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MASALA QANDAY YECHILADI?

(Uslubiy qo‘llanma)

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Ushbu qo‘llanma ko‘p yillardan beri o‘tkazilib kelinayotgan talabalar o‘rtasidagi ichki va respublika matematika fan olimpiadalari dasturidan o‘rin olgan jami 155 ta misol va masalalardan tashkil topgan. Qo‘llanmadagi misol va masalalarning 125 tasi yechilib, to‘la isboti bilan berilgan. Qolgan 30 tasi mustaqil yechish uchun tavsiya etilgan. Bulardan tashqari, olimpiadalarda taklif etilgan jami 150 ta test materiallari kalitlari ham o‘rin olgan. Qo‘llanma olimpiadaga tayyorgarlik ko‘rayotgan talabalar va ularning ustozlari uchun tayyor material hisoblanadi.

Qo‘llanma akademik litsey va kasb-hunar kollejlari hamda oliy o‘quv yurtlari talabalari uchun mo‘ljallangan.

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SO‘ZBOSHI

Respublikamiz ta’lim-tarbiya tizimida qator islohiy o‘zgarishlar amalga oshirilgan bo‘lib, ularning asosiy maqsadi yosh avlodni layoqati, qobiliyati, iqtidorini aniqlash, ochish va ularning rivojlanishi uchun shart-sharoit va imkoniyatlarni yaratishdan iboratdir.

Bular o‘quvchilarni fanga bo‘lgan qiziqishini shakllantirish, faolligini oshirish va ularni rag‘batlantirish bilan bog‘liqdir. Olimpiada ishtirokchilarning ijodiy faolligini o‘stirishda, tafakkur jarayonlarini shakllantirishda mantiqiy fikrlash, matematikadan nostandart masalalarni yechish alohida ahamiyat kasb etadi.

Iqtidorli talabalarning fanlardan egallagan bilimlari qanchalik chuqur va mustahkamligini, ularning ijodiy fikrlash doirasining kengligini, quvvaiy hofizasining kuchliligini aniqlovchi mezonlardan biri olimpiadadir.

Olimpiadada qatnashishni orzu qilgan har bir talaba chuqur mas’uliyatni his qilgan holda, oliy o‘quv yurtida olgan bilimlari bilan cheklanib qolmay, maxsus adabiyotlardan dastlabki tayyorgarlikni o‘tashi kerak.

Qo‘llanmada taniqli matematik D. Poyaning masala yechish texnologiyalari bo‘yicha bildirgan fikrlaridan keng foydalanilgan.

Mualliflar

1-§. Masala qanday yechiladi?

Biz masala bilan tanishamiz

Nimadan boshlashim kerak? Masalaning formulirovkasidan boshlang. Nima qilishim mumkin? Siz masalani shunday tasavvur qilingki, u, iloji boricha, to‘liq, aniq, ravshan bo‘lsin. Hozircha detallariga e’tibor qilmang.

Bu bilan men nimaga erishaman? Masalaga e’tiboringizni qaratib, sizga nima foydali ekanligiga, undan qanday natijalar kelib chiqishiga o‘z xotirangizni tayyorlaysiz.

1.1. Biz masalani tushunib yetamiz

Men nimadan boshlashim kerak? Yana masalaning formulirovkasidan boshlang. Shunday boshlangki, bu paytda masala sizning shu qadar ongingizda mustahkam o‘rnashgan bo‘lsin va siz vaqtinchalik undan uzoqlashishingiz, hech qanday qiyinchiliksiz uni unutishingiz mumkin.

Nima qilishim mumkin? Masalani bosh elementlarga ajrating. Bosh elementlar boshlang‘ich va oxirgi shartlarni ko‘rsatsa (ifodalasa) “isbotlanadiganlar, masalalar, noma’lumlar, berilganlar hamda shartlarni ko‘rsatadigan bosh elementlar topiladigan masalalar”. Masalangizdagi bosh elementlarni o‘rganing, ularni alohida-alohida qarab chiqing, undan keyin birini boshqasi orqali ketma-ket, keyin esa xilma-xil tarzda, har detalni boshqa bir detal orqali ko‘rib chiqib, butun masalani idrok etamiz.

Bu bilan nimaga erishaman? Siz masalaning detallarini tushunib olasiz, bu detallar keyinroq masalani yechish davomida muhim rol o‘ynaydi.

1.2 Biz samarali (foydali) g‘oyalar izlaymiz

Nimadan boshlashim kerak? Bosh elementlarni qarab chiqishdan boshlang. Shunday boshlangki, bajariladigan ishning natijasida sizning bu ishni yaxshi tushunishingiz va uni bir aniq sistemaga solib olganingiz ko‘rinsin. Shu vaqtda boshlangki, unda sizning xotirangiz deyarli aniq va sizga bo‘ysunadugan bo‘lsin.

Nima qilishim mumkin? Siz egallagan bilimlar bilan bu masalaning tutashadigan nuqtalarini aniqlang. Har xil elementlarni tekshiring, bitta detalni bir necha marta tekshiring, ammo har xil nuqtayi nazardan tk-shiring, detallarni har xil solishtiring, ularga turli tomondan yondashing. Masalani butunligicha yangicha talqin qiling.

Bunda men nimaga tayanishim mumkin? Samarali, foydali g‘oyaga, bu biz maqsadga erishishimiz uchun yo‘l ko‘rsatadigan hal qiluvchi g‘oya bo‘lishi ham mumkin.

G‘oyaning samaradorligi nimadan tashkil topgan bo‘lishi mumkin? Bunday g‘oya sizga butun yo‘lni yoki uning qismini ko‘rsatadi, u sizga ko‘proq yoki kamroq aniqlikda, qanday harakat qilish kerakligini ko‘rsatadi. G‘oyalar, odatda, deyarli to‘liq yoki to‘liqmas bo‘ladi. Siz omadlisiz, agarda sizda, hech bo‘lmaganda, qandaydir g‘oya bo‘lsa.

To‘liqmas g‘oya bilan nima qlilishim mumkin? Uni ko‘rib chiqish kerak. Agar u qandaydir ma’noda foydaliroq ekanligini bilsangiz, u holda mukammal qarab chiqish lozim. Agar sizga unga ishonish mumkinday tuyulsa, siz uning yordami bilan nimalarga ega bo‘lishingiz mumkinligini tekshirib ko‘rishingiz kerak. Sizda endi foydali g‘oya borligidan, sharoit o‘zgaradi. Sodir bo‘lgan holatni har tomondan qarab chiqing va uning o‘zingizdagi bilim va oldin egallagan bilimlaringiz bilan to‘qnashadigan nuqtalarni aniqlang. Shunga o‘xshash holatlarda sizga nima yordam berganini eslashga harakat qiling.

Bu bilan nimaga erishaman? Sizga omad kulishi mumkin, siz yana yangi g‘oyaga duch kelib qolishingiz mumkin. Balki, keyingi g‘oya sizni to‘g‘ri yechim sari olib borar, balki, sizga yana bir necha omadli g‘oyalar kerak bo‘lib qolar. Nima bo‘lganda ham, siz barcha yangi g‘oyalardan minnatdor bo‘lishingiz kerak, shu qatorda oddiy, yordam bergan aniqlashtirgan va uncha omadli bo‘lmagan g‘oyalardan ham. Hozir sizga, balkim, qandaydir bebaho yangi g‘oya duch kelmagandir, lekin siz masalaning to‘liqroq, bog‘liqroq, asosiyroq qismiga o‘ta boshlaganingizdan qoniqishingiz lozim.

1.3 Biz rejani amalga oshiramiz

Nimadan boshlashim kerak? Sizni yechim sari yetaklovchi baxtli g‘oyadan boshlang. Shunday boshlangki, siz u paytda asosiy va bosh fikrni mahkam ushlagan, hamda o‘zingiz har bir detalni mohiyatini tushunib yeta oladigan holatda bo‘lishingiz kerak. Bu detallar oldinda sizga kerakli bo‘ladi.

Nima qilishim mumkin? Omadingizni mustahkamlang. Barcha detallar ustida algebraik va geometrik amallar bajaring. Har bir qadamning to‘g‘riligiga mantiqiy tasavvur, fikrlar orqali yoki boshqa yo‘llar bilan ishonch hosil qilasiz.

Agar masala juda qiyin bo‘lsa, unda siz “katta” va “kichik” qadamlardan ajratib, har bir katta qadamni bir nechta kichik qadamlarga bo‘lishingiz mumkin. Avval, katta qadamlarni tekshirib ko‘ring, keyin, kichik qadamlarga o‘tasiz.

Bu bilan nimaga erishaman? Shunga erishasizki, sizning qo‘lingizda yechim bo‘lib, bunda har bir qadam inkorsiz to‘g‘ri bo‘ladi.

1.4 Biz orqaga nazar solamiz

Nimadan boshlashim kerak? Yechimdan, sizga ma’lum bo‘lgan detallarning to‘liq va to‘g‘riligidan boshlaysiz.

Nima qilishim mumkin? Yechimni turli yomondan sinchiklab o‘rganing va o‘zingizning bilim va ko‘nikmalariningizga tayangan holda, yaqinlashish nuqtasini toping.

Yechimning detallarini topib chiqing, uni maksimal darajada oddiylashtirishga harakat qiling: yechimning ifodalanish qismiga e’tibor qiling va, iloji boricha, qisqaroq qilishga harakat qiling.

Yechimning katta va kichik qismlarini ajratishni va ularni yig‘ib, yaxlit bir butun yechim qilishni va uni yana-da aniqroq bo‘lishini maqsad qilib qo‘ying. Bu metodga, ya’ni sizni yechimga olib borgan metodga qarab, unda nima asosiy hisoblanadi va uni boshqa masalalarga qanday qo‘llash xususida o‘ylab ko‘ring.

Bundan nimaga erishaman? Siz yangi, eng yaxshi yechimni topasiz hamda yangi qiziqarli ma’lumotlarga ega bo‘lasiz. Har qanday holda ham agarda siz qarab, o‘rganib chiqish va baholashni olingan yechimga yuqorida ko‘rsatilgandek qo‘llay olsangiz, sizning masalani yechish mahoratingiz oshib boradi.

2-§. Geometrik masala yechish bo‘yicha qisqa evristik lug‘at

Analogiya – yaqinlashish turi. Yaqinlashuvchi predmetlar bir-biri bilan qandaydir ma’noda munosabatda bo‘ladi, analogik predmetlar ularning mos keluvchi qismlari bilan ayrim munosabatlarda bo‘ladi.

1. To‘g‘ri burchak to‘g‘ri burchakli parallelepiped bilan analogikdir. Haqiqatdan ham, to‘g‘ri burchakning tomonlari orasidagi munosabati parallelepipedning yoqlari orasidagi munosabatga yaqin. To‘g‘ri burchakning har bir tomoni parallel va teng boshqa bir tomoniga nisbatan va qolganlariga perpendikulyar. To‘g‘ri burchakning tomonini va parallelepipedning yoqlarini “chegaraviy element” deb atashga kelishib olamiz. Shu tarzda bularning ikkalasiga bir xil tarzda qarab boshlaymiz. Har bir chegaraviy element birgina chegaraviy elementga teng va parallel va boshqa chegaraviy elementga perpendikulyar. Shu tarzda biz shunday munosabatni hosil qildikki, ikkala taqqoslanayotgan obyekt sistemasi uchun bir xil ya’ni umumiy – to‘g‘ri burchakning tomoni va to‘g‘ri burchakli parallelepipedning yoqlari. Bu ikkala sistema o‘rtasidagi analogiya munosabatlarning umumiyligi bilan tugallanadi.

2. Analogiyaga bizning barcha fikrlashlarimiz (tafakkurimiz) kiradi; bizning kundalik nutqimiz va trivial aql-idrokimiz, adabiy tilimiz va yuqori (oliy) fan yutuqlarimiz. Analogiya darajasi turli xilda bo‘lishi mumkin. Insonlar doimo analogiyaning aniqmas, ikki hayollik to‘liqmas yoki deyarli to‘liq turlarini qabul qiladi. Lekin bizga analogiyaning hech qanday ko‘rinishini qabul qilish muhim emas, ularning har biri yechimni izlash jarayonida muhim rol o‘ynashi mumkin.

3. Agarda biz berilgan masalaning yechimini izlashga uringanimizda, bizga oddiygina analogik masala yo‘liqsa, unda bizga omad kulib boqdi, deb hisoblaymiz.

Biz quyidagi masalani qaraylik. Tetraedrning og‘irlik markazini toping? Bu masala oson masalalardan emas. Agarda uni yechishga integral hisobni va mexanikani chala bilib kirishsak, hech nimaga ega bo‘la olmaymiz. Bu o‘z zamonida Arximed yoki Galileylar davrida jiddiy ilmiy muammolardan biri bo‘lgan. Masalani tekislikda tasavvur qilsak, darhol ko‘z o‘ngimizga quyidagi masala keladi:

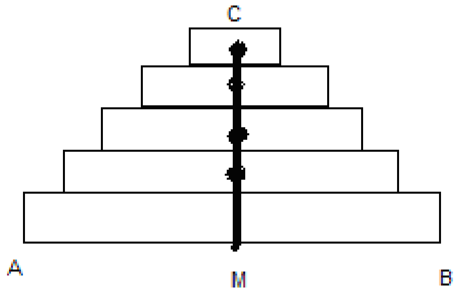
Bir jinsli uchburchakning og‘irlik markazini toping.

Endi bizda bir savolning o‘rniga ikkita savol bor. Lekin bu ikki savolga javob berish, bittasiga javob berishdan ham oson bo‘lishi mumkin, agarda bu ikki savol bir-biri bilan bog‘langan bo‘lsa.

4. 1-masalani vaqtinchalik to‘xtatib qo‘yib, diqqat-e’tiborimizni oddiyroq analogik masala – uchburchak haqidagi masalaga qaratamiz. Bu masalani yechish uchun og‘irlik markazi haqida ozgina bo‘lsa ham, bilishimiz kerak. Hozirgi prinsip bizga tog‘ri va aniq ko‘rinadi. Agar S-sistemalar massasi qismlardan tashkil topgan bo‘lib, masalalar markaz-

lari shu tekislikda joylashgan bo‘lsa, shu tekislikda butun S-sistemaning ham masalalar markazi yotadi. Bu prinsip bizga uchburchak masalasini yechishda juda qo‘l keladi.

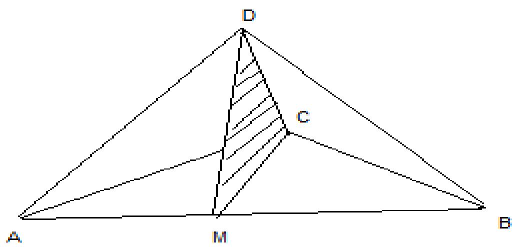
Birinchidan, bunda uchburchakning og‘irlik markazi uchburchak tekisligida yotadi, ikkinchidan, biz uchburchak to‘g‘ri chiziqli qatlamlardan – “cheksiz ingichka (top) – parallelogramlardan tashlkil topgan”, deb hisoblashimiz mumkin.



Har bir polasaning og‘irlik markazi uning markazi bilan ustma-ust tushib turibdi. Polasaning M bilan C ni tutashtiruvchi kesimda joylashgan. CM orqali o‘tuvchi ixtiyoriy tekislik, uchburchakni tashkil qiluvchi barcha parallel polasalarning og‘irlik markazini saqlaydi.

Shunday qilib, biz quyidagi xulosaga kelamiz: uchburchakning og‘irlik markazi medianalarning kesishgan nuqtasida bo‘lar ekan.

5. Tetraedrning og‘irlik markazini topish uncha qiyinchgilik tug‘dirmaydi, chunki biz undan qiyinroq bo‘lgan masalani yechdik hamda namuna sifatida undan foydalanamiz. Biz yuqoridagi ishimizda, analogik masalani yechish davomida *ABC* uchburchak qatlamlaridan tashkil topgan bo‘lib, bu qatlamlar bir-biriga parallel joylashgan, bir tomonga nisbatan. Masalan, endi biz faraz qilamizki, *ABCD* tetraedr ham parallel qatlamlardan (ya’ni, qirralaridan birga, masalan, *AB* ga) tashkil topgan.



Uchburchak tashkil qiluvchi polasalarning o‘rasida *AB* tomonning o‘rtasi M ni qarama-qarshi C bilan tutashtiruvchi medianada yotadi.

6. Tetraedr tashkil qiluvchi qatlamning o‘rtasi M, qirraning o‘rtasidan, AB dan va qarama-qarshi CD qirra bilan tekislikda yotadi. MCS tekislikni biz teraedrning “mediana tekisligi” deb atashimiz mumkin. Uch-

burchak og‘irlik markazini topish masalasida biz 3 ta medianaga ega edik, ularning har biri uchburchakning og‘irlik markazini o‘z ichiga oladi (qamrab oladi).

Shuning uchun bu 3 mediana bir nuqtada, ya’ni uchburchakning og‘irlik markazida kesishishi shart. Tetraedr masalasida biz 6 ta medianaviy tekisliklarga egamiz, ular qirralarning (qarama-qarshi) o‘rtasidan o‘tadi. Ulardan har biri tetraedrning og‘irlik markazi masalasi yechilgan hisoblanadi. Masalani yechishning oxirida geometrik yo‘l bilan hech qanday taxminlarsiz tetraedrning 6 ta mediana tekisliklari faqat 1 ta nuqtadan o‘tishini ko‘rsatish kerak. Bir jinsli uchburchakning og‘irlik markazi haqidagi masalada biz uning 3 ta medianasi bitta nuqtada kesishishiga ishonch hosil qilgandik. Bu masala uchun uchburchak analogik masalasini qo‘llaymiz. 3 ta medianaviy tekislikni, 3 ta DA, DB, DC qirradan o‘tuvchi tekisliklarni qaraymiz.

Ulardan har biri qarama-qarshi qirraning o‘rtasidan o‘tadi (masalan, DC dan o‘tadigan medianaviy tekislik M nuqtadan o‘tadi). Shunday qilib, bu 3 ta medianaviy tekisliklar ΔABC tekisligida uning 3 ta mediana kesishadi. Bu nuqta, xuddi D nuqta kabi, bir vaqtda 3 ta medianaviy tekisliklarga tegishli bo‘ladi. Ikki nuqtadan o‘tuvchi to‘g‘ri chiziqlar har biri 3 medianaviy tekisliklarga tegishli bo‘ladi.

Bu 3 ta medianaviy tekisliklar (D nuqtadan o‘tuvchi) bitta va faqat bitta to‘g‘ri chiziqdan o‘tishini ko‘rsatdik. Bu, albatta, A nuqtadan B va C nuqtalardan o‘tuvchi medianaviy tekisliklar uchun ham o‘rinli.

Bu faktlarga asoslanib, 6 ta medianaviy tekisliklarning bitta nuqtadan o‘tishini ko‘rsatish qiyin emas. ΔABCning tomonidan o‘tuvchi 3 ta medianaviy tekisliklar bitta nuqtada kesishadi, lekin biz har bir kesishish nuqtasidan yana bitta medianaviy tekislik o‘tishi mumkinligini isbotladik.

