6}{6!}-\frac{i\varphi^7}{7!}+\cdots=\left(1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\frac{\varphi^6}{6!}+\cdots\right)+i\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\frac{\varphi^7}{7!}+\cdots\right)=\cos\varphi+i\sin\varphi$$

$$3)\lim_{x\to0}\frac{e^{-\frac{x^2}{2}}-\cos x}{x^2\sin x}=\lim_{x\to0}\frac{\left[1-\frac{x^2}{2}+\frac{x^4}{8}+0(x^n)\right]-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\right]}{x^2[x-0(x)]}=$$

$$\lim_{x\to0}\frac{\frac{x^4}{8}-\frac{x^4}{24}+0(x^4)}{x^4+0(x^4)}=\frac{1}{8}-\frac{1}{24}=\frac{1}{12}.$$

19-mavzu. Funksiyani to‘liq tekshirish

19.1. Funksiya monotonligini tekshirish

Teorema. Agar f(x) funksiya (a;b) intervalda chekli hosilaga ega, $f'(x)\geq 0$ ($f'(x)\leq 0$) bo‘lsa, f(x) funksiya bu intervalda o‘suvchi (kamayuvchi) bo‘ladi.

Isboti. $f'(x)\geq 0$ holni qaraymiz. $x_1, x_2 (x_1 < x_2)\in(a;b)$ *nuqtalar* uchun Lagranj teoremasiga ko‘ra $\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c), c\in(x_1;x_2)$ o‘rinlidir. $f'(c)\geq 0, x_2 > x_1$ *ekanligidan* $f(x_2) > f(x_1)$ kelib chiqadi, ya’ni funksiya o‘suvchi (kamaymaydigan)dir.

$\delta > 0$ *bo'lganda* x_0 nuqtaning biror $(x_0-\delta; x_0+\delta)$ atrofini qaraymiz.

Ta’rif. Agar barcha $(x_0-\delta; x_0+\delta)$ nuqtalar uchun $f(x)\leq f(x_0)$ [$f(x)\geq f(x_0)$] o‘rinli bo‘lsa, x_0 nuqta f(x) funksiyaning maksimum (minimum) nuqtasi deyiladi. Bu nuqtalar birgalikda ekstremum nuqtalari deyiladi. Ularga mos funksiyaning qiymatlari max$f(x_0)$, min$f(x_0)$ tarzida yoziladi.

Funksiyaning bunday qiymatlari shu oraliqdagi eng kata (kichik) qiymatlar bo‘lganligi uchun, Ferma teoremasiga ko‘ra $f'(x_0)=0$ bo‘ladi. Lekin, aksinchasi doimo o‘rinli emas, ya’ni hosila nol bo‘ladigan barcha nuqtalarda ham ekstremum bo‘lavermaydi. Bundan tashqari, hosila mavjud bo‘lmagan nuqtalarda ham ekstremum bo‘lishi mumkin. Masalan, $y=|x|$ funksiya x=0 nuqtada minimumga ega, lekin hosilasi bu nuqtada mavjud emas.

Aniqlanish sohasiga kirgan, funksiya hosilasi nol yoki mavjud bo‘lmaydigan nuqtalar kritik (yoki statsionar) nuqtalar deyiladi.

Yuqoridagilardan quyidagi xulosa kelib chiqadi.

Teorema (ekstremum topishning 1-qoidasi). Agar f(x) funksiya $(x_0-\delta; x_0+\delta)$ atrofida chekli hosilaga ega, $f'(x_0)=0$ bo‘lib, x_0 nuqtada hosila o‘z ishorasini + dan - ga (-dan +ga) o‘zgartirsa, u holda funksiya $x=x_0$ nuqtada maksimum (minimum) qiymatga erishadi.

Funksiya monotonligi, ekstremumlarini 1-qoida asosida topishda jadvaldan foydalanish qulay. Misol. $y=\frac{x^3}{3}-x^2$ funksiya ekstremumlarini toping.

$y' = x^2 - 2x = x(x-2)$ dan $x_1=0$, $x_2=2$ nuqtalar kritik nuqtalardir. Ular yordamida aniqlanish sohasini bo‘laklarga ajratamiz, hosila ishorasini tekshiramiz, ekstremumlarini aniqlaymiz. Bularning barchasi quyidagi jadval yordamida oson hal etiladi:

x	$(-\infty;0)$	0	$(0;2)$	2	$(2;+\infty)$
y'	+	0	-		+
y	↗	0	↘	$-\frac{4}{3}$	↗
		max		min	

Demak, $f_{max}(0)=0$, $f_{min}(2) = -\frac{4}{3}$.

Iqtisodiyot nasariyasida hosila My(x) – **marjinal limit kattalik** deyiladi.

Agar x-sotilgan tovarlar soni, R(x)-foyda funksiyasi, C(x)-ishlab chiqarishga ketgan harajatlar funksiyasi bo‘lsa, u holda sof foyda funksiyasi P(x)=R(x)-C(x) bo‘ladi. Maksimal foyda bu funksiya hosilasi nolga tenglashganda bo‘ladi. Bundan quyidagi qonun kelib chiqadi: **Sof foyda va sarflangan mablag‘ teng holatda foyda maksimal bo‘ladi.**

19.2. Funksiya grafigining botiq-qavariqligi Ekstremum topishning 2-qoidasi

Biror (a;b) oraliqda f(x) funksiya chekli $f'(x)$ hosilaga ega bo‘lsa, u holda funksiyaga bu oraliqda $e(x)$ urinma mavjud.

Ta’rif. Agar ixtiyoriy $x \in (a;b)$ *uchun* $e(x) \le f(x)$ $[e(x) \ge f(x)]$ o‘rinli bo‘lsa, funksiya bu oraliqda botiq (qavariq) deyiladi.

Teorema. Agar (a;b) oraliqda funksiya ikkinchi tartibli f'' hosilaga ega va $f''(x) \ge 0$ $[f''(x) \le 0]$ bo‘lsa, funksiya grafigi bu oraliqda botiq (qavariq) bo‘ladi.

Isboti. Aniqlik uchun, (a,b) da $f''(x) \ge 0$ bo‘lsin. Ixtiyoriy $c \in (a;b)$ da E(c;f(c)) nuqtadan o‘tadigan urinma $y = f(c) + f'(c)(x-c)$ tenglamaga ega. Qaralayotgan f(x) funksiyaning c nuqtadagi Teylor formulasi bo‘yicha yoyilmasi esa $Y = f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(\xi)}{2!}(x-c)^2$; $\xi \in (x;c)$ deyish mumkin. Ularni solishtirib, $Y - y = \frac{f''(\xi)}{2!}(x-c)^2$; $f''(\xi) \ge 0$ bo‘lganda urinma grafikdan pastda joylashishini, ya’ni grafik botiq ekanligini topamiz.

Teorema (ekstremum topishning 2-qoidasi). Agar $x_0 \in (a;b)$ kritik $f'(x_0)=0$ nuqta bo‘lib, $f''(x_0)>0$ $[f''(x_0)<0]$ bo‘lsa, bu nuqtada funksiya minimum (maxsimum) qiymatiga erishadi.

Tar’if. Agar $E(x_0, f(x_0))$ nuqtada f(x) funksiyaga o‘tkazilgan urinmaning bir qismi f(x) dan yuqori, ikkinchi qismi pastda joylashsa, x_0 nuqta **funksiyaning egilish nuqtasi** deyiladi.

Egilish nuqtasida botiqlik qavariqlikka, yoki qavariqlik botiqlikka o‘zgaradi. Demak, x_0 egilish nuqtasi bo‘lsa, $f''(x_0)=0$. Ekstremum topishning 2-qoidasi ham jadval yordamida tekshiriladi.

Misol. 1) $y = \frac{2x^3}{3} - \frac{x^5}{5}$ funksiya egilishi nuqtalari, botiqlik, qavariqlik sohalari, ekstremumlarini toping. $y' = 2x^2 - x^4 = x^2(2-x^2) = 0$ dan $x_1 = -\sqrt{2}$, $x_2 = 0$, $x_3 = \sqrt{2}$ nuqtalar kritik nuqtalardir.

$y'' = 4x - 4x^3 = 4x(1-x^2) = 0$ dan $x_1 = -1$ $x_2 = 0$, $x_3 = 1$ nuqtalar egilish nuqtalaridir.

$y''(-\sqrt{2}) = +4\sqrt{2} > 0$, demak, $f_{min}(-\sqrt{2}) = \frac{8\sqrt{2}}{15}$; $y''(0)=0$, egilish nuqtasi xolos.

$y''(\sqrt{2}) = -4\sqrt{2} < 0$, demak, $f_{max}(\sqrt{2}) = \frac{8\sqrt{2}}{15}$ funksiya $(-\infty; -1) \cup (0;1)$ oraliqda botiq, $(-1;0) \cup (1;\infty)$ oraliqda qavariqdir.