7. 5- va 6-bandlarda biz oddiy analogik masalalarni: uchburchak masalasi va tetraedr masalalarini qarab chiqdik. Uni yechish bevosita uchburchak masalasi bilan bog‘liq. Bu ikkita misol bir-biridan juda muhim munosabatda farq qiladi. 5-bandda biz oddiy analogik masala metodini qo‘lladik. 6-bandda biz shu oddiy analogik masalaning natijasidan, bu natija qanday olinganiga qiziqmagan holda foydalandik. Ba’zi hollarda masalani yechishda biz bir vaqtning o‘zida ham metod va natijalardan (oddiy analitik masalaning) foydalanishimiz mumkin.

Bizning masalamiz tipik. Bu masalani yechish davomida biz, undan ko‘ra oddiyroq analogik masalaning yechimidan foydalanishimiz mumkin.

Xususiy holda analogik masalaning yechimi berilgan masalaning yechimiga asos bo‘la olmaydi. U holda masalani qarayotgan masala yechimiga asos bo‘ladigan holda, o‘zgartirishlar, formaga keltirish orqali qarab chiqish kerak.

8. Bizga bir jinsli uchburchakning og‘irlik markazi uning 3 ta balandligining og‘irlik markazi bilan to‘g‘ri kelishi ma’lum. Buni bilgan holda, biz bir jinsli teraedrning og‘irlik markazi uning 4 ta balandligining og‘irlik markazi bilan to‘g‘ri kelishini aytishimiz mumkin. Bu oxirgi taxmin “analogiya natijasi”dir. Buni bilgan holda, uchburchak tetraedr bir-biriga ko‘pgina munosabatlarda o‘xshash. Biz shunday aytamizki, ular bir-biriga yana bir munosabatda o‘xshash.

Analogiyaning xulosasi sifatida sifatida ehtimol qilish (faraz qilish) sanaladi. U bizni kamroq yoki ko‘proq to‘g‘ri bo‘lgan taxminlarga (bo‘lar yoki bo‘ladi yoki bo‘lmaydi) olib boradi.

9. Analogiya – asosining natijasi ko‘pgina parallel faktlarni bog‘lab turadi. Ularning sonidan sifati ustun qo‘yiladi.

Biz yana xuddi shunday holni ko‘rib chiqamiz. Bu bir jinsli kesma holidir.

Kesmalar orasidagi analogiyada, biz yuqoridagi uchburchak va tetraedr orasidagi analogiyadan foydalangan holda, bu figuralarni turli xil nuqtai nazardan taqqoslaymiz. Kesma biror to‘g‘ri chiziqqa tegishli, uch burchak – tekislikka, tetraedr – fazoga tegishli.

To‘g‘ri chiziq kesmasi oddiy bir o‘lchovli chegaralangan figura, uchburchak oddiy ko‘pburchak, tetraedr – oddiy ko‘pyoq. Kesma 2 ta elementga ega: 2 ta chegaraviy nuqtalar. Kesmaning ichki nuqtalari bir o‘lchovli to‘plamni tashkil qiladi.

Uchburchak 3 ta nol o‘lchovli va 3 ta bir o‘lchovli chegaraviy elementlar (3-balandlik, 3-tomon)ga ega. Ichki nuqtalari 2 o‘lchovli to‘plam tashkil qiladi.

Tetraedr 4 ta nol o‘lchovli, 6 ta bir o‘lchovli va 4 ta 2 o‘lchovli elementlar (4 balandlik, 6 ta qirra, 4 ta yoq)ga ega. Ichki nuqtalari 3 o‘lchovli to‘plam tashkil qiladi. Bu raqamlardan quyidagi jadvalni tuzib olamiz:

Ketma-ketlik ustunlarda 0, 1, 2 va 3 o‘lchovli elementlarning raqamlari, ketma-ketlik satrlarda bo‘lsa, kesma, uchburchak, tetraedr elementlariga taalluqli.

2	1		
3	3	1	
4	6	4	1

Biz bilamizki, bu Paskal uchburchagining qismini ifodalab turibdi. Biz bundan kesma, uchburchak va tetraedrni bir-biri bilan bog‘lovchi ajoyib qonuniyatga ega bo‘ldik.

Bir jinsli kesmaning og‘irlik markazi uning 2 ta chegaraviy nuqtalarning og‘irlik markazi bilan ustma-ust tushadi. Bir jinsli uch burchakning og‘irlik markazi uning 3 ta balandliklarining og‘irlik markazi bilan ustma-ust tushadi.

Biz bundan bir jinsli tetraedrning og‘irlik markazi uning 4 ta balandligining og‘irlik markazi bilan ustma-ust tushishi mumkinligi haqida taxmin qilishimiz mumkin.

Aniqroq qilib aytganda, bir jinsli kesmaning og‘irlik markazi uning chegaraviy nuqtalari bilan 1:1 kabi nisbatda bo‘ladi. Uchburchakning og‘irlik markazi uning ixtiyoriy balandligi va qarama-qarshi tomonning o‘rtasi bilan 2:1 nisbatga bo‘ladi. Bunda biz bir jinsli tetraedrning og‘irlik markazi ixtiyoriy balandligi va qarama-qarshi yoq og‘irlik markazi bilan 3:1 nisbatni hosil qiladi. Biz bu yuqorida qilgan farazlarimiz, $n=1, 2, 3$ sonlar uchun to‘g‘ri bo‘lsa n ning ko‘p qiymatlarida to‘g‘ri bo‘lishini matematik induksiya metodi yordamida ko‘satishimiz mumkin.

Biz yuqorida ko‘rib o‘tgan mulohazalarni qisqacha muhim hollarni ko‘rib chiqish bilan yakunlaymiz.

1-hol. S va S' 2 ta matematik obyektlar sistemasini qarab chiqamiz.

Obyektlar orasidagi munosabatda S bo‘ysunadigan qonunga, obyektlari orasida aniqlangan munosabatga ko‘ra S' ham bo‘ysunadi.

2-hol. S va S' sistema obyektlari orasida bir qiymatli moslik aniqlangan. Bu shuni bildiradiki, agar shunday munosabatga bir sistemaning elementlari ham xuddi shunday munosabatga ega bo‘ladi.

Bunday 2 ta obektlar sistemasi orasidagi bog‘liqlik analogiyaning aniq bir ko‘rinishi hisoblanadi va u izomorfizm deb ataladi.

Bu matematikaning, ayniqsa, gruppalar nazariyasida muhim rol o‘ynaydi. Lekin hozir biz uni batafsil o‘rganmaymiz. Ajoyib g‘oya yoki omadli fikr masalaning yechimiga olib boruvchi omillardir. Bunday g‘oyalarning tug‘ilishi tasvirlashga qiyinchilik tug‘diradi, bu hammaga ham ma’lumdir.

Aristotel kabi, qadimiy olimning tasvirlashlarida biz uni ko‘ra olishimiz mumkin. Ko‘pchilik shunday fikrlaydiki, ajoyib g‘oyaning tug‘ilishi bizning “pronitsatelnosti”ga bog‘liq. Aristotel buni quyidagicha aniqlaydi:

“Pronitsatelnosti” – bu qisqa vaqt mobaynida narsalarning haqiqiy bog‘lanishini o‘ylab topish yo‘liga ega bo‘lish qobilyati. Masalan, agar siz kimdir boy odam bilan gaplashayotganini ko‘rib qoldingiz. Siz darrov payqab, sizni bu inson undan qarzga pul olmoqchi yoki oyning yorug‘lashgan tarafi bilan quyoshga qaraganini kuzata turib, siz birdaniga buning sababi: oy quyosh yorug‘ligining aksi bilan yoritishini sezib qolasiz. Birinchi misol yomon emas, lekin yetarlicha trial: bu holda nimadir topishda, masalan, badavlat kishi va pul masalasida unchalik “pronitsatelnosti” talab qilinmaydi: bu vaqtda o‘ylangan fikrni juda ham ajoyib deb sanab bo‘lmaydi, ya’ni 2-misolimiz har bir odamda, haqiqatan ham, chuqur taassurot qoldiradi.

Biz hamma vaqt Aristitelning zamondoshiga vaqtni, sutkani bilishi uchun quyosh va yulduzlarni kuzatganini yodimizda saqlashimiz kerak. Unga oyning fazolarini o‘rganish to‘g‘ri kelgan, agar u tunda yo‘lga chiqmoqchi bo‘lgan bo‘lsa. Albatta, bu vaqtda ko‘cha fanarlari bo‘lmagan. U zamonaviy shaharlikdan ko‘ra, yulduzli osmon haqida ko‘proq bilimlarga ega bo‘lgan. U oyni bir disk misolida ko‘rgan, uni quyoshga o‘xshatgan, lekin yorug‘ligi kamligini o‘ylagan. U oyning uzluksiz formalarining va joylashishini ko‘rib ajablangan. Unga oyni kunduz kuni quyoshning chiqishi yoki botishi paytida uzatishga ham to‘g‘ri kelgan.

Bunda u quyidagicha xulosa chiqargan “oyning yorug‘ qismi doimo quyoshga qarab turadi”. Bu, albatta, uning e’tiboriga molik yutug‘idir. Nihoyat, u oyning fazosi juda ham bir tasvirni eslata, unda biz shartning turli xil tomonlaridan qaraganimizda yarmiga yon tomondan yorug‘lik tushib turishini anglagan. Endi u quyosh va oyni disklar ko‘rinishida emas, balki dumaloq tana ko‘rinishida tasavvur qiladi. Bulardan biri o‘zi yorug‘lik, chiqaradi, boshqasi bo‘lsa, yorug‘likni qaytaradi. Keyinchalik uning tasavvurida o‘zgarishlar paydo bo‘lib, uning kallasiga ajoyib fikr keldi, endi bu fikrni biz trivial deb atay olmaymiz.

Bolsano – Bernard (1781–1843) logik va matematik. Bo‘lajak matematik masala yechish davomida keng fikrli bo‘lishi kerak, lekin bu kam. Shunday payt keladiki, unga jiddiy matematik muammolarni yechish masasalasi to‘g‘ri keladi. Shuning uchun, hamma narsadan oldin, unga masalalarni yechishda hammasidan ko‘ra uning tabiat in’omi bo‘lmish qobilyati kerakligini e’tirof etish lozim shuning uchun ishning eng muhim qismi olingan yechimni qayta ko‘rib chiqish hisoblanadi. Ishning tartibini va yechimini analiz qilish davomida, u ko‘pgina qiziqarli-narsalarni o‘rganib olishi mumkin. Balki, u masalaning qiyinlik darajasi va uning asosiy g‘oyasi to‘g‘risida ham o‘ylab chiqishi mumkin. Ya’ni u o‘ziga nimalar xalaqit bergani va oxirida nima unga yordam berganini aniqlashga urinishi mumkin. U e’tiborini oddiy intuntiv g‘oyalarga: natijani bir qarashda ko‘rib chiqish mumkinmi? U yana turli metodlardan ham foydalanishi mumkin. Bu natijani boshqacha yo‘l bilan ham olish mumkinmi?

U berilgan masalani chuqurroq tushunishga, uni oldingi yechilgan masalalar bilan taqqoslashga urinib ko‘radi. U yana yangi masalalar o‘ylashga urinishi mumkin, ular hozir bajarilganish, ya’ni oldingi masala qanday yechilgan bo‘lsa, ular ham shu asosida yechilishi kerak. Boshqa biror masalada olingan natijani yoki yechish metodini qo‘llash mumkinmi?

Bo‘lajak matematik, barcha insonlar singari tajriba yordamida o‘qib o‘rganadi. U yaxshi o‘qituvchining ishlarini kuzatishi, qobilyatlar do‘stlari bilan musobaqalashishi kerak. Yana, u o‘zini faqat darsliklar bilan chegaralab qoymasdan, balki yaxshi mualliflar kitoblari bilan qiziqishi lozim. Uni barcha oddiy, go‘zal narsalar quvontira olishi kerak. Bularni barini u izlashi shart. U o‘z qiziqishlari asosida masala tanlab yechishi va yechimni tekshirish va boshqa masalalarga qo‘llashi kerak. Shu yo‘l va boshqa barcha yo‘llar bilan u o‘zining birinchi muhim kashfiyotini – o‘zini bilish, unga nima yoqadi, nima yoqmaydi, didini bilish, shaxsiy qiziqishlari kabi savollarga javob topadi.

Masala ko‘rinishini o‘zgartirish

Hasharot deraza oynasi orqali uchib ketishga harakat qilmoqda va bu o‘zining foydasiz ishini yana va yana takrorlamoqda. U to‘g‘ridagi ochiq turgan, o‘zi uchib kirgan deraza orqali uchib ketishga harakat ham qilmaydi.

Sichqon bo‘lsa, birmuncha yaxshiroq harakat qiladi: qafasga tushib, u qafasning temirlari orasidan o‘tishga urinadi, keyin boshqalarining orasidan ham va hokazo. U o‘zining sinashlarining har xil variantlarini tekshirib boshlaydi.

Inson masalani yechish jarayonida uning mohiyatini keng tushunib, har xil variantlarini tekshira oladi, uning yo‘nalishlarini o‘zgartirib, aqlli yechim topa oladi. Inson o‘zining xatolik va yetishmovchiliklarni topa olishi kerak. “Urin, yana urin” – maqolda donishmand xalqning nasihati namoyon bo‘lgan. Hasharot, sichqon va inson bu nasihatga rioya qiladi. Agar bittasi omadliroq bo‘lsa, boshqalarga qaraganda, bu shuning uchun o‘zining masala ko‘rinishini yaxlit ravishda o‘zgartiradi. Agarda ularning birida, boshqalariga qaraganda yetarlicha omadli chiqsa, u holda buning sababi uning masalani oqilona yo‘l bilan ko‘rinishni o‘zgartira olganidir.

Ish oxirida biz masala yechimini topgan vaqtimizda biz uni dastlabki vaqtdagiga qaraganda aniq va tushunarli ekanini anglaymiz. Bizning dastlabki tasavvurlarimizdan keyin, biz to‘liq tasavvurlarga ega bo‘lamiz va uni turli nuqtayi nazardan qarab chiqib, turli tomondan o‘rganamiz.

Masala yechimining to‘g‘ri bo‘lishi unga to‘g‘ri yo‘l bilan yondashuvga bog‘liq bo‘ladi. Qanday yo‘l bilan yondashuv omadliroq ekanligini bilish uchun biz, masalani turli nuqtayi nazardan, turli tomondan qarab chiqamiz.

1. Katta tavakkal qilmasdan, ko‘p urinmasdan jiddiy masalani yechishga umid qilmasak ham bo‘ladi. Bir xil tizimda o‘zgarishsiz boradigan masalalar bizda charchash uyg‘otadi. Shuning uchun e’tiborni yo‘qotmaslik uchun masala to‘g‘ri yo‘naltirilgan va doimiy o‘zgarib turishi kerak. Agar bizning masalamiz omadliroq bo‘lsa, ya’ni biz uning yechimini izlab topa oladigan bo‘lsak, u holda yangi detallarni o‘rganish imkoniyatiga ega bo‘lamiz, bizning e’tiborimiz band va barcha qiziqishimiz shu ishga yo‘naltirilgan bo‘ladi.

2. Agarda qilayotgan ishimiz natijaga ega bo‘lmay qolsa, u holda bizning diqqat e’tiborimiz susayadi, qiziqish kamayib, fikrlarimiz torroq bo‘la boshlaydi va qo‘rqinchlisi, biz, umuman, masalani yecha olmaymiz. Bundan qutulishimiz uchun shu narsalar bilan bog‘liq savolga javob izlashni o‘z oldimizga maqsad qilib qo‘yishimiz kerak. Bu yangi savol bizning qiziqishimizni yana tiklaydi. Ko‘rinishi o‘zgaradigan masalaning yangi tomonlarini, qirralarini ochib beradi.

3. Misol. Kesik piramidaning hajmini toping. Uning pastki asosi a va katta asosi b bo‘lib, balandligi h ga teng bo‘lsin. Bu masala prizma va

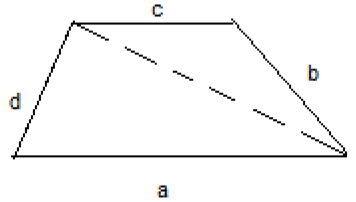
piramidaning hajmini bilgan sinfga qo‘yilishi mumkin. Kesik piramidaning avval $a>b$ holatini qarab chiqamiz. Agar biz b tomonni u a ga teng bo‘lguncha kattalashtirsak, bu holda kesik piramida prizmaga aylanadi va bizni qiziqtirgan hajm $V=\frac{a^2h}{3}$ ga teng bo‘ladi. Bunday variantlar, birinchi navbatda, berilgan masalaga nisbatan qiziqish uyg‘otadi. Undan tashqari, ular bizni prizma va piramida hajmini topish masalasida qo‘llanadigan usulllarni, natijalarni qanday qilib bu masalaga qo‘llasa bo‘larkin, degan g‘oyaga turtki beradi. Barcha hollarda ham biz oxirgi natijaga ega bo‘ldik va uni topdik.

Oxirgi formula shunday ko‘rinishda bo‘lish kerakki, uni $b=a$ bo‘l--ganda, $V=a^2h$ ko‘rinishga, $b=0$ bo‘lganda $V=\frac{a^2h}{3}$ ko‘rinishga keltirish mumkin bo‘lsin. Biz agar oxirgi formulani topa olsak, uni tekshirib ko‘rish imkoniyatiga egamiz.

4. Misol. Agar *a, b, c* va *d* tomonlari berilgan bo‘lsa, trapetsiyani yasang.

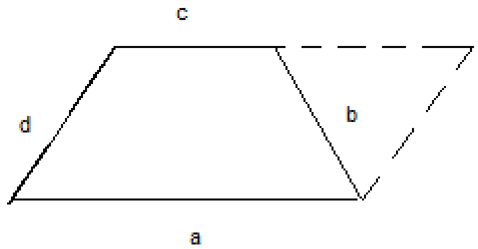
a – pastki (kichik asos), *c* – katta asos bo‘lsin. *a* va *c* bir biriga parallel, ammo $a\neq c$, *b va d* parallel emas. Agar biz hech narsani o‘ylab topa olmasak, yuqoridagi usullarimizdan foydalanishimiz mumkin. Agar trapetsiyada $a>c$ bo‘lsa, u holda *c* tomonni 0 gacha kichiklashtirsak, nima hosil bo‘ladi? Bunda trapetsiya uchburchakka aylanib qoladi. Uchburchak bo‘lsa – tanish va elementar figura. Biz uni berilganlarga qarab yasay olishimiz mumkin.

Shu uchburchak bizga trapetsiyani ko‘rishimizda yordam berishi mumkin. Bu holda biz trapetsiya diaganallaridan foydalanishga urinib ko‘ramiz.



Bizga ikki tomon: *a* va *d* ma’lum. Lekin biz uchta berilganlarga ega bo‘lishimiz lozim. Shuning uchun boshqa bir usuldan foydalanishimiz kerak. Agarda biz trapetsiyaning *c* tomonini kattalashtirib, uni *a* ga teng qilsak, nima hosil bo‘ladi? Bunda trapetsiya parallelogrammga aylanadi

Biz bundan foydalanishimiz mumkinmi? Rasmni ko‘zdan kechirar ekanmiz, e’tiborimizni uchburchakka, ya’ni oldingi trapetsiyadan parallelogram hosil qilinganda, paydo bo‘lgan uchburchakka qaratamiz.