2) $y = x^2 - \frac{x^4}{2}$ funksiya kritik, egilish nuqtalari, botiqlik-qavariqlik sohalarini toping. $y' = 2x - 2x^3 = 2x(1-x^2)$dan $x_1=0$, $x_2=1$, $x_3=-1$ kritik nuqtalari, $y'' = 2 - 6x^2 = 6(\frac{1}{3} - x^2)$dan $x_4 = -\frac{1}{\sqrt{3}}$, $x_5 = \frac{1}{\sqrt{3}}$ egilish nuqtalaridir.

$y''(-1)<0$, $y''(1)<0$, ekanligidan $f_{max}(-1) = \frac{1}{2}$, $f_{max}(1) = \frac{1}{2}$, $y''(0)=2>0$ ekanligidan $f_{min}(0)=0$.

Botiqlik-qavariqlik sohalarini quyidagi jadval yordamida topish qulay:

X	$(-\infty;-\frac{1}{\sqrt{3}})$	$-\frac{1}{\sqrt{3}}$	$(-\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}})$	$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}};0)$
y''	-	0	+	0	-
Y	∩	$\frac{5}{18}$	∪	$\frac{5}{18}$	∩

19.3. Ekstremum topishda yuqori tartibli hosiladan foydalanish

Ba'zi funksiyalar uchun $x_0 \in (a;b)$da

$f'(x_0)=f''(x_0)=f'''(x_0)=...=f^{(n-1)}(x_0)=0$, $f^{(n)}(x_0)\neq 0$ bo'ladi. Bu holda qoldiq hadli Teylor formulasi bo'yicha (x-x_0) darajali bo'yicha yoyilma f(x)- $f(x_0)=\frac{f^n(x_0)+\alpha(x)}{n!}(x-x_0)^n$ ko'rinishiga ega bo'ladi, x→ x_0 da $\alpha(x)\to 0$ ekanligidan f(x)- $f(x_0)$ ishorasini $f^{(n)}(x_0)$ ishorasi hal qiladi.

Bunda ikki xil holat bo'lishi mumkin:

1) n-toq son, n=2k+1 x ning x_0 dan kichik qiymatlaridan katta qiymatlariga o'tganda $(x-x_0)^{2k+1}$ ishorasini o'zgartiradi va f(x)- $f(x_0)$ ham ishorasini o'zgartiradi. Demak, bu holda ekstremum mavjud emas.
2) n-juft son, n=2k. Bu holda $(x-x_0)^{2n}>0$ bo'lganligi uchun, f(x)- $f(x_0)$ ishorasi o'zgarmaydi, $f^n(x_0)$ ishorasi bilan bir xil bo'ladi.

Bulardan quyidagi qoida kelib chiqadi (K.Makloren, 1942).

Teorema (ekstremum topishning 3-qoidasi). Agar hosilalar ichida x_0 da nolga teng bo'lmaganlaridan birinchisi toq tartibli bo'lib qolsa, bu x_0 nuqtada ekstremum bo'lmaydi. Agar bu hosila juft tartibli bo'lsa, $f^n(x_0)<0$ da maksimum, $f^n(x_0)>0$ da minimumga ega bo'ladi.

Misollar. 1) $y=x^3$ uchun $y'=3x^2$, $y''=6x$, $y'''=6>0$ bo'lganligi uchun x=0 nuqtada ekstremum mavjud emas.
2) $y=x^4$ uchun $y'=4x^3$, $y''=12x^2$, $y'''=24x$, $y^{IV}=24>0$ bo'lganligi uchun $f_{min}(0)=0$ mavjud;
3) f(x)= e^x+e^{-x}+2cosx funksiya uchun x=0 kritik nuqtadir, chunki $f'(0)=e^x+e^{-x}$-2sinx=0

Endi $f''(0)=e^x+e^{-x}$-2cosx=0, $f'''(0)=e^x+e^{-x}$+2sinx=0,
$f^{IV}(0)=e^x+e^{-x}$+2cosx=0, $f^{V}(0)=4>0$ ekanligidan $f_{min}(0)=4$.

19.4. Funksiya grafigining asimptotalari

Funksiya xarakterini x→ ±∞ da, 2-tur uzilish nuqtalari yaqinida tekshirganda funksiya grafigi biror to'g'ri chiziqqa yaqinlashadi. Bunday to'g'ri chiziqlar **asimptotalar** deyiladi.

Asimptotalar uch xil bo'ladi: vertikal, gorizontal va og'ma.

1-ta'rif. Agar $\lim\limits_{x\to a\pm} f(x)=\pm\infty$ bo'lsa, x=a to'g'ri chiziq f(x) funksiyaga **vertikal asimptota** deyiladi.

Bu holda E(x;f(x)) nuqtadan x=a gacha masofa d=$\sqrt{(x-a)^2+(f(x)-f(x))^2}$ bo'lib, x→ a da d→ 0.

2-ta'rif. Agar $\lim\limits_{x\to a\pm\infty} f(x)=A$ bo'lsa, y=A to'g'ri chiziq f(x) funksiya grafigiga **gorizontal asimptota** deyiladi.

Bu holda E(x,f,(x)) nuqtadan y=A gacha masofa

d=$\sqrt{(x-x)^2+(f(x)-A)^2}=|f(x)-A|$ bo'lib x→ ∞ va d→ 0.

Misol. y=$\frac{1}{x}$ funksiya x=0 vertikal, y=0 gorizontal asimptotalarga egadir.

3-ta'rif. Agar $\lim\limits_{x\to\infty}[f(x)-(kx+b)]=0$ bo'lsa, u holda y=kx+b to'g'ri chiziq f(x) funksiyga **og'ma asimptota** deyiladi.

K=0 da og'ma asimptota gorizontal asimptota bo'lib qoladi.

$\lim\limits_{x\to\infty}\left[\frac{f(x)}{x}-(k+\frac{b}{x})\right]=0$ munosabatdan k=$\lim\limits_{x\to\infty}\frac{f(x)}{x}$, ta'rifdan b=lim $[f(x)-kx]$ tarzida topilishi kelib chiqadi.

Misol 1. y=$\frac{x^2}{1-x^2}$ funksiya asimptotalarini toping.

II tur uzilish nuqtalari 1-x^2=0 dan x=±1 ekanligi kelib chiqadi.

x=1 va x-1 to'g'iri chiziqlar vertikal asimptotalardir.

$\lim\limits_{x\to\infty}\frac{x^2}{1-x^2}=\lim\limits_{x\to\infty}\frac{1}{\frac{1}{x^2}-1}=-1$ bo'lganligi y=-1 to'g'iri chiziq gorizontal asimptota ekanligini ko'rsatadi.

k=$\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\frac{x}{1-x^2}=0, b=\lim\limits_{x\to\infty}[f(x)-kx]=\lim\limits_{x\to\infty}\frac{x^2}{1-x^2}=-1$

demak, og'ma asimptota gorizontal asimptota bo'lib qolgan.

Misol 2. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ giperbola og'ma asimptotalari $y=\pm\frac{b}{a}x$ to'g'ri chiziqlar ekanligini isbotlang.

$y=\pm\frac{b}{a}\sqrt{x^2-a^2}$ bo'lganligi uchun,

k=$\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\left[\pm\frac{b}{a}\sqrt{1-(\frac{a}{x})^2}\right]=\pm\frac{b}{a}$,

b=$\lim\limits_{x\to\infty}[f(x)-kx]=\lim\limits_{x\to\infty}\left[\pm\frac{b}{a}\sqrt{1-(\frac{a}{x})^2\pm\frac{b}{a}x}\right]=$

$\lim\limits_{x\to\infty}\left[\pm\frac{b}{a}\left(\sqrt{x^2-a^2}\right)-\pm\frac{b}{a}x\right]=\pm\frac{b}{a}\lim\limits_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x^2}=0$

Demak, y=$\pm\frac{b}{a}$x to'g'ri chiziqlar giperbola og'ma asimptotalari ekan.

Funksiya grafigini to‘liq tekshirishi sxemasi

Funksiyalarni tekshirish, grafigini chizish quyidagi qoidalar bo‘yicha amalga oshiriladi:
1. Funksiya aniqlanishi iloji bo‘lganda o‘zgarishi sohalarini topish.
2. Funksiya uzluksizligini tekshirish, uzilish nuqtalarini topish.
3. Funksiyaning juft-toqligi, davriyligini aniqlash.
4. Funksiyani jadval yordamida monotonlikka, ekstremumga tekshirish.
5. Funksiyani jadval yordamida qavariq-botiqlikka tekshirish, egilish nuqtalarini topish.
6. Funksiyaning abssissa va ordinata o‘qlari bilan kesishgan nuqtalari – nollarini topish.
7. Funksiya grafigi asimptotalarini topish.
8. Funksiya grafigini chizish.

Misol 1. $y=\dfrac{x^2}{x-1}$ funksiyani to‘liq tekshiring.

1) Funksiya $x\neq 1$ da, ya’ni $(-\infty;1)\cup(1;+\infty)$ da aniqlangan.