Bu uchburchakni uning uchta *b, d, a-c* tomonlari orqali osongina ko‘rib olish mumkin. Oldingi masaladan biz yordamchi masala uchburchak masalasiga keldik. Bu yordamchi masalaning natijalaridan foydalanish orqali biz osongina dastlabki masalani yechishimiz mumkin bo‘ladi. Bizning yuqorida keltirgan misolimiz tipik sanaladi. Bunda bizning birinchi qadamimiz omadsizroq bo‘ldi. Yechish jarayoniga qaytib, shuni ta’kidlab aytamizki, birinchi variant unchalik foydadan xoli emas edi. U o‘zining ma’nosiga ega edi. Bunda u bizni masalani yechishda uchburchakdan foydalanish g‘oyasiga turtki bo‘ldi.

Biz *c* tomonni dastlab kichiklashtirib, keyin kattalashtiramiz, bu uning asosini tashkil etdi. Masala yechimini topdi.

3-§. Masala yechish bo‘yicha tavsiyanoma

Masalani mohiyatini aniq tushunib yetish kerak

Nimalar noma’lum? Nimalar berilgan? Masalaning sharti qanday tuzilgan? Masala shartini qanoatlantirish mumkinmi? Noma’lumlarni aniqlash uchun berilgan shartlar yetarlimi yoki aksincha keragidan ortiqmi? Yoki masalani yechimini izlashga qarama qarshi holatda berilganmi? Chizgilar qiling va unga tegishli belgilashlarni kirgizing, shartlarni qismalarga bo‘ling va ularni yozishga harakat qiling.

Ma’lumlar va noma’lumlar orasidagi bog‘lanishni topish kerak

Sizga bu masala oldin uchramaganmidi? Hech bo‘lmaganda, boshqacha ko‘rinishda? Sizga shunga o‘xshash masala ma’lummi? Sizga foydali bo‘lgan teoremalarni bilasizmi?

Noma’lumlarni ko‘rib chiqing va xuddi shu noma’lum yoki shunga o‘xshash noma’lumlar bilan berilgan masalani eslashga harakat qiling.

Mana, masala, berilganlar bilan qardosh va yechilgan. Undan foydalansa bo‘ladimi? Uni natijasini qo‘llasa bo‘ladimi? Oldingi masaladan foydalanish uchun qoshimcha, yordamchi elementlarni kiritish zarurmi?

Masalani boshqacha formulirovkasini berish mumkin emasmi? Va yana boshqacha ta’riflarga va ko‘rsatmalarga qayting.

Bu holatda ham masalani yechishga muvaffaq bo‘lmasangiz, unga yaqinroq bo‘lgan masalani yechishga harakat qiling. Yana ham yaxshiroq bo‘lgan masalani topishga yana bir marta urinib ko‘ring. Mazkur masala mohiyatiga bog‘langan umumiy masalani, shuningdek, xususiy masalani hamda analitik masalani toping. Topilgan masalalarni qiyosiy masalalarga ajrating va yechishga harakat qiling. Masala shartining bir qismini saqlab qolganda noma’lumlarni topish oydinlashadimi?

Berilganlardan shunday foydaliroq bo‘lgan narsalarni ajratib oling. Noma’lumlarni aniqlash uchun boshqa bir berilganlarni o‘ylab topish haqida fikr yuriting masala shartida berilganlarni, izlanganlarni zarur bo‘lagan holatda har ikkalasini o‘zgartirishga harakat qiling. Masala shartida berilganlarning barcha imkoniyatlaridan to‘liq foydalanildimi, fikr qiling. Masala mazmunida berilgan barcha imkoniyatlardan to‘liq foydalanildimi-yo‘qmi, yana bir o‘ylab ko‘ring. Gar bu bog‘liqlikni topish mumkin bo‘lmasa, qo‘shimcha masaladan foydalanish maqsadga muvofiq. Oxir natijada masala yechish rejasini tuzishga kelinadi.

Masala yechish uchun tuzilgan rejani amalga oshirish

Tuzilgan rejani amalga oshirishda har bir qadamingizni nazorat qiling. Yechim tomon qo‘yilgan har bir qadam to‘g‘riligini oydinlashtiring. Qo‘yilgan har bir qadamning to‘g‘riligini isbot qiling.

Topilgan yechimni o‘rganish kerak

Natijani tekshirib ko‘rish mumkinmi? Yechish yo‘lini tekshirib ko‘rish mumkinmi? Xuddi shunday natijani boshqa yo‘l bilan olish mumkinmi? Masalaga bir qarashda uni yechimini ko‘rish mumkin emasmi? Olingan natijadan boshqa bir masalani yechishda foydalanish mumkinmi yoki yechish usulidan?

Yechimni qanday izlash kerak

1. Tavsiya qilingan masalani tushunish.
2. Oraliq (qo‘shimcha) masalalarni ko‘rish orqali noma’lumlardan ma’lumlarga yo‘l topish (tahlil).
3. Topilgan g‘oyani, yechimni topish uchun amalga oshirish (sintez).
4. Yechimni tekshirish va uni tanqidiy baholash.

Tavsiya qilingan masalani tushunish

Masala to‘la o‘rganildi, masalaning sharti, talabi yoki masalada berilganlar va izlanganlar, shuningdek, ma’lumlar va noma’lumlar aniqlanadi. Masalada noma’lumlarni topish uchun berilganlar yetarlimi yoki yoki ular keragidan ortiqmi? Shuningdek, ular yetishmaydimi? Berilgan masala bilan oldindan yechimi ma’lum bo‘lgan birorta masala o‘rtasida bog‘liqlik mavjudmi? Shuningdek, masala yechilishi soddaroq bo‘lgan boshqa bir masala bilan bog‘langanmi yoki yechimi ma’lum masala bilan yuqorida qayd qilinganlarni har bir oraliq masalalarni yoki qismiy masalalarni yechish jarayonida takrorlash lozim bo‘ladi.

Yana bir marta masala shartida berilgan barcha imkoniyatlardan to‘la fo‘ydalanildimi-yo‘qmi?

Oraliq (qo‘shimcha) masalalarni ko‘rish orqali noma’lumlardan ma’lumlarga yo‘l topish (tahlil)

– ma’lumlar va noma’lumlar orasidagi munosabatlarni bayon qilish;
– noma’lum elementlarni almashtirish;
– berilganlarga yaqinroq bo‘lgan yangi o‘zgaruvchilarni kiritish;
– masalada berilgan elementlarni almashtirish;
– shu yo‘l bilan yangi berilgan noma’lumlarga yaqinroq bo‘lgan elementlarni hosil qilish;
– masalani bir qismini yechish;
– ayrim shartlarni qanoatlantiruvchi qismiy yechimlar berish va bular orqali mumkin qadar noma’lumni topishga yaqinlashish;
– umumlashtirish. Xususiy hollarni ko‘rish va topilgan o‘xshashliklarni tatbiq qilish.

Topilgan g‘oyani yechimni topish uchun amalga oshirish (sintez)

– har bir qadamning to‘g‘riligiga ishonch hosil qilish. Faqat “to‘la aniqlikda ko‘rilayotgan va to‘la ishonch bilan kiritilayotganini” qabul qilish orqali (Dekard);

– atamalarni ta’riflar bilan almashtirish.

Yechimni tekshirish va uni tanqidiy baholash

Natija to‘g‘rimi, nima uchun? Tekshirishni amalga oshirish mumkinmi? Hosil bo‘lgan natijaga olib keluvchi boshqa yo‘l yo‘qmi? Qisqaroq, osonroq yo‘l yo‘qmi? Mazkur yechish orqali yana qanday natijalar olish mumkin?

“… dalil va aniqlik haqiqiy ilmga xosdir”.
Abu Rayhon Beruniy

4-§. Olimpiada masalalari

1. (*2010-yil. Koen tengsizligi*[1]) Agar $x_1, x_2, ..., x_n > 0$ sonlari $x_1 \cdot x_2^2 \cdot ... \cdot x_n^n = 1$ shartni qanoatlantirsa, quyidagi tengsizlikni isbotlang.

$$\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} \geq \frac{n(n+1)}{2}$$

2. Agar $n \geq 2$ va $n \in N$ bo‘lsa, quyidagi tengsizlikni isbotlang.

$$\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq n\left(1 - \sqrt[n]{\frac{1}{2}\left(1 + \frac{1}{n}\right)}\right)$$

3. $n \geq 3$ $(n \in N)$ larda $(n+1)^n < n^{(n+1)}$ ni isbotlang.

4. Agar $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + 1$ ko‘phad n ta haqiqiy ildizga ega bo‘lsa, $P(n) \geq 3^n$ ekanini isbotlang. Bu yerda $a_1, a_2, ..., a_n \geq 0$.

5. $a, b \geq 0$ shartlarni qanoatlantiruvchi $\forall a, b$ lar uchun $ab \leq e^{a-1} + b \ln b$ tengsizlikni isbotlang.

6. $f(x) \in C[0,1]$, $f(x) > 0$ bo‘lsa, $\lim\limits_{n \to \infty}\left(\int_0^1 \sqrt[n]{f(x)}dx\right)^n$ ni hisoblang.

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[1] Kamalov Ne’matjon.

7. Agar uchburchak tomonlari uchun $a^2 + b^2 = c^2$ tenglik bajarilsa, $\forall n \geq 3\,(n \in N)$ soni uchun $a^n + b^n < c^n$ tengsizlikning o‘rinli ekanini isbotlang.

8. Agar $n \geq 2,\ |x| < 1$ bo‘lsa, $(1-x)^n + (1+x)^n < 2^n$ tengsizlikni isbotlang.

9. $a > 0, b > 0, c > 0$ bo‘lsa,quyidagi tengliksizlikni isbotlang.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}$$

10. Agar $a_1 > 0,\ a_2 > 0,\ \ldots a_n > 0$ bo‘lsa, $\left(a_1^{a_1} \cdot a_2^{a_2} \cdot \ldots \cdot a_n^{a_n}\right)^n \geq (a_1 a_2 \ldots a_n)^{a_1 + a_2 + \ldots + a_n}$ tengsizlikni isbotlang.

11. $x, y, z, p, q, r > 0$ sonlari uchun $\begin{matrix} p + q + r = 1 \\ x^p y^q z^r = 1 \end{matrix}$ shartlar o‘rinli bo‘lsa, quyidagi tengsizlikni isbotlang.

$$\frac{p^2 x^2}{qy + rz} + \frac{q^2 y^2}{px + rz} + \frac{r^2 z^2}{px + qy} \geq \frac{1}{2}$$

12. $f(x)$ funksiya quyidagi shartlarni bajarsin:

1) $f(x) \in C^{n-1}[x_0, x_n]$;

2) (x_0, x_n) intervalda n-darajali hosilaga ega.

3) $x_0 < x_1 < \ldots < x_n$ uchun $f(x_0) = f(x_1) = \ldots = f(x_n)$ tenglik o‘rinli.

U holda $\exists\, \varepsilon(\varepsilon \in (x_0, x_n))$ topilishini isbotlangki, $f^{(n)}(\varepsilon) = 0$ bo‘lsin.

13. Agar $f(x)$ funksiya $[a,b]$ da uzluksiz hamda (a,b) intervalda chekli hosilaga ega bo‘lib, chiziqli funksiya bo‘lmasa, u holda (a,b) da $\exists c \in (a,b)$ topiladiki,

$|f'(c)| > \left|\dfrac{f(b) - f(a)}{b - a}\right|$ tengsizlik o‘rinli bo‘ladi. Shuni isbotlang.

14. Agar $f(x)$ funksiya $[a,b]$ segmentda 2-tartibli hosilaga ega bo‘lib, $f'(a) = f'(b) = 0$ bo‘lsa, u holda $\exists c \in (a,b)$ topilishini isbotlangki, quyidagi munosabat o‘rinli bo‘lsin.

$$|f''(c)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)|$$

15. ABC uchburchakning ichida ixtiyoriy M nuqta olingan va bu nuqtadan AM, BM, CM to‘g‘ri chiziqlar o‘tkazilgan. Bu to‘gri chiziqlar uchburchak tomonlarini mos ravishda A_1, B_1, C_1 nuqtalarda kesib o‘tadi. Quyidagi tengsizlikni isbotlang.

$$\frac{AM}{A_1 M} + \frac{BM}{B_1 M} + \frac{CM}{C_1 M} \geq 6$$

16. Agar $|x|<1$ bo‘lsa, $1+2x+3x^2+4x^3+...$ yig‘indining qiymati nimaga teng?

17. $\forall A,B\in \mathbf{C}[m\times m]$ matritsalar uchun quyidagi tenglik o‘rinli ekanini isbotlang.

$$\det(I+AB)=\det(I+BA)$$

18. $f:R\to R$ akslantirishlar orasida ixtiyoriy $x,y\in R$ lar uchun $(f(x)-f(y))^2\le|x-y|^3$ shartni qanoatlantiruvchi o‘zgarmaslardan farqli funksiya mavjudmi?

19. $f(x)$ uzluksiz va irratsional qiymat qabul qilmaydi. Agar $f(\frac{1}{4})=\frac{1}{5}$ tenglik o‘rinli bo‘lsa, $f(\frac{1}{2010})$ ni toping.

20. Agar n-tartibli D_n determinantning elementlari faqat 1 va -1 lardan iborat bo‘lib, $n\ge 3$ bo‘lsa, u holda $|D_n|\le(n-1)(n-1)!$ tengsizlikni isbotlang.

21. ABC uchburchakning ichidagi M nuqta qanday joylashganda ushbu $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$ yig‘indi eng kichik qiymat qabul qiladi. Bu yerda a,b,c uch burchak tomonlarining uzunliklari va x,y,z lar mos ravishda BC,AC,AB tomonlargacha bo‘lgan masofalar.

22. Aytaylik A va B n-tartibli kvadrat matritsalar va A teskarilanuvchi. $AB-BA=A$ tenglik o‘rinli bo‘lishi mumkinmi?

23. $f(x)$funksiya (0,1)da differensiallanuvchi. Agar $f(0)=0,\quad f(1)=1$ bo‘lsa, u holda $\exists a,b\in(0,1)\quad(a\ne b)$ topilishini isbotlangki $f'(a)f'(b)=1$ bo‘lsin.

24. ABC uchburchak ichida ixtiyoriy O nuqta olingan va bu nuqtadan uchburchak tomonlariga parallel to‘g‘ri chiziqlar o‘tkazilgan. $AB//DE, BC//MN, AC//FK$. Bu yerda $F,M\in AB;\quad E,K\in BC;\quad D,N\in AC$. U holda quyidagi tenglikni isbotlang.

$$\frac{AF}{AB}+\frac{BE}{BC}+\frac{CN}{AC}=1$$

25. $\forall n\ge 2,\ a_1,a_2,...,a_n\ge 0$ sonlari uchun quyidagi tengsizlikni isbotlang.

$$a_1+a_2+...+a_n-n\sqrt[n]{a_1a_2...a_n}\ge(\sqrt{a_1}-\sqrt{a_2})^2$$

26. $x+\frac{1}{x}=2\cos\alpha$ bo‘lsa, $x^n+\frac{1}{x^n}$ nimaga teng?

27. Agar $a\ge 0,b\ge 0,c\ge 0$ bo‘lsa, quyidagi tengsizlikni isbotlang.

$$a^3+b^3+c^3+6abc \geq \frac{1}{4}(a+b+c)^3$$

28. $x^y+1=z$ tenglamani tub sonlarda yeching.

29. Agar $f(x)=\frac{1}{1-x^2}$ $|x| \neq 1$ bo‘lsa, u holda $f^{(n)}(0)$ ning qiymatini hisoblang.

30. $a_i>0, b_i>0$ $(i=1,2,...,n)$ sonlari uchun quyidagi tengsizlikni keltirib chiqaring.

$$\sqrt{(a_1+a_2+...+a_n)^2+(b_1+b_2+...+b_n)^2} \leq \sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+...+\sqrt{a_n^2+b_n^2}$$

31. Agar $f(x)$ funksiya $[x_1,x_2]$ $(0<x_1<x_2)$ segmentda differensiallanuvchi bo‘lsa, u holda $\exists\xi \in (x_1,x_2)$ nuqta topiladiki,

$\frac{1}{x_2-x_1}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(\xi)-\xi f'(\xi)$ tenglik o‘rinli bo‘ladi. Shuni isbot qiling.

32. Agar $f(x)$ funksiya ikkinchi tartibli differensiallanuvchi bo‘lsa, u holda ixtiyoriy simmetrik oraliqda $\exists\xi \in (x_0-r, x_0+r)$ topilishini isbotlangki, quyidagi tenglik o‘rinli bo‘lsin.

$$f''(\xi)=\frac{3}{r^3}\int_{x_0-r}^{x_0+r}\left(f(x)-f(x_0)\right)dx$$

33. Isbotlang. $\overline{abc}\cdot\overline{bca}\cdot\overline{cab} \geq \overline{aaa}\cdot\overline{bbb}\cdot\overline{ccc}$ bu yerda $\overline{xyz}$-uch xonali son.

34. $\{x_n\}$ ketma-ketlik quyidagi shartlarni qanoatlantiradi:

$x_1=2012,\ x_{n+1}=\frac{1}{4-3x_n}$ u holda $\lim\limits_{n\to\infty} x_n$ ni hisoblang.

35. Agar $f(x)$ funksiya $[a,b]$ segmentda uzluksiz differensiallanuvchi bo‘lib, $f(a)=f(b)=0$ bo‘lsa, u holda $\exists\xi \in (a,b)$ topilishini isbotlangki, quyidagi tengsizlik o‘rinli bo‘lsin. $|f'(\xi)| \geq \frac{4}{(b-a)^2}\int_0^1 |f(x)|dx$

36. $\begin{vmatrix} a & b & b & b...b \\ c & a & b & b...b \\ c & c & a & b...b \\ \ldots & \ldots & \ldots & \ldots \\ c & c & c & c...a \end{vmatrix}$ n-tartibli determinantni hisoblang.

37. Qanday uchburchakda $\frac{a\cos\alpha+b\cos\beta+c\cos\gamma}{a\sin\beta+b\sin\gamma+c\sin\alpha}=\frac{P}{9R}$ tenglik bajariladi. Bu yerda a,b,c lar uchburchak tomonlari. α,β,γ lar tomonlarga mos burchaklar. P perimetr va R uchburchakka tashqi chizilgan aylana radiusi.

38. Ixtiyoriy uch burchak uchun quyidagi tengsizlikni isbotlang. $\frac{2R}{r} \geq \frac{1}{\sin\frac{\alpha}{2}(1-\sin\frac{\alpha}{2})}$ bu yerda R va r lar mos ravishda uch burchakka tashqi va ichki chizilgan aylana radiuslari. α uchburchakning biror burchagi.

39. Tenglamani yeching.

$$\begin{vmatrix} x & c_1 & c_2 & \dots & c_n \\ c_1 & x & c_2 & \dots & c_n \\ \dots & \dots & \dots & \dots & \dots \\ c_1 & c_2 & c_3 & \dots & x \end{vmatrix} = 0$$

40. ABC uchburchakning tomonlari a, b, c bo‘lsin. M nuqta uchburchak tekisligidagi ixtiyoriy nuqta bo‘lsin. Ushbu $|MA|^2 + |MB|^2 + |MC|^2$ ifodaning eng kichik qiymatini toping.

41. Fibonachchi sonlari deb 1,2 dan boshlanuvchi shunday sonlar qatoriga aytiladiki, har bir keyingi had oldingi ikkita hadning yig‘indisiga teng. 1, 2, 3, 5, 8, 13,...