2) x=1 da II tur uzilishga ega, chunki

$$\lim_{x\to 1-0}\frac{x^2}{x-1}=-\infty, \qquad \lim_{x\to 1+0}\frac{x^2}{x-1}=+\infty,$$

3) y(-x)=$\dfrac{(-x)^2}{-x-1}=-\dfrac{x^2}{x+1}\neq\pm y(x)$, ya’ni funksiya juft ham, toq ham emas. Funksiya davriy emas.

4) $y'=\dfrac{x(x-2)}{(x-1)^2}$ bo‘lganligi uchun $x_1=0$, $x_2=2$ kritik nuqtalardir.

X	$(-\infty;0)$	0	(0;1)	(1;2)	2	$(2;+\infty)$
y’	+	0	-	-	0	+
Y	↗	0	↘	↘	4	↗
		max			min	

Ekstremum topishning 1-qoidasiga ko‘ra $f_{max}(0)=0$, $f_{min}(2)=4$.

5) $y''=\dfrac{2}{(x-1)^3}\neq 0$ bo‘lganligi uchun egilish nuqtalari mavjud emas, lekin x=1 nuqtada y’’ ishorasini o‘zgartiradi.

x	$(-\infty;1)$	$(1;+\infty)$
y’’	-	+
y	$\cap$	$\cup$

6) Funksiya uchun x=1 to‘g‘ri chiziq vertikal asimptota ekanligi 2) da tekshirildi.

$$k=\lim_{x\to\infty}\frac{x^2}{x-1}\cdot\frac{1}{x}=\lim_{x\to\infty}\frac{x}{x-1}=1,$$

$$b=\lim_{x\to\infty}\left[\frac{x^2}{x-1}-x\right]=\lim_{x\to\infty}\frac{x}{x-1}=1$$

bo‘lganligi uchun y=x+1 og‘ma asimptota bo‘ladi.

7) x=0 da y=0 va aksincha bo‘lganligi uchun, funksiya grafigi son o‘qlarini faqat 0(0;0) nuqtada kesib o‘tadi.

8). Topilganlar yordamida funksiya grafigi chiziladi.

19.5. Funksiyani eng katta va eng kichik qiymatlarini topish bo‘yicha masalalar

Agar f(x) funksiya chekli yopiq [a;b] oraliqda aniqlangan, uzluksiz bo‘lsa, uning eng katta (kichik) qiymatlari maksimum, minimum qiymatlarda yoki sohaning chegarasida bo‘lishi mumkin.

Demak, bunday qiymatlarni topish uchun kritik va chegaraviy nuqtalarda funksiya qiymatlarini topish va ularni o‘zaro solishtirish kifoya.

Misol: 1) f(x)=x^2-4x+6 funksiyaning [-3;10] kesmadagi eng katta va eng kichik qiymatlarini toping.

f'(x)=2x-4=0 dan x=2 kritik nuqta [-3;10] kesmaga tegishli ekanligini topamiz.

$f(-3)=(-3)^2-4(-3)+6=9+12+6=27$, f(2)= $2^2-4\,2+6=2$, f(10)=$10^2-4\,10+6$=66.

Demak, $f_{\text{э.kat}}(10)=66$, $f_{\text{э.kich}}(2)=2$.

Turli sohalarda funksiyaning eng katta, eng kichik qiymatlarini izlashga keltiriladigan masalalar ko‘p. Bunday masalalarda funksiya ekstremumga erishadigan nuqtalar muhimdir.

Masalalar. 1) Tomoni a ga teng kvadrat shaklidagi tunukaning burchaklaridan teng kvadratchalar qirqilib, chetlarini qayirib, ochiq to‘g‘ri to‘rtburchakli quticha yasaladi. Qanday qilib eng katta sig‘imli quti yasash mumkin?

Agar kesilgan kvadratchalar tomoni x bo'lsa, quticha asosi tomonlari a-2x, balandligi x bo'ladi. Hajmi y=x $(a-2x)^2$ funksiya bilan ifodalanadi, bunda $x \epsilon \left(0; \frac{a}{2}\right)$ bo'lishi mumkin.

$y' = (a-2x)(a-6x) = 0$ dan, faqatgina x=$\frac{a}{6}$ ildiz $(0;\frac{a}{2})$ oraliqdan ekanligini topamiz.

$y'' = -8a + 24x$ bo'lib, $y''\left(\frac{a}{6}\right) = -4a < 0$ ekanligidan, 2-qoidaga ko'ra, $y_{max}\left(\frac{a}{6}\right) = \frac{2a^3}{27}$.