Fibonachchi sonlarining n-hadi quyidagi determinantga tengligini isbotlang.

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{vmatrix}$$

42. Agar $f(x)$ funksiya a nuqtada chekli hosilaga ega bo‘lsa, $\lim\limits_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n$ limitni hisoblang.

43. $x > 0$ bo‘lganda ushbu $\left|\int_x^{x+1} \sin t^2 dt\right| \leq \frac{1}{x}$ tengsizlikni isbotlang.

44. Quyidagi limitni hisoblang.

$$\lim_{x\to 0}\left(\lim_{n\to\infty}\frac{1}{x}\left(A^n - E\right)\right)$$

bu yerda $A = \begin{pmatrix} 1 & \frac{x}{n} \\ -\frac{x}{n} & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

45. a,b,c lar to‘g‘ri burchakli uchburchakning tomonlari bo‘lsa, (c gipotenuza)

$ab(a+b+c)<\frac{5}{4}c^3$ tengsizlikni isbotlang.

46. Agar $A=\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}$ bo‘lsa, e^A ni hisoblang.

47. a_{ij} koeffitsiyentlar butun son bo‘lsa, quyidagi tenglamalar sistemasi yagona yechimga ega bo‘lishini ko‘rsating.

$$\begin{cases}\frac{1}{2}x_1 = a_{11}x_1 + \quad + a_{1n}x_n\\ \frac{1}{2}x_2 = a_{21}x_1 + \quad + a_{2n}x_n\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \frac{1}{2}x_n = a_{n1}x_1 + \quad + a_{nn}x_n\end{cases}$$

48. ABCD qavariq to‘rtburchakning BC va DA qarama-qarshi tomonlarida M va N nuqtalar shunday olinganki, bunda ushbu $\frac{|BM|}{|MC|}=\frac{|AN|}{|ND|}=\frac{|AB|}{|CD|}$ tenglik o‘rinli. MN to‘g‘ri chiziq AB va CD tomonlar yordamida hosil qilingan burchak bissektrisasiga parallel bo‘lishini isbotlang.

49. $\int \sin(\ln x)dx$ aniqmas integralni hisoblang.

50. $\int e^x \sin x dx$ aniqmas integralni hisoblang.

51. $x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)$ bo‘lsin, bu yerda $x_0=2$. $\{x_n\}$ ketma-ketlik yaqinlashuvchi ekanini isbotlang.

52. $n!<\left(\frac{n+1}{2}\right)^n$ $(n\geq 2)$ tengsizlikni isbotlang.

53. $(a+b)^n=\sum_{k=0}^{n} C_n^k a^{n-k} b^k$ Nyuton binomi formulasini isbotlang.

54. Agar $f(x)$ funksiya [0,1]da differensiallanuvchi bo‘lib, $f'(0)=1, f'(1)=0$ bo‘lsa, $\exists c\in(0,1)$ uchun $f'(c)=c$ ekanini isbotlang.

55. $f(x)\in[0,1]$ va (0,1) da differensiallanuvchi. Agar $f(0)=f(1)=0$ bo‘lsa, $\exists x\in(0,1)$ uchun $f'(x)=f(x)$ ekanini isbotlang.

56. $P(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+....+\frac{x^n}{n!}$ ko‘phad karrali ildizga ega emasligini isbotlang.

57. $a_1 = 0, a_n = \frac{a_{n-1}+3}{4}$ ketma-ketlikning yaqinlashuvchi ekanini isbotlang va uning limitini toping.

58. $[x]+[2x]+[3x]=6$ tenglamani yeching, bu yerda [x]-x sonning butun qismi.

59. $\int \frac{\sin x}{\sin x + \cos x + \sqrt{2}} dx$ aniqmas integralni hisoblang.

60. Quyidagi integrallarni hisoblang.

a) $\int \sqrt{tgx} dx$;

b) $\int \frac{dx}{\sin x}$;

c) $\int_0^1 x f(x^2) dx$ bu yerda $\int_0^1 f(x) dx = a$.

61. $f(x) = x^3 - 3x^2 + 1$ tenglamaning ildizi $0.6 < x_0 < 0.7$ oraliqda ekanini isbotlang.

62. $f(x) = x^2 - 8x + 1$ tenglamaning ildizi $x_0 \in (0,1)$ bo‘lsa, bu ildizni verguldan keyin o‘ndan bir aniqlikda toping.

63. Ixtiyoriy $x_1, x_2, ..., x_n \in R$ uchun $|x_1 + x_2 + ... + x_n| \le |x_1| + |x_2| + ... + |x_n|$ tengsizlikni isbotlang.

64. 2^{nx} va 5^{nx} sonlar ketma-ket yozilsa hosil bo‘lgan sonning raqamlari soni $nx+1$ ekanini isbotlang. Bu yerda $n \in N$ shartni qanoatlantiruvchi ixtiyoriy natural son.

65. $f(x) \in C[a,b]$ bo‘lsin. Agar $x_1, x_2, x_3 \in [a.b]$ bo‘lsa, $\exists \xi \in (a.b)$ topiladiki, $f(\xi) = \frac{f(x_1) + f(x_2) + f(x_3)}{3}$ tenglik o‘rinli bo‘ladi. Shuni isbotlang.

66. $f(x) \in C^1[0,\infty), |f^1(x)| \le M$ bo‘lsin. U holda $f(x)$ funksiya $[0,\infty)$ da tekis uzluksiz bo‘lishini isbotlang.

67. i^i ni hisoblang. Bu yerda $i^2 + 1 = 0$.

68. $\forall z_1, z_2 \in C$ kompleks sonlar uchun $|z_1 + z_2| \le |z_1| + |z_2|$ tengsizlikni isbotlang.

69. $\forall z_1, z_2, ... z_n \in C$ kompleks sonlar uchun $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + .. + |z_n|$ tengsizlikni isbotlang.

70. $\int_0^{+\infty} e^{x^2} dx$ xosmas integralni hisoblang.

71. Faqat 2013 ta nuqtada uzluksiz va qolgan nuqtalarda uzilishga ega bo‘lgan funksiya quring.

72. $f(x)$ va $g(x)$ funksiyalar berilgan bo'lsin. Ular integrallanuvchi bo'lmasa, ularning yig'indisi integrallanuvchi bo'ladimi? Integrallanuvchi bo'lsa, shu funksiyalarni toping.

73. Agar $f'(\sin^2 x)=1+\cos^2 x$ bo'lsa $f(x)$ ni toping.

74. $\lim\limits_{t\to 0}\int\limits_{-1}^{1}\frac{t}{t^2+x^2}\cos x dx$ limitni hisoblang.

75. $f_1(x)=x^2$ va $f_2(x)=x-1$ funksiyalar grafiklari orasidagi eng qisqa masofani toping.

76. Musbat x,y,z sonlar

$\frac{1}{\sqrt{2}}\le z<\frac{1}{2}\min\left\{x\sqrt{2},y\sqrt{3}\right\},\ x+z\sqrt{3}\ge\sqrt{6},\ y\sqrt{3}+z\sqrt{10}\ge 2\sqrt{5}$ shartlarni qanoatlantirsa, $P(x,y,z)=\frac{1}{x^2}+\frac{2}{y^2}+\frac{3}{z^2}$ ifodaning eng katta qiymatini toping.

77. $n\in N,\ n>1$ uchun $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}<n-n^{\frac{n-1}{n}}$ tengsizlikni isbotlang.

78. $x_1,x_2,\ldots,x_{2011}$ musbat sonlari $\sum\limits_{i=1}^{2011}\frac{1}{1+x_{i_i}^2}=1$ shartni qanoatlantirsa, $x_1\cdot x_2\cdots x_n\ge 2010^{2011}$ tengsizlikning o'rinli ekanini isbotlang.

79. $x_1,x_2,...,x_n\ (n\ge 2)$ musbat sonlari

$\frac{1}{x_1+2011}+\frac{1}{x_2+2011}+....+\frac{1}{x_n+2011}=\frac{1}{2011}$ tenglikni qanoatlantirsa,

$\frac{\sqrt[n]{x_1x_2..x_n}}{n-1}\ge 2011$tengsizlikni isbotlang.

80. Agar $x_i\in\left(0;\frac{\pi}{2}\right)\ (i=1,2,...,n)$ sonlari $\sum\limits_{i=1}^{n}tgx_i\le n$ shartni qanoatlantirsa, $\sin x_1\cdot\sin x_2\cdot...\cdot\sin x_n\le 2^{-\frac{n}{2}}$ tengsizlikni isbotlang.

81. Istalgan natural n soni $(n\ge 2)$ uchun

$\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2n}\right)$ tengsizlikni isbotlang.

82. a) Agar a,b,c sonlari uchburchak tomonlarining uzunliklari bo'lib, $a+b+c=1$ shartni qanoatlantirsa, $n\in N$, $n\ge 2$ uchun

$\sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n}<1+\frac{\sqrt[n]{2}}{2}$ tengsizlikni isbotlang;

b) Agar musbat x, y, z sonlari $xyz = x + y + z + 2$ tenglikni qanoatlantirsa, $5(x + y + z) + 18 \geq 8(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})$ tengsizlik o‘rinli bo‘lishini isbotlang.

83. $\left(a^n - b^n\right) = (a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right)$ tenglikni isbotlang.

84. Haqiqiy sonlardan tuzilgan chegaralangan $\{a_n\}_{n=1}^{\infty}$ ketma-ketlik uchun $\lim_{n\to\infty}(a_n - 2a_{n+1} + a_{n+2}) = 0$ tenglikning bajarilishi ma’lum bo‘lsa, u holda ushbu $\lim_{n\to\infty}(a_n - a_{n+1}) = 0$ tenglikning ham bajarilishini isbotlang.

85. Ixtiyoriy n natural son uchun quyidagi tengsizlikni isbotlang.

$$|\sin nx| \leq n|\sin x|$$

86. Ixtiyoriy n natural sonlarda $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n+1} > 1$ tengsizlikni isbotlang.

87. Quyidagi tengsizlikni isbotlang. $\frac{5^n}{n!} \leq \frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5}, \quad \forall n \in N : n \geq 6.$

88. $S_n = arctg\frac{1}{2} + arctg\frac{1}{8} + ... + arctg\frac{1}{2n^2}, \forall n \in N$ yig‘indini toping.

89. Tengsizlikni isbotlang.

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} < \ln n + 1, \ \forall\ n \in N, \ n \geq 2$$

90. Ixtiyoriy juft n natural son uchun quyidagi tengsizlikni isbotlang.

$$x - \frac{x^3}{3!} + ... - \frac{x^{2n-1}}{(2n-1)!} \leq \sin x \leq x - \frac{x^3}{3!} + ... + \frac{x^{2n+1}}{(2n+1)!}, 0 \leq x \leq \frac{\pi}{2}$$

91. Ixtiyoriy n natural son uchun $\left(10^n + 10^{n-1} + ... + 1\right) \cdot \left(10^{n+1} + 5\right) + 1$ sonni to‘la kvadrat ekanligini isbotlang.

92. Ixtiyoriy natural m, n, p larda $x^{3m} + x^{3n+1} + x^{3p+2}$ ko‘phad $x^2 + x + 1$ ko‘phadga qoldiqsiz bo‘lishini isbotlang.

93. Qanday natural m, n, p larda $x^{3m} - x^{3n+1} + x^{3p+2}$ ko‘phad $x^2 - x + 1$ ko‘phadga qoldiqsiz bo‘linadi.

94. Qanday natural m, n, p larda larda $x^{3m} + x^{3n+1} + x^{3p+2}$ ko‘phad $x^4 + x^2 + 1$ ko‘phadga qoldiqsiz bo‘linadi.

95. m ning qanday qiymatlarida $x^{2m} + x^m + 1$ ko‘phad $x^2 + x + 1$ ko‘phadga qoldiqsiz bo‘linadi.

96. Ixtiyoriy natural k va $a_1, a_2, ..., a_k$ larda $x^{ka_1} + x^{ka_2+1} + ... + x^{ka_k+k-1}$ ko‘phadning $x^{k-1} + x^{k-2} + ... + 1$ ko‘phadga qoldiqsiz bo‘linishini isbotlang.

97. m ning qanday natural qiymatlarida $(x+1)^m - x^m - 1$ ko‘phad $x^2 + x + 1$ ko‘phadga qoldiqsiz bo‘linadi.

98. m ning qanday natural qiymatlarida $(x+1)^m + x^m + 1$ ko‘phad $x^2 + x + 1$ ko‘phadga qoldiqsiz bo‘linadi.

99. m ning qanday natural qiymatlarida $(x+1)^m - x^m - 1$ ko‘phad $(x^2 + x + 1)^2$ ko‘phadga qoldiqsiz bo‘linadi.

100. m ning qanday natural qiymatlarida $(x+1)^m + x^m + 1$ ko‘phad $(x^2 + x + 1)^2$ ko‘phadga qoldiqsiz bo‘linadi.

101. $\forall n \in N$ lar uchun quyidagi tengsizlikni isbotlang**.**

$$\frac{e}{2n+1} < e - \left(1 + \frac{1}{n}\right)^n$$

102.(Tyoplitsa teoremasi) Agar $\lim_{n\to\infty} x_n = a$ bo‘lib, $\sum_{k=1}^{n} P_k = 1, P_k \geq 0, k = \overline{1,n}$ bo‘lsa, $\sum_{k=1}^{n} P_k x_k$ ketma-ketlik yaqinlashuvhi va $\lim_{n\to\infty} \sum_{k=1}^{n} P_k x_k = a$ bo‘lishini isbotlang.

103. Quyidagi limitni hisoblang.

$$\lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdot ...\cdot(2n-1)}{2\cdot 4\cdot 6\cdot ...2n}\right)$$

104. $\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x.....\cos nx}{x^2}$ limitni hisoblang.

105. Taqqoslang. e^{π} va π^{e}

106. (Urganch, 2011) $a > b > 1$ sonlari uchun $a^{b^a} > b^{a^b}$ tengsizlik o’rinli ekanini isbotlang.

107. $\forall n \geq 2$ $(n \in N)$ da $\log_n(n+1) > \log_{n+1}(n+2)$ tengsizlikni isbotlang.

108. Agar $a,b > 0$ va m butun son bo‘lsa, $\left(1+\frac{a}{b}\right)^m + \left(1+\frac{b}{a}\right)^m \geq 2^{m+1}$ tengsizlikni isbotlang.

109. $x_1, x_2, ..., x_n$ sonlari $[a,b]$ kesmada yotadi, bunda $0 < a < b$. Quyidagi tengsizlikni isbotlang.

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

110. $a,b,c>0$ va $a^2+b^2+c^2=\frac{5}{3}$ bo‘lsa, $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}<\frac{1}{abc}$ tengsizlikni isbotlang.

111. Bizga haqiqiy koeffitsiyentli $Q(x)=x^m+b_1x^{m-1}+...+b_m$ va $P(x)=x^n+a_1x^{n-1}+...+a_n$ ko‘phadlar berilgan. Agar $P(x)$ ko‘phad turli xil $x_1,x_2,...,x_n$ n ta ildizga ega bo‘lsa, u holda $\sum_{j=1}^{n}\frac{Q(x_j)}{P'(x_j)}$ ni toping.

112. $\left(1+\frac{1}{n}\right)^{n+1}>\left(1+\frac{1}{n+1}\right)^{n+2}, \forall n\in N$ tengsizlikni isbotlang.

113. Agar $f(x)$ funksiya $[a,b]$ segmentda aniqlangan va uzluksiz, shu bilan birga, qavariq bo‘lsa, u holda

$$(b-a)\frac{f(a)+f(b)}{2}\le\int_a^b f(x)dx\le(b-a)f\left(\frac{a+b}{2}\right)$$

tengsizlikni isbotlang.

114. Agar $a,b\in R$ va $n\in N$ bo‘lsa, $\lim_{n\to\infty}\int_0^{2\pi}|a\cos nx-b\sin nx|dx$ ni hisoblang.

115. Tenglamani natural sonlarda yeching. $x^y=y^x$ bu yerda $x\neq y$.

116. $\forall n>1,(n\in N)$ da quyidagi tenglikni isbotlang.

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot...\cdot\sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}}$$

117. Quyidagi funksiyalarning berilgan oraliqda tekis uzluksiz emasligini isbotlang.

a) $f(x)=\frac{1}{x}$ (0.1) da;

b) $f(x)=x\sin x$ $[0,\infty)$ da;

c) $f(x)=\sin\frac{1}{x}$ (0.1) da.

118. $A=(a_{ij})_{i,j=1}^{n}$ matritsa uchun $|a_{ii}|\ge\sum_{i,j=1}^{n}|a_{ij}|$ bo‘lsa, A matritsa teskarilanuvchi ekanini isbotlang.

119. $f(x)$ funksiya $[0,+\infty)$ da monoton o‘suvchi bo‘lsa, $\int_1^n f(x)dx$ integralni yuqoridan va quyidan $f(1),f(2),...,f(n)$ lar yordamida baholang.

120. $f(x)\in C^1[a,b]$ bo‘lib, $a\le x_n\le b$ va $|f'(x)|\le q<1$ $(q\in R)$ bo‘lsin. Agar $x_n = f(x_{n-1})$ (bu yerda $f(x_0)\in[a,b]$) bo‘lsa, x_n yaqinlashuvchi ketma-ketlik ekanini isbotlang.

121. (Shtolts teoremasi) Bizga $\{x_n\}$ va $\{y_n\}$ sonli ketma-ketliklar berilgan. Ular quyidagi shartlarni qanoatlantirsin.

1) $y_{n+1} > y_n$ $(n\in N)$;

2) $\lim\limits_{n\to\infty} y_n = +\infty$;

3) $\lim\limits_{n\to\infty}\dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}$ limit mavjud.

U holda $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}=\lim\limits_{n\to\infty}\dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}$ tenglikni isbotlang.

122. Quyidagi yig‘indilarni toping.

a) $C_n^0+C_n^2+C_n^4+C_n^6+\ldots$;

b) $C_n^1+C_n^3+C_n^5+C_n^7+\ldots$.

123. Quyidagi yig‘indilarni toping.

a) $C_n^0-C_n^2+C_n^4-C_n^6+C_n^8-C_n^{10}+\ldots$;

b) $C_n^1-C_n^3+C_n^5-C_n^7+C_n^9-C_n^{11}+\ldots$.

124. Quyidagi yig‘indilarni hisoblang.

a) $C_n^0+C_n^3+C_n^6+\ldots$;

b) $C_n^2+C_n^5+C_n^8+\ldots$;

c) $C_n^1+C_n^4+C_n^7+\ldots$.

125. Quyidagi yig‘indilarni hisoblang.

a) $C_n^0+C_n^4+C_n^8+\ldots$;

b) $C_n^1+C_n^5+C_n^9+\ldots$;

c) $C_n^2+C_n^6+C_n^{10}+\ldots$;

d) $C_n^3+C_n^7+C_n^{11}+\ldots$.

5-§. Yechimlar va ko‘rsatmalar

1. Mashhur Koshi tengsizligini qo‘llaymiz:

$$\frac{1}{x_1}+\frac{2}{x_2}+\ldots+\frac{n}{x_n}=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_3}+\frac{1}{x_3}+\ldots+\underbrace{\frac{1}{x_n}+\frac{1}{x_n}+\ldots+\frac{1}{x_n}}_{n\,marta}\ge$$

$$\ge\frac{n(n+1)}{2}\cdot\sqrt[\frac{n(n+1)}{2}]{\frac{1}{x_1x_2{}^2x_3{}^3\ldots x_n{}^n}}=\frac{n(n+1)}{2}$$

isbot tugadi.

Natija. Agar $a>0, b>0, c>0$ sonlari uchun $ab^2c^3=1$ tenglik o‘rinli bo‘lsa,

$\frac{1}{a}+\frac{2}{b^2}+\frac{3}{c^3}\geq 6$ tengsizlikni isbotlang.

Bu tengsizlik yuqoridagi tengsizlikning $n=3$ holiga tushadi.

2. Buning uchun Koshi tengsizligini bir marta qo‘llash kifoya, ya’-ni

$$n-\left(\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}\right)=\left(1-\frac{1}{2^2}\right)+\left(1-\frac{1}{3^2}\right)+...+\left(1-\frac{1}{n^2}\right)\geq$$

$$\geq n\sqrt[n]{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)}=n\sqrt[n]{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)}=$$

$$=n\sqrt[n]{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot...\cdot\frac{n-1}{n}\cdot\frac{n+1}{n}}=n\sqrt[n]{\frac{1}{2}\left(1+\frac{1}{n}\right)}$$

bundan berilgan tengsizlik kelib chiqadi.

3. Biz $x\geq 3$ da $f(x)=\frac{\ln x}{x}$ funksiyani qaraylik.

$$f'(x)=\frac{1-\ln x}{x^2}<0 \qquad (x\geq 3)$$

Bu esa funksiyaning $(x\geq 3)$ oraliqda kamayuvchi ekanini bildiradi. U holda $\forall n, n+1\in[3,+\infty)$ sonlariga funksiyani ta’sir ettirsak:

$$n<n+1 \Rightarrow f(n)>f(n+1) \Rightarrow \frac{\ln n}{n}>\frac{\ln(n+1)}{n+1}\Rightarrow$$

$(n+1)\ln n>n\ln(n+1)\Rightarrow n^{n+1}>(n+1)^n$ ekanligi kelib chiqadi.

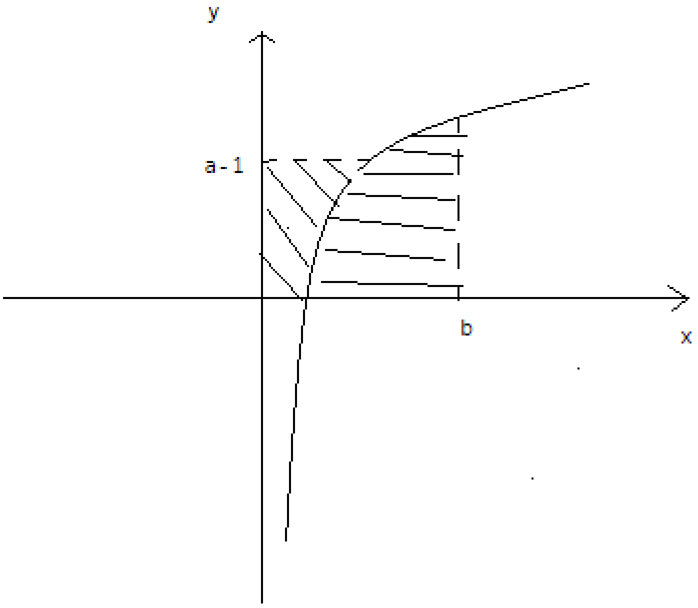
4. Uning ildizlarini $\alpha_1,\alpha_2,...,\alpha_n$ deb belgilaylik. U holda $P(x)=(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$ tenglik o‘rinli. $\alpha_i=-\beta_i$ almashtirish bajarsak, $\left(i=\overline{1,n}\right)$

$$P(x)=(x+\beta_1)(x+\beta_2)...(x+\beta_n)\Rightarrow P(2)=(2+\beta_1)(2+\beta_2)...(2+\beta_n)=$$

$$=(1+1+\beta_1)(1+1+\beta_2)...(1+1+\beta_n)\geq 3\sqrt[3]{\beta_1}\cdot 3\sqrt[3]{\beta_2}\cdot...\cdot 3\sqrt[3]{\beta_n}=3^n\sqrt[3]{\beta_1\beta_2...\beta_n}=3^n$$

ya’ni $P(n)\geq 3^n$. Chunki, Viet teoremasiga ko‘ra, $\beta_1\beta_2...\beta_n=1$.

5. $y=\ln x \quad (x>0)$ funksiyani qaraymiz.



$y = \ln x \Rightarrow x = e^y$

$$(a-1)b \le \int_0^{a-1} e^y dy + \int_1^b \ln x dx = e^y\Big|_0^{a-1} + (x\ln x - x)\Big|_1^b = e^{a-1} - 1 + b\ln b - b \Rightarrow$$

$$\Rightarrow \quad ab \le e^{a-1} + b\ln b$$

6. $x_n = \left(\int_0^1 \sqrt[n]{f(x)}dx\right)^n$ ketma-ketlikni tuzib olamiz. $f(x) > 0 \Rightarrow x_n > 0$ ekani aniq. $\frac{1}{n} = \alpha$ deb almashtirish bajarsak, $n \to \infty \Rightarrow \alpha \to 0$

$$\ln x_n = n\ln\int_0^1 (f(x))^{\frac{1}{n}}dx = \frac{\int_0^1 (f(x))^{\frac{1}{n}}dx}{\frac{1}{n}}$$ $\ln x_n = t_\alpha$ desak,

$$t_\alpha = \frac{\ln\int_0^1 f^\alpha(x)dx}{\alpha}$$ bunga Lopital qoidasini qo‘llasak,

$$\lim_{\alpha\to 0} t_\alpha = \lim_{\alpha\to 0}\frac{\ln\int_0^1 f^\alpha(x)dx}{\alpha} = \lim_{\alpha\to 0}\frac{\int_0^1 f^\alpha(x)dx\int_0^1 \ln f(x)dx}{\int_0^1 f^\alpha(x)dx} = \int_0^1 \ln f(x)dx$$ U holda

$$\lim_{\alpha\to 0} t_\alpha = \lim_{n\to\infty}\ln x_n = \int_0^1 \ln f(x)dx \Rightarrow \lim_{n\to\infty} x_n = e^{\int_0^1 \ln f(x)dx}$$

Javob: $e^{\int_0^1 \ln f(x)dx}$

7.

$a < c \;\Rightarrow\; a^{n-2} < c^{n-2}$

$b < c \;\Rightarrow\; b^{n-2} < c^{n-2}$

$a^n + b^n = a^2 \cdot a^{n-2} + b^2 \cdot b^{n-2} < a^2 \cdot c^{n-2} + b^2 \cdot c^{n-2} = c^{n-2} \cdot \left(a^2 + b^2\right) = c^{n-2} \cdot c^2 = c^2$

$a^n + b^n < c^n$

Natija: $n \ge 3$ da $(\sin\alpha)^n + (\cos\alpha)^n < 1$.

8. $|x| < 1$ bo‘lgani uchun $x = \cos\alpha$ deb belgilab olamiz. Bu yerda $\cos\alpha = 1, \cos\alpha = -1$ qiymatlarni qabul qilmaydi, deb olamiz. U holda $(1-\cos\alpha)^n + (1+\cos\alpha)^n = 2^n\left(\left(\sin\frac{\alpha}{2}\right)^{2n} + \left(\cos\frac{\alpha}{2}\right)^{2n}\right) < 2^n$ chunki, 7-misolning natijasiga ko‘ra, $2n \ge 4$ bo‘lgani uchun $\left(\sin\frac{\alpha}{2}\right)^{2n} + \left(\cos\frac{\alpha}{2}\right)^{2n} < 1$. Bundan esa berilgan tengsizlikning isboti kelib chiqadi.

9. Buning uchun $x > 0, y > 0, z > 0$ sonlariga $x^2 + y^2 + z^2 \ge xy + xz + yz$ tengsizlikni 3 marta qo‘llasak, berilgan tengsizlik hosil bo‘ladi.

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} = \frac{a^5}{b^3c^3} + \frac{b^5}{a^3c^3} + \frac{c^5}{a^3b^3} \ge \sqrt{\frac{a^5b^5}{a^3b^3c^6}} + \sqrt{\frac{a^5c^5}{a^3b^6c^3}} + \sqrt{\frac{b^5c^5}{a^6b^3c^3}} = \frac{ab}{c^3} + \frac{ac}{b^3} + \frac{bc}{a^3} \ge$$

$$\ge \sqrt{\frac{a^2bc}{b^3c^3}} + \sqrt{\frac{ab^2c}{a^3c^3}} + \sqrt{\frac{abc^2}{a^3b^3}} = \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \ge \sqrt{\frac{ab}{abc^2}} + \sqrt{\frac{ac}{ab^2c}} + \sqrt{\frac{bc}{a^2bc}} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

10. Biz $x > 0$ oraliqda $f(x) = \ln x^x$ funksiyani qaraylik. $f'(x) = \ln x + 1$ $\quad f''(x) = \frac{1}{x} > 0$ $\quad (x > 0)$ $f(x)$ ga Iensen tengsizligini qo‘llaymiz ya’ni, $a_1 > 0,\ a_2 > 0,\ \dots a_n > 0$ va $p_1 \ge 0, p_2 \ge 0, \dots, p_n \ge 0$ $\quad p_1 + p_2 + \dots + p_n = 1$ shartlarni qanoatlantiruvchi a_i, p_i $\left(i = \overline{1,n}\right)$ lar uchun $f''(x) \ge 0$ bo‘lsa, $f(p_1a_1 + \dots + p_na_n) \le p_1 f(a_1) + \dots + p_n f(a_n)$ o‘rinli, bundan

$$\ln\left(p_1a_1 + \dots + p_na_n\right)^{(p_1a_1+\dots+p_na_n)} \le p_1 \ln a_1^{a_1} + \dots + p_n \ln a_n^{a_n} = \ln a_1^{a_1p_1} + \dots + \ln a_n^{a_np_n} = \ln\left(a_1^{a_1p_1} \cdot \dots \cdot a_n^{a_np_n}\right) \Rightarrow$$

$\Rightarrow \quad (p_1a_1 + \dots + p_na_n)^{(p_1a_1+\dots+p_na_n)} \le a_1^{a_1p_1} \cdot \dots \cdot a_n^{a_np_n}$ endi $p_1 = p_2 = \dots = p_n = \frac{1}{n}$ deb olsak,

$\left(a_1^{a_1}a_2^{a_2}\dots a_n^{a_n}\right)^{\frac{1}{n}} \ge \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^{\frac{a_1+a_2+\dots+a_n}{n}}$ Koshi tengsizligiga ko‘ra,

$$\left(a_1^{a_1}a_2^{a_2}\dots a_n^{a_n}\right)^{\frac{1}{n}} \ge \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^{\frac{a_1+a_2+\dots+a_n}{n}} \ge \left(\left(a_1^{a_1}a_2^{a_2}\dots a_n^{a_n}\right)^{\frac{1}{n}}\right)^{\frac{a_1+a_2+\dots+a_n}{n}} \Rightarrow$$

$$\Rightarrow \left(a_1^{a_1} \cdot a_2^{a_2} \cdot \dots \cdot a_n^{a_n}\right)^n \ge (a_1a_2\dots a_n)^{a_1+a_2+\dots+a_n}.$$

11. $\forall x > 0$ uchun $x \ge \ln x + 1$ ekanidan foydalanamiz. $x^py^qz^r = 1$ ning ikkala tomonini natural logarifm ostiga olamiz, natijada quyidagiga ega bo‘lamiz:

$p\ln x + q\ln y + r\ln z = 0 \quad p\ln x + q\ln y + r\ln z \le p(x-1) + q(y-1) + r(z-1) = px + qy + rz - (p+q+r)$
$\Rightarrow\ px + qy + rz \ge p + q + r = 1$

Endi $a,b,c,d > 0$ sonlari uchun $\frac{a^2}{b} + \frac{c^2}{d} \ge \frac{(a+c)^2}{b+d}$ ekanini ko‘rsatamiz.
$(a^2 d + c^2 b)(b+d) \ge bd(a+c)^2 \Rightarrow a^2 bd + a^2 d^2 + c^2 b^2 + c^2 bd \ge bda^2 + 2abcd + bdc^2 \Rightarrow$
$\Rightarrow (ad - bc)^2 \ge 0$

Shunga ko‘ra,

$$\frac{p^2x^2}{qy+rz} + \frac{q^2y^2}{px+rz} + \frac{r^2z^2}{px+qy} \ge \frac{(px+qy)^2}{px+2rz+qy} + \frac{r^2z^2}{px+qy} \ge \frac{(px+qy+rz)^2}{2(px+qy+rz)} = \frac{px+qy+rz}{2} \ge \frac{1}{2}$$

12. $[x_k, x_{k+1}], \forall k = \overline{0, n-1}$ segmentda $f(x)$ funksiya Roll teoremasini qanoatlantiryapti, u holda $\exists \xi_k \in (x_k, x_{k+1})$ topiladiki, $f'(\xi_k) = 0$ bo‘ladi. Bu fikr har bir segment uchun o‘rinli. Ya’ni, $f'(\xi_1) = f'(\xi_2) = ... = f'(\xi_k) = 0$. Endi $[\xi_0, \xi_1], [\xi_1, \xi_2], ..., [\xi_i, \xi_{i+1}]...$ $i = \overline{0.k-1}$ segmetlar uchun esa $f'(x)$ funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. U holda $\exists \eta_i \in (\xi_i, \xi_{i+1})$ topiladiki, $f''(\eta_i) = 0$ bo‘ladi. n - 2 qadamdan keyin biror $[\varsigma_1, \varsigma_2]$ segmentda $f^{(n-1)}(x)$ funksiya Roll teoremasini qanoatlantirishini matematik induksiya yordamida ko‘rish qiyin emas. U holda

$f^{(n-1)}(\varsigma_1) = f^{(n-1)}(\varsigma_2) = 0 \Rightarrow \exists \varepsilon \in (\varsigma_1, \varsigma_2)$ topilib, $f^{(n)}(\varepsilon) = 0$ tenglik o‘rinli bo‘ladi.

13. Biz $[a,b]$ ni n ta bo‘lakka bo‘lamiz. $a = x_0 < x_1 < ... < x_n = b$ har bir $[x_k, x_{k+1}], \forall k = \overline{0, n-1}$ segmentda $f(x)$ funksiya Lagranj teoremasini qanoatlantiyapti. U holda $\exists \xi_k \in (x_k, x_{k+1})$ topiladiki, $f'(\xi_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$ tenglik o‘rinli. $x_{k+1} - x_k = \Delta x_k$ deb belgilasak,

$$f(b) - f(a) = \sum_{k=0}^{n-1} (f(x_{k+1}) - f(x_k)) = \sum_{k=0}^{n-1} f'(\xi_k)\Delta x_k \quad (1)$$

Endi $|f'(c)| = \max_k\{f'(\xi_k)\}$, $c \in (a,b)$ deb belgilash kiritib va (1) dan

$$|f(b) - f(a)| = \left|\sum_{k=0}^{n-1} f'(\xi_k)\Delta x_k\right| \le \sum_{k=0}^{n-1} |f'(\xi_k)|\Delta x_k \le |f'(c)|\sum_{k=0}^{n-1}\Delta x_k = |f'(c)||b-a| \Rightarrow |f'(c)| \ge \left|\frac{f(b) - f(a)}{b-a}\right|$$

kelib chiqadi.

14. 1-hol. $f(x) = const$ bo‘lsin. U holda $\forall c \in (a,b)$ uchun tenglik bajariladi.

2-hol. $f(x)$ chiziqli funksiya bo‘lsin. U holda $f'(a) = f'(b) = 0$ shart o‘rinli bo‘lmay qoladi.

3-hol. $[a,b]$ segmentni teng ikkiga bo‘lamiz. $[a,\frac{a+b}{2}]$ segmentda $\varphi(x)=\frac{(x-a)^2}{2}$ va $[\frac{a+b}{2},b]$ segmentda $\psi(x)=\frac{(x-b)^2}{2}$ yordamchi funksiyalarni qaraymiz. $f(x)$ va $\varphi(x)$ funksiyalar $[a,\frac{a+b}{2}]$ segmentda, $f(x)$ va $\psi(x)$ funksiyalar esa $[\frac{a+b}{2},b]$ sgmentda Koshi teoremasining barcha shartlarini qanoatlantiradi. Ya’ni, $\xi_1\in\left(a,\frac{a+b}{2}\right)$ topiladiki,

$$\frac{f'(\xi_1)}{\varphi'(\xi_1)}=\frac{f\left(\frac{a+b}{2}\right)-f(a)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)}=\frac{8\left(f\left(\frac{a+b}{2}\right)-f(a)\right)}{(b-a)^2}$$ hamda $\xi_2\in\left(\frac{a+b}{2},b\right)$

topiladiki, $\frac{f'(\xi_2)}{\psi'(\xi_2)}=\frac{f(b)-f\left(\frac{a+b}{2}\right)}{\psi(b)-\psi\left(\frac{a+b}{2}\right)}=\frac{8\left(f(b)-f\left(\frac{a+b}{2}\right)\right)}{(b-a)^2}$ tengliklar o‘rinli bo‘ladi. $\varphi'(\xi_1)$ va $\psi'(\xi_2)$ larga qiymatini qo‘yamiz

$$+\begin{cases}\frac{f'(\xi_1)}{\xi_1-a}=\frac{8\left(f\left(\frac{a+b}{2}\right)-f(a)\right)}{(b-a)^2}\\ \frac{f'(\xi_2)}{\xi_2-b}=\frac{8\left(f(b)-f\left(\frac{a+b}{2}\right)\right)}{(b-a)^2}\end{cases}\Rightarrow\ \frac{8(f(b)-f(a))}{(b-a)^2}=\frac{f'(\xi_1)}{\xi_1-a}-\frac{f'(\xi_2)}{\xi_2-b}=$$

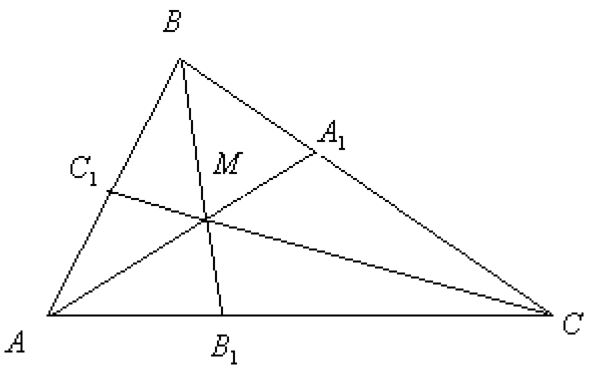
$$=\frac{f'(\xi_1)-f'(a)}{\xi_1-a}-\frac{f'(\xi_2)-f'(b)}{\xi_2-b}$$

Bu ifodalardan $a<\xi_1<\frac{a+b}{2}$ ekanini hisobga olsak, $f'(x)$ funksiya $[a,\xi_1]$ da Lagranj teoremasini qanoatlantiryapti. Xuddi shunga o‘xshash, $\frac{a+b}{2}<\xi_2<b$ dan $[\xi_2,b]$ segment uchun ham. U holda $\exists c_1\in(a,\xi_1)$ topiladiki, $f''(c_1)=\frac{f'(\xi_1)-f'(a)}{\xi_1-a}$ va $\exists c_2\in(\xi_2,b)$ topiladiki, $f''(c_2)=\frac{f'(\xi_2)-f'(b)}{\xi_2-b}$ lar o‘rinli. Agar $f''(c)=\max\{f''(c_1);f''(c_2)\}$ deb olsak:

$$\left|\frac{8(f(b)-f(a))}{(b-a)^2}\right| = \left|f''(c_1)+f''(c_2)\right| \le \left|f''(c_1)\right|+\left|f''(c_2)\right| \le 2\left|f''(c)\right| \Rightarrow$$

$$\Rightarrow \left|f''(c)\right| \ge \frac{4}{(b-a)^2}\left|f(b)-f(a)\right|.$$

15.