2) 1 mlrd.so'm miqdoridagi kapital bankka yiliga 50% foydaga qo'yilishi yoki daromadidan p% soliq olinadigan ishlab chiqarishga 100% foydaga ijaraga berilishi mumkin. p ning qanday qiymatlarida kapitalni ishlab chiqarishga berish bankda saqlashdan foydaliroq bo'ladi?

Yechish. Faraz qilaylik, kapitalning x qismi ishlab chiqarishga ijaraga, (1-x) qismi bankka qo'yilsin. Bir yildan so'ng bankdagi kapital (1-x)(1+$\frac{50}{100}$) $= \frac{3}{2} - \frac{3}{2}x$, ishlab chiqarishga ajratilgan kapital esa 2x bo'ladi, lekin unda sarf-xarajat αx^2 ko'rinishda bo'lsa, foyda 2x-αx^2 bo'lib, undan (2x-αx^2)$\frac{p}{100}$ qismi soliqqa ketadi, sof daromad (1-$\frac{p}{100}$)$(2x-\alpha x^2)$ ko'rinishda bo'ladi. Demak, 1 yildan so'ng kapital:

$$y(x)=\frac{3}{2}-\frac{3}{2}x+\left(1-\frac{p}{100}\right)(2x-\alpha x^2)=\frac{3}{2}+\left[\frac{2\left(1-\frac{p}{100}\right)-}{\frac{3}{2}}\right]x-\alpha\left(1-\frac{p}{100}\right)x^2$$

miqdorida bo'ladi. Uning [0;1] kesmadagi maksimal qiymatini topish zarur. $y'(x) = 2\left(1-\frac{p}{100}\right) - \frac{3}{2} - 2\alpha\left(1-\frac{p}{100}\right)x = 0$ dan kritik nuqta $x_0 = \frac{2\left(1-\frac{p}{100}\right)-\frac{3}{2}}{2\alpha(1-\frac{p}{100})}$ kelib chiqadi.

$y''(x) = -2\alpha\left(1-\frac{p}{100}\right) < 0$ ekanligi, 2-qoidaga ko'ra, topilgan x_0 nuqtada maksimum bor ekanligini bildiradi. Uning [0;1] kesmaga tegishli bo'lishidan $0 < 2\left(1-\frac{p}{100}\right) - \frac{3}{2}$<1 yoki p<25 ekanligini topamiz.

Shunday qilib, p>25 bo'lsa, mablag'ni bankka qo'yish, p<25 da ishlab chiqarishga berishi ma'qul,

$$Y(x_0)=\frac{3}{2}+\frac{[2\left(1-\frac{p}{100}\right)-\frac{3}{2}]^2}{4\alpha(1-\frac{p}{100})}>\frac{3}{2}=y(0).$$

Iqtisodiy jarayonlar, asosan, 6 turdagi funksiya bilan ifodalanadi:

1) Bir xil tezlik bilan o'suvchi: y'>0, y''=0.
2) Monoton kamayuvchi tezlik bilan o'suvchi: y'>0, y''<0.
3) Monoton o'suvchi tezlik bilan o'suvchi: y'>0, y''>0.
4) Bir xil tezlik bilan kamayuvchi: y'<0, y''=0.
5) Monoton o'suvchi tezlik bilan kamayuvchi: y'<0, y''>0.
6) Monoton kamayuvchi tezlik bilan kamayuvchi: y'<0, y''<0.

Bu jarayonlar doimo bir xil xarakterdagi funksiya bilan ifodalanmaydi, egilish nuqtalari yordamida biridan ikkinchisiga otadi, aks holda, iqtisodiy inqirozlar yuz bermaydigan zamonlarda yashayotgan bo'lar edik.

Mavzuga doir misol va masalalar

1. f(x)=$\sqrt[3]{x^2-1}$ funksiya (-1;1)dan x=0 da eng kichik qiymatiga erishadi, lekin Ferma teoremasi o'rinli emas. Nima uchun?
2. f(x)=x(x^2-1) funksiya uchun [-1;1],[0;1] oraliqlarda Roll teoremasi shartlarini tekshiring.
3. Lagranj teoremasidan foydalanib, isbotlang:
1) $\frac{x}{1+x} < \ln(1+x) < x, x > 0$ 2) $e^x > ex, x > 1$
3) $|sinx - siny| \le |x-y|$.
4. f(x)=$x^2, g(x)$=x^3 funksiyalar uchun [-1;1] oraliqda Koshi teoremasi o'rinlimi?
5. **Lopital** qoidalari yordamida limitlarni toping.