Umumiylikka zarar yetkazmasdan, tengsizlikning har ikkala tomoniga 3 ni qo‘shamiz: $\frac{AM}{A_1M}+1+\frac{BM}{B_1M}+1+\frac{CM}{C_1M}+1 \ge 6+3 \Rightarrow \frac{AA_1}{A_1M}+\frac{BB_1}{B_1M}+\frac{CC_1}{C_1M} \ge 9$ ni isbotlash kifoya. $S_{BMC}=\frac{1}{2}A_1M\cdot BC\cdot\sin MA_1B,\ S_{ABC}=\frac{1}{2}AA_1\cdot BC\cdot\sin MA_1B \Rightarrow \frac{AA_1}{A_1M}=\frac{S_{ABC}}{S_{AMC}}$ xuddi shunga o‘xshash $\frac{BB_1}{B_1M}=\frac{S_{ABC}}{S_{AMC}},\ \frac{CC_1}{C_1M}=\frac{S_{ABC}}{S_{AMB}}$ endi

$$\frac{AA_1}{A_1M}+\frac{BB_1}{B_1M}+\frac{CC_1}{C_1M}=S_{ABC}\left(\frac{1}{S_{AMC}}+\frac{1}{S_{BMC}}+\frac{1}{S_{AMB}}\right)=(S_{AMC}+S_{AMB}+S_{BMC})\left(\frac{1}{S_{AMC}}+\frac{1}{S_{BMC}}+\frac{1}{S_{AMB}}\right)\ge 9$$

oxirgi tengsizlik har bir qavs ichiga Koshi tengsizligining n=3 holini qo‘llash orqali hosil qilinadi. Isbot tugadi.

16. Bu yig‘indini S deb belgilab olamiz va tenglikning har ikkala tomonini x ga ko‘paytirib, quyidagilarga ega bo‘lamiz:

$$\begin{cases} S=1+2x+3x^2+4x^3+\dots \\ Sx=x+2x^2+3x^3+4x^4+\dots \end{cases}$$

1-ifodadan 2-ni ayiramiz

$S(1-x)=1+x+x^2+x^3+x^4+\dots \Rightarrow S(1-x)=\frac{1}{1-x} \Rightarrow S=\frac{1}{(1-x)^2}$ ekani kelib chiqadi. Bu yerda $|x|<1$.

17. Quyidagi hollarni qaraymiz.

1-hol. A-teskarilanuvchi ya’ni, $\det A\neq 0$.

Ravshanki, $I + AB = AA^{-1} + AB = A(A^{-1} + B)$ va

$I + BA = A^{-1}A + BA = (A^{-1} + B)A$

Bu tengliklardan quyidagiga ega bo‘lamiz:

$$\det(I + AB) = \det A(A^{-1} + B) = \det A \cdot \det(A^{-1} + B)$$

$$\det(I + BA) = \det(A^{-1} + B)A = \det(A^{-1} + B) \cdot \det A$$

Demak, $\det(I + AB) = \det(I + BA)$.

2-hol. $\det A = 0$. Yuqoridagi teoremaga ko‘ra, $\exists \lambda_1, \lambda_2, ..., \lambda_m \in C$ va $A_\lambda \in \mathbf{C}[m \times m]$ topilib, $\forall \lambda \in \mathbf{C} \setminus \{\lambda_1, \lambda_2, ...\lambda_m\}$ uchun $\det A \neq 0$ va $\lim\limits_{\lambda \to 0} \det A_\lambda = \det A$ bo‘ladi. A_λ teskarilanuvchiligidan, 1-holga ko‘ra,

$\det(I + A_\lambda B) = \det(I + BA_\lambda)$ tenglik o‘rinli.

Determinant uzluksiz funksiya ekanidan foydalanib, $\lambda \to 0$ da limitga o‘tamiz va quyidagilarni hosil qilamiz:

$$\lim_{\lambda \to 0} \det(I + A_\lambda B) = \det \lim_{\lambda \to 0}(I + A_\lambda B) = \det(I + A\ B)$$

$$\lim_{\lambda \to 0} \det(I + BA_\lambda) = \det \lim_{\lambda \to 0}(I + BA_\lambda) = \det(I + BA\ )$$

Demak, $\det(I + AB) = \det(I + BA)$

18. Misol shartiga ko‘ra: ixtiyoriy $x, y \in R$ lar uchun

$(f(x) - f(y))^2 \leq |x - y|^3$ bu yerdan

$x \neq y \Rightarrow \dfrac{(f(x) - f(y))^2}{|x - y|^2} \leq |x - y|$

$x = y + \Delta y$

$\dfrac{(f(y + \Delta y) - f(y))^2}{|y + \Delta y - y|^2} \leq |y + \Delta y - y|$, bundan $\left| \dfrac{(f(y + \Delta y) - f(y))}{\Delta y} \right|^2 \leq |\Delta y|$.

Endi $\Delta y \to 0$ da limitga o‘tsak,

$\lim\limits_{\Delta y \to 0} \left| \dfrac{(f(y + \Delta y) - f(y))}{\Delta y} \right|^2 \leq \lim\limits_{\Delta y \to 0} |\Delta y| = 0 \quad \Rightarrow \quad [f'(y)]^2 \equiv 0$. Bundan esa $f(x) = const$ ekani kelib chiqadi, chunki hosilasi nolga teng, uzluksiz funksiya faqat o‘zgarmas funksiyadir. Demak, o‘zgarmasdan farqli va misol shartini qanoatlantiruvchi funksiya mavjud emas ekan.

19. $f : R \to R$ uzluksizdir, biz $f(\frac{1}{2010}) = const = \frac{1}{5}$ ekanini ko‘rsatamiz.

Faraz qilaylik $f(\frac{1}{2010}) = x \neq \frac{1}{5}$ bo‘lsin, bundan haqiqiy sonlar to‘plami to‘liqligidan x va $\frac{1}{5}$ sonlari orasidan irratsional ξ son topiladi. Bolsano-

Koshining 2-teoremasiga asosan, $f(x)$ uzluksizligidan shu irratsional ξ sonning asli mavjud. Bu esa $f(x)$ funksiya irratsional qiymat qabul qilmaydi, degan shartga zid. Demak, farazimiz noto‘g‘ri. Shunday qilib, biz quyidagi xulosaga kelamiz. Agar funksiya uzluksiz bo‘lib faqat ratsional sonlar qabul qilsa, bu funksiya o‘zgarmas bo‘ladi. Demak,

$f(\frac{1}{2010}) = \frac{1}{5}$.

20. Matematik induksiya metodi orqali isbotlaymiz.

$$n = 3 \quad |D_3| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \le \left| a_1 \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} \right| + \left| a_2 \begin{vmatrix} a_4 & a_6 \\ a_7 & a_9 \end{vmatrix} \right| + \left| a_3 \begin{vmatrix} a_4 & a_5 \\ a_7 & a_8 \end{vmatrix} \right| \qquad (1)$$

$a_i = \pm 1, \quad i = \overline{1,9}$

D_3 ni 1-satr elementlari bo‘yicha yoyamiz.

A_1, A_2, A_3 algebraik to‘ldiruvchilardan kamida biri nolga teng bo‘ladi.

$$A_1 = \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_4 & a_6 \\ a_7 & a_9 \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_4 & a_5 \\ a_7 & a_8 \end{vmatrix}$$

Haqiqatan ham, aksincha, $A_1 \ne 0,\ A_2 \ne 0,\ A_3 \ne 0$ desak,

$$A_1 = a_9 a_5 - a_6 a_8$$
$$A_2 = a_4 a_9 - a_7 a_6$$
$$A_3 = a_4 a_8 - a_5 a_7$$

A_1, A_2, A_3 nolga teng bo‘lmasligi uchun $a_9 a_5$ va $a_6 a_8$ ishorasi har xil bo‘lishi kerak, agar bir xil bo‘lsa, xuddi shunday $a_4 a_9$ va $a_7 a_6$, $a_4 a_8$ va $a_5 a_7$ lar ham ishorasi har xil bo‘lishi kerak. Aks holda,

$a_9 a_5 = a_6 a_8 = \pm 1 \Rightarrow a_9 a_5 - a_6 a_8 = 0$ ziddiyatga kelamiz, bundan

$a_9 a_5 a_6 a_8 < 0$, $a_4 a_9 a_7 a_6 < 0$, $a_4 a_8 a_5 a_7 < 0$ bo‘lishi kelib chiqadi va ularni ko‘paytirsak $(a_4 a_5 a_6 a_7 a_8 a_9)^2 < 0$ ziddiyatga kelamiz. Demak, farazimiz noto‘g‘ri. A_1, A_2, A_3 lardan kamida bittasi nolga teng. Shunday qilib, determinantimizni baholaymiz:

$$|D_3| = |a_1 A_1 + a_2 A_2 + a_3 A_3| \le |a_1 A_1| + |a_2 A_2| + |a_3 A_3| \le |A_1| + |A_2| + |A_3|$$

Yuqorida isbotlab o‘tganimizdek, qo‘shiluvchilardan kamida bittasi 0 ga teng. Masalan, $A_2 = 0$ bo‘lsin, u holda

$D_3 \le |A_1| + |A_3|$

$$\begin{cases} |A_1| = |a_9 a_5 - a_6 a_8| \le 2 \\ |A_3| = |a_4 a_8 - a_5 a_7| \le 2 \end{cases}$$

$D_3 \le |A_1| + |A_3| \le 4$

Demak, $n = 3$ bo‘lganda $D_3 \le (3-1)(3-1)! = 4$ tengsizlik bajarildi.

$n = k$ uchun tengsizlik to‘g‘ri, deb faraz qilamiz. $D_k \le (k-1)(k-1)!$

$n = k+1$ da to‘g‘riligini ko‘rsatamiz.

D_{k+1} ni birinchi satr elementlari bo‘yicha yoyib chiqsak,

$D_{k+1} = |a_1 A_1 + ... + a_n A_n| \le |a_1 A_1| + ... + |a_n A_n| \le |A_1| + |A_2| + ... + |A_n|$

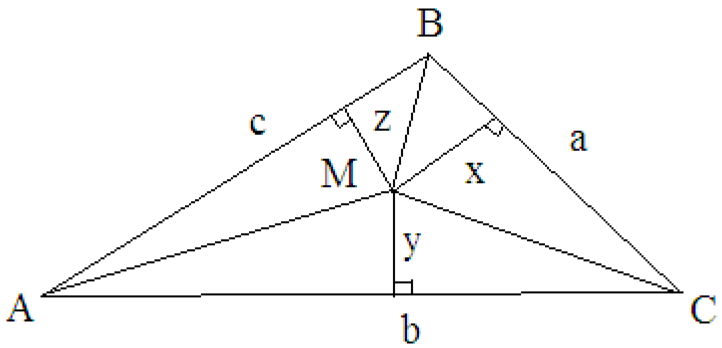
A_i lar $n-$tartibli determinant bo‘lib, elementlari -1 va 1 lardan iborat, bundan

$|A_i| \le (k-1)(k-1)!$

$D_{k+1} \le (k-1)(k-1)!(k+1) = (k^2-1)(k-1)! \le k^2(k-1)! = kk!$

Tengsizlik to‘liq isbot bo‘ldi.

21.



$S_{AMB} = \frac{1}{2}cz, \quad S_{AMC} = \frac{1}{2}by, \quad S_{CMB} = \frac{1}{2}ax, \quad S_{ABC} = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c \Rightarrow \frac{x}{h_a} + \frac{y}{h_b} + \frac{z}{h_c} =$

$= \frac{S_{AMB} + S_{AMC} + S_{CMB}}{S_{ABC}} = 1$

ekanidan foydalanamiz. Endi $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)\left(\frac{x}{h_a} + \frac{y}{h_b} + \frac{z}{h_c}\right)$

ko‘paytmaga Koshi-Bunyakovskiy tengsizligini qo‘llaymiz, u holda

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)\left(\frac{x}{h_a} + \frac{y}{h_b} + \frac{z}{h_c}\right) \ge \left(\sqrt{\frac{a}{h_a}} + \sqrt{\frac{b}{h_b}} + \sqrt{\frac{c}{h_c}}\right)^2 = \left(\frac{a}{\sqrt{2S}} + \frac{b}{\sqrt{2S}} + \frac{c}{\sqrt{2S}}\right)^2 =$$

$$= \frac{(a+b+c)^2}{2S} = \frac{(a+b+c)^2}{2 \cdot \frac{1}{2}(a+b+c)r} = \frac{a+b+c}{r}$$

Koshi-Bunyakovskiy tengsizligida tenglik belgisi

$$\frac{\frac{a}{x}}{\frac{x}{h_a}}=\frac{\frac{b}{y}}{\frac{y}{h_b}}=\frac{\frac{c}{z}}{\frac{z}{h_c}}\Rightarrow\frac{ah_a}{x^2}=\frac{bh_b}{y^2}=\frac{ch_c}{z^2}\Rightarrow x=y=z=r$$ da bajariladi. U holda M nuqta uchburchakning bissektrisalari kesishgan nuqtada bo‘lar ekan.

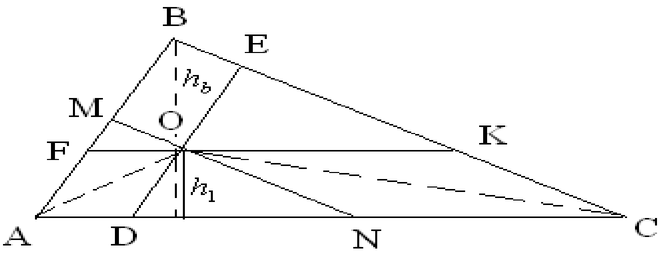
22. Mumkin deb faraz qilaylik. $AB-BA=A$ ni o‘ng tomondan A^{-1} ga ko‘paytiramiz. $ABA^{-1}-BAA^{-1}=AA^{-1}\Rightarrow ABA^{-1}-B=E$ E-birlik matritsa oxirgi tenglikdan ABA^{-1}va B matritsalarning xarakteristik ko‘phadlari ust-ma-ust tushadi va ularning izlari ham teng. Ya’ni:

$tr(ABA^{-1}-B)=tr(ABA^{-1})-trB=0$ lekin $tr(E)=n$ Ziddiyat. Faraz noto‘g‘ri. Mumkin emas ekan.

23. Biz $g(x)=f(x)+x-1$ funksiyani $[0,1]$ segmentda qaraymiz. $g(0)=-1,\ g(1)=1$ u holda Bolsano-Koshining 1-teoremasiga ko‘ra, $\exists c\in(0,1)$ topiladiki $g(c)=0$ tenglik o‘rinli bo‘ladi. Bundan esa $f(c)=1-c$ ekani kelib chiqadi. Endi $[0,1]$ segmentni $[0,c]$ va $[c,1]$ bo‘lgan ikkita segmentga ajratamiz. Masala shartiga ko‘ra, $f(x)\in C[0,c]\cap C'(0,c)$ Lagranj teoremasiga ko‘ra, $\exists a\in(0,c)$ mavjudki, $f'(a)=\frac{f(c)-f(0)}{c-0}$ tenglik o‘rinli. Xuddi shunday $f(x)\in C[c,1]\cap C'(c,1)$ yana Lagranj teoremasiga ko‘ra, $\exists b\in(c,1)$ topilib $f'(b)=\frac{f(1)-f(c)}{1-c}$ o‘rinli bo‘ladi. Demak, $a\neq b$ uchun

$f'(a)f'(b)=\frac{1-c-0}{c}\cdot\frac{1-(1-c)}{1-c}=1$. Isbot tugadi.

24.



$$\frac{S_{AOC}}{S_{ABC}}=\frac{\frac{AC}{2}\cdot h_1}{\frac{AC}{2}\cdot h_b}=\frac{h_1}{h_b}=\frac{AF}{AB}$$ chunki, $\sin A=\frac{h_b}{AB}=\frac{h_1}{OD}=\frac{h_1}{AF}$ xuddi shunga o‘xshash

$$\frac{S_{BOC}}{S_{ABC}}=\frac{CN}{AC}\ \text{va}\ \frac{S_{AOB}}{S_{ABC}}=\frac{BE}{BC}\Rightarrow\frac{AF}{AB}+\frac{BE}{BC}+\frac{CN}{AC}=\frac{S_{AOC}+S_{AOB}+S_{BOC}}{S_{ABC}}=1$$

25. Quyidagicha belgilash kiritamiz:

$$A_n = a_1 + a_2 + ... + a_n - n\sqrt[n]{a_1 a_2 ... a_n} \Rightarrow A_{n+1} - A_n = a_{n+1} + n\sqrt[n]{a_1 a_2 ... a_n} - (n+1)\sqrt[n+1]{a_1 a_2 ... a_n a_{n+1}} =$$

$$= a_{n+1} + n\sqrt[n]{a_1 a_2 ... a_n} - (n+1)\left(\sqrt[n]{a_1 a_2 ... a_n a_n}\right)^{\frac{n}{n+1}} \cdot (a_{n+1})^{\frac{1}{n+1}} \Rightarrow x = (a_{n+1})^{\frac{1}{n+1}},\ y = \left(\sqrt[n]{a_1 a_2 ... a_n a_n}\right)^{\frac{1}{n+1}} \Rightarrow$$

$$\Rightarrow A_{n+1} - A_n = x^{n+1} + ny^{n+1} - (n+1)xy^n = y^{n+1}\left(\left(\frac{x}{y}\right)^{n+1} - (n+1)\frac{x}{y} + n\right) \Rightarrow \frac{x}{y} = 1 + z \Rightarrow A_{n+1} - A_n =$$

$$= y^{n+1}\left((1+z)^{n+1} - 1 - (n+1)z\right)$$

Bernulli tengsizligiga ko‘ra, $(1+z)^{\alpha} > 1 + \alpha z$ ekanidan

$$A_{n+1} - A_n = y^{n+1}\left((1+z)^{n+1} - 1 - (n+1)z\right) > y^{n+1}\left(1 + (n+1)z - 1 - (n+1)z\right) = 0 \Rightarrow A_{n+1} > A_n .$$

Xuddi shunga o‘xshash, $A_{n+1} > A_n > A_{n-1} > ... > A_2 > A_1$ $A_2 = \left(\sqrt{a_1} - \sqrt{a_2}\right)^2$ ekanini e’tiborga olsak, $A_n \geq \left(\sqrt{a_1} - \sqrt{a_2}\right)^2$ ya’ni berilgan tengsizlikning isboti kelib chiqadi. Tenglik ishorasi faqat $a_1 = a_2 = ... = a_n \geq 0$ da bajariladi.