1) $lim_{x\to 0}\frac{sin^2 ax}{sin^2 bx}$ 2) $lim_{x\to 0}\frac{tgx-x}{x-sinx}$ 3) $lim_{x\to\frac{\pi}{2}}\frac{tg3x}{tgx}$
4) $lim_{x\to 0}\cdot\frac{a^x-a^{sinx}}{x^2}(a>0)$ 5) $lim_{x\to 0}\frac{\ln(gosax)}{ln(cosbx)}$
6) $lim_{x\to 0}\frac{\cos(sinx)-cosx}{x^4}$ 7) $lim_{x\to 0}x^{\frac{k}{1+\ln x}}$ 8) $lim_{x\to\frac{\pi}{4}}(tgx)^{tg2x}$
9) $lim_{x\to a}\frac{a^x-x^a}{x-a}$(a> 0) 10)$lim_{x\to+\infty}(thx)^x$ 11) $lim_{x\to 0}(\frac{1+e^x}{2})^{cthx}$ 12) $lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

6. Makloren formulasi bo'yicha $0(x^2)$ hadgacha yoying:
1) y=e^{tgx} 2) y=lncosx 3) y=ln$\frac{1+2x}{1-x}$
7. Teylor formulasi bo'yicha $0((x-x_0)^2)$ hadgacha yoying.
1) y=$\frac{1}{x}$, x_0=2 2) y=xe^{2x}, x_0=1 3) y=$\frac{2x}{1-x^2}$, x_0=2.
8. Makloren yoyilmalaridan foydalanib, limitlarini hisoblang.

1) $\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}$ 2) $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$ 3) $\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}$

4) $\lim_{x\to 0}\frac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$ 5) $\lim_{x\to 0}\frac{e^x\sin x-x(1+x)}{x^3}$ 6) $\lim_{x\to 0}(\frac{1}{x}-\frac{1}{\sin x})$

7) $\lim_{x\to 0}\frac{a^x+a^{-x}-2}{x^2}$ (a>0) 8) $\lim_{x\to 0}\frac{1-(\cos x)^{\sin x}}{x^2}$

9. Funksiya monotonlik oraliqlarini aniqlang.

1) y=4+x-x^2 2) y=3x-x^3 3) y=$\frac{\sqrt{x}}{x+100}$ 4) y=x+sinx

5) y=$\frac{x^2}{2^x}$ 6) y=x^2lnx 7) y=xe^{-3x} 8) y=arctgx-lnx

10. Funksiya ekstremumlarni toping.

1) y=2+x-x^2 2) y=2x^2 -x^4 3) y=$\frac{x^4}{4}$-2x^3+$\frac{11}{x}$ x^2-6x+3

4) y=xe^{-x} 5) y=$\frac{\ln^2 x}{x}$ 6) y=x+$\sqrt{3-x}$ 7) y=x^x

11. Funksiyaning qavariqlik va botiqlik oraliqlarini toping.

1) y=e^x 2) y=lnx 3) y=x^5-10x^2+3x 4) y=$\frac{\sqrt{x}}{x+1}$ 5) y=e^{-x^2} 6) y=x+sinx

12. Funksiyanining egilish nuqtalarini toping.

1) y=cosx 2) y=1+x^2+$\frac{x^4}{2}$ 3) y=e^{2x-x^2} 4) y=$(x^2$-1$)^3$

5) y=$\frac{\ln x}{\sqrt{x}}$ 6) y=$\sqrt{1-x^3}$

13. Ko'rsatilgan sohada funksiyaning eng katta (kichik) qiymatlarini toping.

1) y=2^x [-1;5] 2) y=x^2-3x+2, [-10;10]

3) y=$\sqrt{5-4x}$, [-1;1] 4) y=6x^2-x^3; 5) y=y=x^2-6x+13 [0;6]

6) y=2sinx-cos2x, [0;$\frac{\pi}{2}$] 7) y=$\sqrt{\frac{1+x}{\ln x}}$, (1;e)

15. Berilgan funksiyani to'liq tekshiring, grafigini chizing.

1) y=5x^2-x^4; 2) y=$\frac{x^3}{3}$-x^5;

3) y=2x^2-8x; 4) $y=\frac{x}{x^2-1}$;

5) y=x-lnx; 6) y=$\frac{\ln x}{x}$

7) y=e^{-x^2} 8) y=$\frac{x^2-1}{x^2+1}$; 9) y=sinx+$\cos^2 x$;

10) y=x+arctgx; 11) y=ln(x+$\sqrt{x^2+1}$); 12) y=x^x.