26. $x + \dfrac{1}{x} = 2\cos\alpha \Rightarrow x^2 - 2x\cos\alpha + 1 = 0 \Rightarrow x = \cos\alpha \pm i\sin\alpha$ bu yerda

$i^2 = -1$ u holda Muavr formulasidan foydalansak,

$$x^n + \frac{1}{x^n} = (\cos\alpha \pm i\sin\alpha)^n + (\cos\alpha \pm i\sin\alpha)^{-n} = \cos n\alpha \pm i\sin n\alpha + \cos(-n)\alpha \pm i\sin(-n)\alpha = 2\cos n\alpha$$

27. Buni sbotlash uchun quyidagi yordamchi tengsizlikdan foydalanamiz. $a \geq 0, b \geq 0, c \geq 0$ sonlari uchun $(a+b-c)(a-b+c)(b+c-a) \leq abc$ buni isbotlash oson. Biz bu tengsizlikning chap tomonidagi qavslarni ochib, uni soddalashtirgandan so‘ng u quyidagi ko‘rinishga keladi:

$a^3 + b^3 + c^3 \geq ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c + 3abc$ bu tengsizlikni ikkala tomonini 3 ga ko‘paytirib, o‘ng tomoniga $27abc$ ni qo‘shamiz. Natijada $3(a^3 + b^3 + c^3) + 18abc \geq 3(ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c)$ hosil bo‘ladi. Endi buning ikkala tomoniga $a^3 + b^3 + c^3 + 6abc$ ni qo‘shamiz:

$$4(a^3 + b^3 + c^3) + 24abc \geq a^3 + b^3 + c^3 + 3(ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c) + 6abc \Rightarrow$$

$$\Rightarrow 4((a^3 + b^3 + c^3) + 6abc) \geq (a+b+c)^3 \Rightarrow a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}(a+b+c)^3$$

28. Javob: (2, 2, 5). Agar $x = 2$ bo‘lsa, u holda $y = 2, z = 5$ bo‘lishi ko‘rinib turibdi. Endi x 2 dan boshqa tub son bo‘lsin. Demak, u toq hamdir. U holda toq sonni har qanday natural darajaga ko‘targanda, yana toq son hosil bo‘ladi, ya’ni tenglikning chap tomoni juft son. z bo‘lsa 2 dan boshqa juft-tub qiymat qabul qila olmaydi. Bu holda tub sonlarda yechim mavjud emas. Tenglama tub sonlarda yagona yechimga ega ekan.

29. $f(x) = \dfrac{1}{1-x^2} = \dfrac{1}{2}\left(\dfrac{1}{1+x} + \dfrac{1}{1-x}\right) = \dfrac{1}{2}\left((1+x)^{-1} + (1-x)^{-1}\right)$

$n = 1$ da $f'(x) = \dfrac{1}{2}\left(-(1+x)^{-2} + (1-x)^{-2}\right)$

$n=2$ da $f''(x)=\frac{1}{2}\left(2(1+x)^{-3}+2(1-x)^{-3}\right)$

$n=3$ da $f'''(x)=\frac{1}{2}\left(-6(1+x)^{-4}+6(1-x)^{-4}\right)$

$n=k$ da $f^{(k)}(x)=\frac{1}{2}\left((-1)^k k!(1+x)^{-(k+1)}+k!(1-x)^{-(k+1)}\right)$ deb faraz qilamiz va bu tasdiqning $n=k+1$ uchun o'rinli ekanini ko'rsatamiz.

$n=k+1$ da $f^{(k+1)}(x)=\left(f^{k}(x)\right)'=\frac{1}{2}\left((-1)^{k+1}(k+1)!(1+x)^{-(k+2)}+(k+1)!(1-x)^{-(k+2)}\right)$. Demak, ixtiyoriy natural n larda $f^{(n)}(x)=\frac{1}{2}\left((-1)^n n!(1+x)^{-(n+1)}+n!(1-x)^{-(n+1)}\right)$ tenglik o'rinli ekan. U holda

$$f^{(n)}(0)=\frac{1}{2}\left((-1)^n n!(1+0)^{-(n+1)}+n!(1-0)^{-(n+1)}\right)=\frac{n!}{2}\left(1+(-1)^n\right)=\begin{cases} n!, & agar\ \ n-juft\ bo'lsa, \\ 0, & agar\ \ n-toq\ bo'lsa. \end{cases}$$

30. $f(x)=\sqrt{1+x^2}$ funksiyaga Iensen tengsizligini qo'llaymiz:

$f'(x)=\frac{x}{\sqrt{1+x^2}}$, $f''(x)=\frac{\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}}{1+x^2}=\frac{1}{(1+x^2)\sqrt{1+x^2}}>0$ demak, Iensen tengsizligiga ko'ra, $f''(x)\geq 0,\ x\in(a,b)$ bo'lsa, ixtiyoriy $x_1,x_2,...,x_n\in(a,b)$ va $p_1+p_2+...+p_n=1$ tenglikni qanoatlantiruvchi ixtiyoriy $p_1\geq 0, p_2\geq 0,..., p_n\geq 0$ sonlari uchun ushbu $f(p_1x_1+p_2x_2+...+p_nx_n)\leq p_1f(x_1)+p_2f(x_2)+...+p_nf(x_n)$ (1) tengsizlik o'rinli. Biz yuqoridagi tengsizlikda $p_i,\ x_i$ $\left(i=\overline{1,n}\right)$ larning ixtiyoriyligidan ularni quyidagicha tanlab olamiz:

$p_i=\frac{a_i}{a_1+a_2+...+a_n}$, $x_i=\frac{b_i}{a_i}$ $\left(i=\overline{1,n}\right)$. Bularni (1) ga qo'yamiz:

$$\sqrt{1+\left(\frac{b_1+b_2+...+b_n}{a_1+a_2+...+a_n}\right)^2}\leq\frac{a_1\sqrt{1+\left(\frac{b_1}{a_1}\right)^2}}{a_1+a_2+...+a_n}+\frac{a_2\sqrt{1+\left(\frac{b_2}{a_2}\right)^2}}{a_1+a_2+...+a_n}+...+\frac{a_n\sqrt{1+\left(\frac{b_n}{a_n}\right)^2}}{a_1+a_2+...+a_n}\Rightarrow$$

$$\Rightarrow\quad\sqrt{(a_1+a_2+...+a_n)^2+(b_1+b_2+...+b_n)^2}\leq\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+...+\sqrt{a_n^2+b_n^2}.$$

Isbot tugadi.

31. Biz quyidagi ikkita yordamchi funksiyani $[x_1,x_2]$ segmentda qaraymiz:

$\varphi(x)=\frac{f(x)}{x}$ va $\psi(x)=\frac{1}{x}$ ko'rinib turibdiki bu funksiyalar $[x_1,x_2]$ segmentda aniqlangan, uzluksiz va differensiallanuvchi hamda $\psi(x)=\frac{1}{x}$ nolga teng emas. U holda biz qarayotgan ikkala funksiya $[x_1,x_2]$ $(0<x_1<x_2)$ da Koshhi

teoremasining barcha shartlarini qanoatlantiradi, ya'ni $\exists\xi\in(x_1,x_2)$ topiladiki, $\frac{\varphi'(\xi)}{\psi'(\xi)}=\frac{\varphi(x_2)-\varphi(x_1)}{\psi(x_2)-\psi(x_1)}$ tenglik o'rinli bo'ladi. $\varphi(x)=\frac{f(x)}{x}$, $\psi(x)=\frac{1}{x}$ larni hisobga olgan holda, $\frac{1}{x_2-x_1}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2)\end{vmatrix}=f(\xi)-\xi f'(\xi)$ ga ega bo'lamiz. Shuni isbotlash talab qilingan edi.

32. Teylor formulasiga ko'ra, $(x\neq x_0)$

$$f(x)-f(x_0)=(x-x_0)f'(x_0)+\frac{(x-x_0)^2 f''(\overline{x_0})}{2}\qquad (x_0-r<\overline{x_0}<x_0+r)$$

bu tenglikni ikkala tarafini ham (x_0-r,x_0+r) oraliqda integrallab, quyidagiga ega bo'lamiz:

$$\int_{x_0-r}^{x_0+r}[f(x)-f(x_0)]=\int_{x_0-r}^{x_0+r}(x-x_0)f'(x_0)dx+\int_{x_0-r}^{x_0+r}\frac{(x-x_0)^2 f''(\overline{x_0})}{2}dx=\int_{x_0-r}^{x_0+r}\frac{(x-x_0)^2 f''(\overline{x_0})}{2}dx$$

O'rta qiymat haqidagi teoremaga ko'ra, $\exists\xi\in(x_0-r,x_0+r)$ topiladiki,

$$\int_{x_0-r}^{x_0+r}[f(x)-f(x_0)]=\int_{x_0-r}^{x_0+r}\frac{(x-x_0)^2 f''(\overline{x_0})}{2}dx=f''(\xi)\int_{x_0-r}^{x_0+r}\frac{(x-x_0)^2}{2}dx=\frac{r^3}{3}f''(\xi)$$

bo'ladi. Bundan

$$f''(\xi)=\frac{3}{r^3}\int_{x_0-r}^{x_0+r}(f(x)-f(x_0))dx$$

tenglik kelib chiqadi.

33. Bizga ma'lumki, uch xonali sonni $\overline{xyz}=100x+10y+z$ ko'rinishda yoyish mumkin. Tengsizlikni quyidagi ko'rinishga keltiramiz:

$$(100a+10b+c)(100b+10c+a)(100c+10a+b)\geq 111^3abc\Rightarrow$$

$$\Rightarrow\quad(\frac{100a+10b+c}{a})(\frac{100b+10c+a}{b})(\frac{100c+10a+b}{c})\geq 111^3\quad\Rightarrow$$

$$\Rightarrow\quad(100+10\frac{b}{a}+\frac{c}{a})(100+10\frac{c}{b}+\frac{a}{b})(100+10\frac{a}{c}+\frac{b}{c})\geq 111^3$$

Oxirgi tengsizlikni isbotlash kifoya. Buning uchun qavslarni ochib chiqamiz:

$$(100+10\frac{b}{a}+\frac{c}{a})(100+10\frac{c}{b}+\frac{a}{b})(100+10\frac{a}{c}+\frac{b}{c})=10^6+10^5(\frac{a}{c}+\frac{c}{b}+\frac{b}{a})+2\cdot10^4(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+$$

$$+.10^3(\frac{a^2}{bc}+\frac{c^2}{ab}+\frac{b^2}{ac})+2\cdot10^2(\frac{a}{c}+\frac{c}{b}+\frac{b}{a})+10(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+3\cdot10^3+1\geq 10^6+3\cdot10^5+6\cdot10^4+$$

$$+6\cdot10^3+6\cdot10^2+3\cdot10+1=(100+10+1)^3=111^3$$

Bu yerda Koshi tengsizligining $n=3$ holidan foydalanildi. Isbot tugadi.

34. n=1 da $x_2=\frac{1}{4-6036}=-\frac{1}{6032}<\frac{1}{3}$

n=2 da $x_3=\dfrac{1}{4+\dfrac{1}{6032}}<\dfrac{1}{3}$

n=3 da $x_4=\dfrac{1}{4-3x_3}<\dfrac{1}{4-3\cdot\dfrac{1}{3}}=\dfrac{1}{3}$

n=k da $x_{k+1}<\dfrac{1}{3}$ deb faraz qilib, bu tasdiqning n=k+1 uchun ham o‘rinli ekanini ko‘rsatamiz.

n=k+1 da $x_{k+2}=\dfrac{1}{4-3x_{k+1}}<\dfrac{1}{4-3\cdot\dfrac{1}{3}}=\dfrac{1}{3}$ demak, $\forall n\in N$ uchun $x_n<\dfrac{1}{3}$. Endi ketma-ketlikning monoton o‘suvchiligini ko‘rsatamiz:

$x_{n+1}-x_n=\dfrac{1}{4-3x_n}-x_n=\dfrac{1-4x_n+3x_n^2}{4-3x_n}=\dfrac{3(x_n-1)(x_n-\frac{1}{3})}{4-3x_n}>0$ chunki $\forall n\in N$ uchun $x_n<\dfrac{1}{3}$. U holda $x_{n+1}-x_n>0\Rightarrow x_{n+1}>x_n$, ya’ni ketma-ketlik monoton o‘suvchi. Endi monoton ketma-ketliklar haqidagi teormega ko‘ra, $\{x_n\}$ ketma-ketlik yaqinlashuvchi. Uning limitini biror c deb olamiz. $x_{n+1}=\dfrac{1}{4-3x_n}$ ning ikkala tomonini $n\to+\infty$ da limitga o‘tib, $3c^2-4c+1=0$ dan $c=1$ va $c=\dfrac{1}{3}$ larni topamiz. Yuqoridagilarga ko‘ra, $\lim\limits_{n\to+\infty}x_n=\dfrac{1}{3}$ ekanini topamiz.

35. $f(x)$ funksiya $[a,b]$ segmentning ixtiyoriy nuqtasida uzluksiz differensiallanuvchi bo‘lgani uchun $\forall x\in[a,b]$ da $|f'(x)|$ chegaralangan bo‘ladi va uning maksimal qiymatini M bilan belgilasak, $\exists\varepsilon\in[a,b]$ topiladiki,

$$M=\max_{a\le x\le b}|f'(x)|=f'(\varepsilon)$$

bo‘ladi. $[a,b]$ segmentni teng ikki qismga bo‘lib, $\left[a,\dfrac{a+b}{2}\right]$ va $\left[\dfrac{a+b}{2},b\right]$ segmentlarning har birida $f(x)$ differensiallanuvchi bo‘lgani uchun quyidagini yoza olamiz:

$$\exists\xi\in(a,x),\ a\le x\le\frac{a+b}{2}\qquad f(x)=f'(\xi)(x-a)\le M(x-a)$$

$$\exists\eta\in(x,b),\ \frac{a+b}{2}\le x\le b\qquad f(x)=f'(\eta)(b-x)\le M(b-x)$$

Bularni e’tiborga olgan holda, quyidagi mulohazani yoza olamiz:

$$\int_a^b f(x)dx=\int_a^{\frac{a+b}{2}} f(x)dx+\int_{\frac{a+b}{2}}^b f(x)dx=\int_a^{\frac{a+b}{2}} f'(\xi)(x-a)dx+\int_{\frac{a+b}{2}}^b f'(\eta)(b-x)dx\le$$

$$\le M\int_a^{\frac{a+b}{2}}(x-a)dx+M\int_{\frac{a+b}{2}}^b(b-x)dx=M\,\frac{(b-a)^2}{4}=|f'(\varepsilon)|\cdot\frac{(b-a)^2}{4}\Rightarrow$$

bundan

$$|f'(\varepsilon)|\ge\frac{4}{(b-a)^2}\int_0^1|f(x)|dx$$

tengsizlik kelib chiqadi.

36. Biz $W(x)=\begin{vmatrix} a+x & b+x & b+x & b+x ... b+x \\ c+x & a+x & b+x & b+x ... b+x \\ c+x & c+x & a+x & b+.x ... b+x \\ .... & & & \\ c+x & c+x & c+x & c+x ... a+x \end{vmatrix}, x\in R$ funksiyani qaraymiz. Uning 1-satr elementlarini -1 ga ko‘paytirib, 2-,3-,...n-satr elementlariga qo‘shib chiqilsa, quyidagicha determinant hosil bo‘ladi:

$W(x)=\begin{vmatrix} a+x & b+x & b+x & b+x ... b+x \\ c-a & a-b & 0 & 0 \;...\; 0 \\ c-a & c-b & a-b & 0 ... \;\; 0 \\ .... & & & \\ c-a & c-b & c-b & c-b ... a-b \end{vmatrix}$ bu determinant hisoblansa, biror $W(x)=Ax+B$ x ga nisbatan xhiziqli funksiya hosil bo‘ladi. Chunki uning 1-satridan boshqa satr elementlarida x ishtirok etmagan. Bu yerdagi A biror n-tartibli determinant, B esa $x=0$ bo‘lganda hosil bo‘ladigan biz izlayotgan determinant. Endi $x=-b$ va $x=-c$ bo‘lgan hollarni qaraymiz:
$W(-b)=-Ab+B=(a-b)^n$
$W(-c)=-Ac+B=(a-c)^n$ birinchi tenglikning ikkala tomonini c ga, ikkinchi tenglikni esa $-b$ ga ko‘paytirib, ikkalasini qo‘shamiz:

$B(c-b)=c(a-b)^n-b(a-c)^n \Rightarrow B=\dfrac{c(a-b)^n-b(a-c)^n}{(c-b)}$ hosil bo‘ladi. Bunda quyidagi hollar bo‘lishi mumkin:

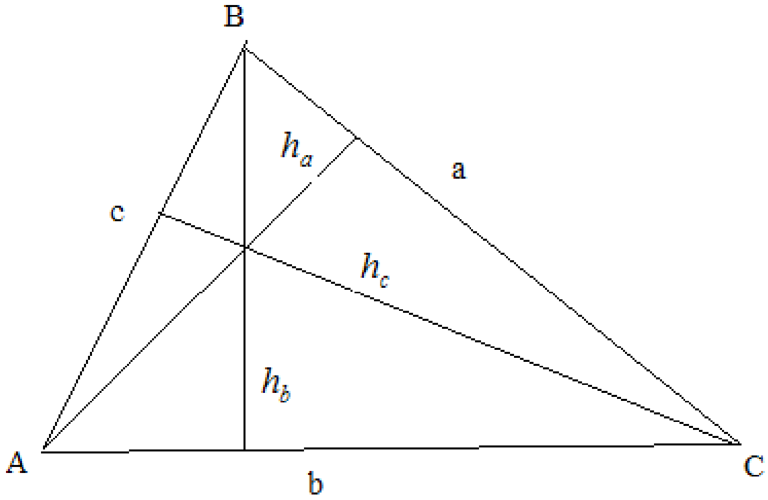
1-hol: $a=b=c$ u holda $B=0$;

2-hol: $a=b\ne c$ u holda $B=b(b-c)^{n-1}$;

3-hol: $a=c\ne b$ u holda $B=c(c-b)^{n-1}$;

4-hol: $a\ne b\ne c$ u holda $B=\dfrac{c(a-b)^n-b(a-c)^n}{(c-b)}$.

37.



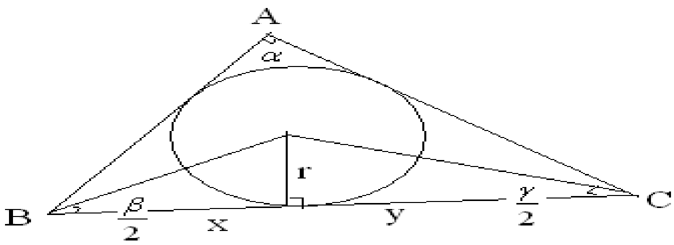
Chizmadan ko‘rinadiki $a\sin\beta = h_c,\ b\sin\gamma = h_a,\ c\sin\alpha = h_b$ tengliklar o‘rinli. Bundan tashqari, $a = 2R\sin\alpha,\ b = 2R\sin\beta,\ c = 2R\sin\lambda$ ekanini hisobga olsak, quyidagiga ega bo‘lamiz:

$$9R(a\cos\alpha + b\cos\beta + c\cos\gamma) = P(a\sin\beta + b\sin\gamma + c\sin\alpha) \quad \Rightarrow \quad P(h_a + h_b + h_c) =$$

$$= 9R(2R\sin\alpha\cos\alpha + 2R\sin\beta\cos\beta + 2R\sin\gamma\cos\gamma) = 9R^2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 18S$$

Bundan tashqari, $P(h_a + h_b + h_c) = (a+b+c)(h_a + h_b + h_c) \geq 9\sqrt[3]{abc}\cdot\sqrt[3]{h_a h_b h_c} = 18S$ bu yerda tenglik belgisi faqat $a = b = c,\ h_a = h_b = h_c$ da bajariladi. Demak, biz izlayotgan uchburchak muntazam uchburchak ekan.