16. Ekstremumga doir quyidagi masalalarni yeching.

1) yig'indisi a bo'lgan ikki musbat son qanday bo'lganda ko'paytmasi eng katta bo'ladi;

2) yuzasi S bo'lgan uchburchaklar ichida perimetri eng kichigini toping;

3) moddiy nuqta S(t)=$-t^3+9t^2$-24t-8 qonun bilan harakatlanadi. Uning maksimal tezligini toping;

4) y=x^2 dan y=2x-4 gacha eng qisqa masofani toping;

5) yon sirti S bo'lgan konsillar ichida hajmi kattasini toping;

6) shar hajmi unga ichki chizilgan eng katta hajmi silindr hajmidan necha marta katta bo'ladi;

7) eni a va b bo'lgan ikki kanal ko'ndalang kavlangan. Qanday uzunlikdagi g'o'lani bir kanaldan ikkinchiga o'tkazish mumkin;

8) to'la sirti S bo'lgan silindrlar ichida hajmi eng kattasi o'lchovlarini toping;

9) V hajmi qopqoqsiz silindirsimon baklar ichidan sirti eng kichigini toping;

10) R radiusli sharga ichkli chizilgan silindrlar orasidan hajmi kattasini toping;

11) R radiusli sharga ichkli chizilgan konuslar orasidan hajmi kattasini toping;

12) Doiradan α burchaklli sektor qirqilib, so'ngra undan konus yasalgan. α burchak kattaligi qanday bo'lganda konusning hajmi eng katta bo'ladi?

13) Eni bir xil uchta taxtadan nov (lotok) yasalmoqda. Nov yon devorlarining asosga og'ish burchaklari qanday bo'lganda nov ko'ndalang kesim yuzi eng katta bo'ladi?

14) Funksiyalar grafiklari asimptotalarini toping.

1) $y=\frac{3-4x}{2+5x}$; 2) y=$\frac{1+x^2}{1-x^2}$; 3) y=$\frac{1-x^2}{1+x^2}$; 4) y=$\frac{x^5}{2+x^4}$.

Mavzuga doir joriy nazorat uchun uy vazifasi

1. [-1;N], [1;N+1] kesmada f(x)= x^{N+1}, g(x)= x^{N+2}; funksiyalar uchun Koshi formulasini tekshiring.

2. Lopital qoidalari bo'yicha limitlarni toping.

1) $\lim_{x\to 0}\frac{(N+1)^x-(N+1)^{\sin x}}{x^2}$; 2) $\lim_{x\to 0}\frac{\ln(\sin Nx)}{\ln(\sin(100-N)x)}$; 3) $\lim_{x\to 0+} x^{\frac{N}{1+\ln x}}$;

3. 0<y<x uchun (N+1)$y^N(x-y)\le x^{N+1}-y^{N+1}\le(N+1)x^N-(x-y)$ tengsizlikni isbotlang.

4. Makloren yoyilmalari yordamida toping.

1) $\lim_{x\to 0}\frac{\cos Nx-e^{-\frac{x^2}{2N}}}{x^4}$ 2) $\lim_{x\to 0}\frac{(N+1)^x-(N+1)^{-x}-2}{x^2}$

5. Quyidagi funksiyalarni to'liq tekshiring, grafigini chizing.

1) y=$\frac{x^{N+1}}{N+1}-Nx$; 2) $y=(x-N)\sqrt{N+x}$; 3) y=xe^{-Nx};

4) y=$\frac{x^k}{x^2-(-1)^N N^2}$, *bunda* k=$\begin{cases}1, agar\ N=3k-2\\ 2, agar\ N=3k-1\\ 3, agar\ N=3k.\end{cases}$

6. To'la sirti N bo'lgan, kvadrat asosli, qopqoqsiz yashik eng katta hajmga ega bo'lishi uchun, uning o'lchamlari qanday bo'lishi kerak?

7. Sirti N bo'lgan, usti ochiq silindrik bak asosi radiusi va balandligi qanday bo'lganda eng katta hajmga ega bo'ladi?

8. Eni 27.N va 64.N bo'lgan ikki kanal ko'ndalang joylashgan. Bir kanaldan ikkinchisiga qanday maksimal uzinlikdagi kemalar o'ta oladi?

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MUNDARIJA

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IQTISODCHILAR UCHUN OLIY MATEMATIKA FANIDAN MA'RUZA VA MASHQLAR

(O'quv qo'llanma)

1-qism

Muharrir M.A.Xakimov

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