38.



Bizga ma’lumki, $a = 2R\sin\alpha \quad \Rightarrow 2R = \dfrac{a}{\sin\alpha}$. Endi r ni ham uch burchakning a tomoni va burchaklari orqali ifodalaymiz. Chizmadan ko‘rinib turibdiki,

$$a = x + y = rctg\frac{\beta}{2} + rctg\frac{\gamma}{2} = \frac{r\sin\dfrac{\beta+\gamma}{2}}{\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}} = \frac{r\cos\dfrac{\alpha}{2}}{\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}} \quad \Rightarrow \quad r = \frac{a\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}}{\cos\dfrac{\alpha}{2}}$$

$$\frac{2R}{r}=\frac{\frac{a}{\sin\alpha}}{\frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}}=\frac{1}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}=\frac{1}{\sin\frac{\alpha}{2}}\cdot\frac{1}{\cos\left(\frac{\beta-\gamma}{2}\right)-\cos\left(\frac{\beta+\gamma}{2}\right)}\geq$$

$$\geq\frac{1}{\sin\frac{\alpha}{2}}\cdot\frac{1}{1-\sin\frac{\alpha}{2}}=\frac{1}{\sin\frac{\alpha}{2}(1-\sin\frac{\alpha}{2})}$$

39. 2-ustundan boshlab, barcha ustun elementlarini 1-ustun elementlariga qo‘shib chiqamiz:

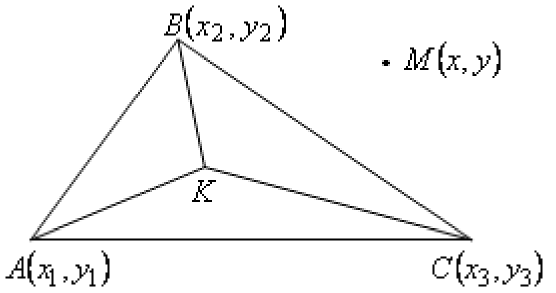
$$\begin{vmatrix} x+c_1+c_2+...c_n & c_1 & c_2 & ... & c_n \\ x+c_1+c_2+...c_n & x & c_2 & ... & c_n \\ .... \\ x+c_1+c_2+...c_n & c_2 & c_3 & ... & x \end{vmatrix}=0 \quad \Rightarrow (x+c_1+c_2+...c_n)\begin{vmatrix} 1 & c_1 & c_2 & ... & c_n \\ 1 & x & c_2 & ... & c_n \\ .... \\ 1 & c_2 & c_3 & ... & x \end{vmatrix}=0$$

endi 1-satr elementlarini -1 ga ko‘paytirib, 2-satr elementlariga, 2-satr elementlarini -1 ga ko‘paytirib 3-satr elementlariga qo‘shamiz va hokazo. (n-1)-satr elementlarini -1 ga ko‘paytirib, n-satr elementlariga qo‘shgandan keyin quyidagiga ega bo‘lamiz:

$$\left(x+c_1+c_2+...c_n\right)\begin{vmatrix} 1 & c_1 & c_2 & ... & c_n \\ 0 & x-c_1 & 0 & ... & 0 \\ 0 & 0 & x-c_2 & ... & 0 \\ .... \\ 0 & 0 & 0 & ... & x-c_n \end{vmatrix}=0$$

$(x+c_1+c_2+...+c_n)(x-c_1)(x-c_2)...(x-c_n)=0$ bu tenglamadan esa $x=-(c_1+c_2+...+c_n), x=c_1, x=c_2, ..., x=c_n$ yechimlarni olamiz.

40.



$$MA^2=(x-x_1)^2+(y-y_1)^2$$
$$MB^2=(x-x_0)^2+(y-y_0)^2$$
$$MC^2=(x-x_3)^2+(y-y_3)^2$$

$$f(x,y)=MA^2+MB^2+MC^2\equiv(x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2+$$
$$+(x-x_3)^2+(y-y_3)^2$$
$$f'_x=2(x-x_1)+2(x-x_2)+2(x-x_3)=0$$
$$6x=2x_2+2x_2+2x_3$$
$$x=\frac{x_1+x_2+x_3}{3}$$
$$f'_y=2(y-y_1)+2(y-y_2)+2(y-y_3)=0$$
$$y=\frac{y_1+y_2+y_3}{3}$$
$$a_{n1}=f''_{xx}=6>0$$
$$a_{12}=f''_{xy}=0 \qquad a_{11}a_{22}-a_{12}^2=6>0$$
$$a_{22}=f''_{yy}=6>0 \qquad a_{12}>0$$

Demak, $(x,y)-$ minimum nuqta

$(x,y)=\left(\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}\right)$ medianalar kesishgan nuqta bo‘ladi.

$$KB=\frac{2}{3}m_b \qquad KC=\frac{2}{3}m_c$$
$$KA=\frac{2}{3}m_a$$
$$KB^2+KA^2+KC^2=\frac{4}{9}m_a^2+\frac{4}{9}m_b^2+\frac{4}{9}m_c^2=\frac{4}{9}\left(m_a^2+m_b^2+m_c^2\right)=\frac{4}{9}\cdot\frac{3}{4}\left(a^2+b^2+c^2\right)=$$
$$=\frac{1}{3}\left(a^2+b^2+c^2\right)$$

41. Buning uchun determinantni biror D_n deb belgilab olib, uni 1- va 2-satrlari bo‘yicha yoyamiz:

$$D_n=\begin{vmatrix}1&1&0&0&\dots&0&0\\-1&1&1&0&\dots&0&0\\0&-1&1&1&\dots&0&0\\\dots&&&&&&\\0&0&0&0&\dots&-1&1\end{vmatrix}=D_{n-1}+\begin{vmatrix}1&0&0&\dots&0&0\\-1&1&1&\dots&0&0\\\dots&&&&&\\0&0&0&0&\dots&-1&1\end{vmatrix}$$

hosil bo‘lgan (n-1)-tartibli determinantni ham 1-satr bo‘yicha yoysak, $D_n=D_{n-1}+D_{n-2}$ tenglik-ka ega bo‘lamiz. Bu Fibonachchi sonlari ketma-ketligining rekkurent formulasini ifodalaydi. Endi buni tekshirib ko‘rish qoldi.

$$D_1=1,\ D_2=\begin{vmatrix}1&1\\-1&1\end{vmatrix}=2,\ D_3=\begin{vmatrix}1&1&0\\-1&1&1\\0&-1&1\end{vmatrix}=3$$

bu esa rekkurent formulani qanoatlantiradi. Demak, determinantning n-hadi Fibonachchi sonlari ketma-ketligining n-hadiga teng ekan.

42. $A=\left(\dfrac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n$ deb belgilash kiritamiz va ikkala tomonini

natural logarifm ostiga olamiz: $\ln A=\ln\left(\dfrac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n=\dfrac{\ln\left(\dfrac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)}{\frac{1}{n}}$ bu esa

$n\to\infty$ da $\frac{\infty}{\infty}$ ko'rinishidagi aniqmaslikni ifodalaydi. $n=x$ deb almashtirib, bunga Lopital qoidasini qo'llaymiz:

$$\lim_{x\to\infty}\frac{\ln\left(\dfrac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)}{\frac{1}{n}}=\lim_{x\to\infty}\frac{\ln\left(\dfrac{f\left(a+\frac{1}{x}\right)}{f(a)}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{-\dfrac{1}{x^2}\cdot\dfrac{f'\left(a+\frac{1}{x}\right)}{f(a)}}{-\dfrac{1}{x^2}\cdot\dfrac{f\left(a+\frac{1}{x}\right)}{f(a)}}=\frac{f'(a)}{f(a)}\Rightarrow A=e^{\frac{f'(a)}{f(a)}}$$

43. Ushbu integralni qaraymiz:

$$\int_x^{x+1} t\sin t^2 dt$$

Bunda aniq integral uchun o'rta qiymat haqidagi teoremani qo'llasak, shunday $\xi,\ x\le\xi\le x+1$ nuqta topilib, ushbu tenglik o'rinli bo'ladi:

$\int_x^{x+1} t\sin t^2 dt=\xi\cdot\int_x^{x+1}\sin t^2 dt$. Natijada ushbu tenglikka kelamiz:

$$\xi\cdot\int_x^{x+1}\sin t^2 dt=\int_x^{x+1} t\sin t^2 dt=\frac{1}{2}[\cos x^2-\cos(x+1)^2]$$

Bundan $\left|\xi\cdot\int_x^{x+1}\sin t^2 dt\right|=\left|\frac{1}{2}[\cos x^2-\cos(x+1)^2]\right|\le 1$

Boshqa tomondan $x\cdot\left|\int_x^{x+1}\sin t^2 dt\right|\le\xi\cdot\left|\int_x^{x+1}\sin t^2 dt\right|=\left|\xi\cdot\int_x^{x+1}\sin t^2 dt\right|\le 1$

Natijada berilgan tengsizlikka ega bo'lamiz: $\left|\int_x^{x+1}\sin t^2 dt\right|\le\frac{1}{x}$

44.

$$\begin{vmatrix} 1-\lambda & \frac{x}{n} \\ -\frac{x}{n} & 1-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)^2 + \frac{x^2}{n^2} = 0$$

$$\lambda - 1 = \pm i\frac{x}{n}$$

$\lambda_1 = 1 + i\frac{x}{n}$, $\lambda_2 = 1 - i\frac{x}{n}$ xos qiymatlari $\lambda_1 = 1 + i\frac{x}{n}$ xos qiymatga mos keluvchi xos vektorni topamiz:

$$\begin{pmatrix} -i\frac{x}{n} & \frac{x}{n} \\ -\frac{x}{n} & -i\frac{x}{n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-i\frac{x}{n}a_1 + \frac{x}{n}a_2 = 0$$

$$a_2 = ia$$

$\bar{e}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ xos vektorlar bo‘ladi.

$\lambda_2 = 1 - i\frac{x}{n}$ xos qiymatga $\bar{e}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ xos vektor mos keladi. Ushbu $C = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ matritsani tuzib olamiz. $C^{-1} = \frac{1}{2}\cdot\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ bo‘ladi.

$$C^{-1}\cdot A\cdot C = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\cdot\begin{pmatrix} 1 & \frac{x}{n} \\ -\frac{x}{n} & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\cdot\begin{pmatrix} 1+i\frac{x}{n} & 1-i\frac{x}{n} \\ -\frac{x}{n}+i & -\frac{x}{n}-i \end{pmatrix} = \begin{pmatrix} 1+i\frac{x}{n} & 0 \\ 0 & 1-i\frac{x}{n} \end{pmatrix} = J$$

Demak, $A = C\cdot J\cdot C^{-1}$

$A^2 = C\cdot J\cdot C^{-1}\cdot C\cdot J\cdot C^{-1} = C\cdot J^2\cdot C^{-1}$,

$A^k = C\cdot J^k\cdot C^{-1}$ bo‘lsin deb faraz qilib, $A^{k+1} = C\cdot J^{k+1}\cdot C^{-1}$ bo‘lishini ko‘rsatamiz:

$$A^{k+1} = A^k\cdot A = C\cdot J^k\cdot C^{-1}\cdot C\cdot J\cdot C^{-1} = C\cdot J^{k+1}C^{-1}$$

Demak, ixtiyoriy n uchun

$$A^n = C\cdot J^n\cdot C^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\cdot\begin{pmatrix} \left(1+i\frac{x}{n}\right)^2 & 0 \\ 0 & \left(1-i\frac{x}{n}\right)^2 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$B=\lim_{n\to\infty}A^n=\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\cdot\begin{pmatrix}e^{ix} & 0\\ 0 & e^{-ix}\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}=\lim_{n\to\infty}\left(\lim_{n\to\infty}\frac{1}{x}\left(A^n-E\right)\right)=\lim_{n\to\infty}\frac{1}{x}\left(B-C\cdot C^{-1}\right)=$$

$$C\cdot\lim_{n\to\infty}\begin{pmatrix}\frac{e^{ix-1}}{x} & 0\\ 0 & \frac{e^{-ix-1}}{x}\end{pmatrix}\cdot C^{-1}=C\cdot\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\cdot i^{-1}=\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\cdot\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}i & -i\\ -1 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}=\frac{1}{2}\cdot\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

Javob: $\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$

45. $ab\le\frac{a^2+b^2}{2}=\frac{c^2}{2}$ va $a.+b=\sqrt{a^2+b^2+2ab}\le\sqrt{c^2+2\cdot\frac{c^2}{2}}=c\sqrt{2}$ endi $ab(a+b+c)\le\frac{c^2}{2}\cdot\left(c\sqrt{2}+c\right)=\frac{\sqrt{2}+1}{2}\cdot c^3=\frac{2\sqrt{2}+2}{4}\cdot c^3<\frac{3+2}{4}\cdot c^3=\frac{5}{4}\cdot c^3$. Isbot tugadi.

46. $\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ differensial tenglamani qaraymiz. Uning xarakteristik tenglamasini yechamiz:

$\det(A-\lambda I)=\begin{vmatrix}3-\lambda & -1\\ 2 & -\lambda\end{vmatrix}=\lambda^2-3\lambda+2=0\Rightarrow\lambda_1=1,\ \lambda_2=2$. Endi har bir xos qiymatga mos bo‘lgan xos vektorni aniqlaymiz.

$$(A-1\cdot I)v_1=\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0\Rightarrow 2a-b=0\Rightarrow a=1,\ b=2\Rightarrow v_1=\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$(A-2\cdot I)v_2=\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0\Rightarrow a-b=0\Rightarrow a=1,\ b=1\Rightarrow v_2=\begin{pmatrix}1\\ 1\end{pmatrix}$$

Sistemaning umumiy yechimi quyidagi ko‘rinishda bo‘ladi:

$u(t)=\begin{pmatrix}x\\ y\end{pmatrix}=C_1\begin{pmatrix}1\\ 2\end{pmatrix}e^t+C_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t}=\begin{pmatrix}C_1e^t+C_2e^{2t}\\ 2C_1e^t+C_2e^{2t}\end{pmatrix}$ bunga $t=0$ ni qo‘yib,

$u(0)=\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}=\begin{pmatrix}C_1+C_2\\ 2C_1+C_2\end{pmatrix}$ ni hosil qilamiz, natijada quyidagilarga ega bo‘lamiz:

$$u(0)=e_1\Leftrightarrow\begin{pmatrix}C_1+C_2\\ 2C_1+C_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}\Rightarrow\begin{cases}C_1=-1\\ C_2=2\end{cases}\Rightarrow u_1(t)=\begin{pmatrix}2e^{2t}-e^t\\ 2e^{2t}-2e^t\end{pmatrix}$$

$$u(0)=e_2\Leftrightarrow\begin{pmatrix}C_1+C_2\\ 2C_1+C_2\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}\Rightarrow\begin{cases}C_1=1\\ C_2=-1\end{cases}\Rightarrow u_2(t)=\begin{pmatrix}e^t-e^{2t}\\ 2e^t-e^{2t}\end{pmatrix}$$

$u_1(t)$ va $u_2(t)$ vektorlardan matritsa tuzamiz. $e^{tA} = \begin{pmatrix} 2e^{2t} - e^t & e^t - 2e^{2t} \\ 2e^{2t} - 2e^t & 2e^t - e^{2t} \end{pmatrix}$ bunga $t = 1$ ni qo‘ysak, $e^A = \begin{pmatrix} 2e^2 - e & e - 2e^2 \\ 2e^2 - 2e & 2e - e^2 \end{pmatrix}$ biz izlayotgan matritsa hosil bo‘ladi.

47. Sistemaning determinanti

$$D = \begin{vmatrix} a_{11} - \frac{1}{2} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \frac{1}{2} & & a_{2n} \\ \dots\dots\dots & \dots\dots\dots\dots & & \\ \dots\dots\dots\dots\dots\dots & & & \\ a_{n1} & a_{n2} & & a_{nn} - \frac{1}{2} \end{vmatrix}$$

Agar biz

$$P(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \dots\dots\dots & \dots\dots\dots\dots & & \\ \dots\dots\dots\dots\dots\dots & & & \\ a_{n1} & a_{n2} & & a_{nn} - \lambda \end{vmatrix}$$

belgilash kiritsak, u holda $D = P\left(\frac{1}{2}\right)$ bo‘ladi. Ikkinchi tomondan,

$$P(\lambda) = (-1)^n \lambda^n + b_1 \lambda^{n-1} + \dots + b_n$$

ko‘rinishda bo‘ladi, bu yerda b_i – butun son $i = \overline{1, n}$

Agarda $P\left(\frac{1}{2}\right) = 0$ bo‘lsa, u holda

$$(-1)^n \frac{1}{2^n} + b_1 \frac{1}{2^{n-1}} + \dots + b_n = 0$$

va bu oxirgi tenglikning ikkala tarafini ham 2^n ga ko‘paytirsak,

$$(-1)^n + 2b_1 + 2^2 b_2 \dots + 2^n b_n = 0 \text{ bo‘ladi.}$$

Ushbu $2b_1 + 2^2 b_2 \dots + 2^n b_n = N$ belgilash kiritsak, u holda

$$(-1)^n + 2N = 0$$

ega bo‘lamiz. Bunday bo‘lishi mumkin emas, chunki N butun.

Demak, $D = P\left(\frac{1}{2}\right) \neq 0$ ekan, u holda sistema yagona yechimga ega va bu yechim $x_1 = x_2 = ... = x_n = 0$ bo‘ladi.

48. $\frac{|BM|}{|MC|} = \frac{|AN|}{|ND|} = \frac{|AB|}{|CD|} = \lambda$ deb olamiz. U holda $\overline{BM} = \lambda\overline{MC}$, $\overline{AN} = \lambda\overline{ND}$ lardan $\overline{BM} = \frac{\lambda}{\lambda+1}\overline{BC}$ va $\overline{AN} = \frac{\lambda}{\lambda+1}\overline{AD}$ kelib chiqadi. Bularga ko‘ra,

$$\overline{MN} = \overline{MB} + \overline{BA} + \overline{AN} = -\frac{\lambda}{\lambda+1}\overline{BC} + \overline{BA} + \frac{\lambda}{\lambda+1}\overline{AD} = \frac{\lambda}{\lambda+1}(\overline{AD} - \overline{BC}) + \overline{BA}.$$

Ushbu $|BA| \cdot \overline{CD}$ va $|CD| \cdot \overline{BA}$ vektorlarning uzunliklari o‘zaro teng bo‘lgani uchun, ularning yig‘indisi, ya’ni
$\overline{p} = |BA| \cdot \overline{CD} + |CD| \cdot \overline{BA} = |CD| \cdot (\lambda\overline{CD} + \overline{BA})$ vektor BA va CD tomonlar yordamida hosil qilingan burchak bissektrisasi bo‘yicha yo‘naladi.
$\overline{CD} = -\overline{BC} + \overline{BA} + \overline{AD}$ bo‘lgani uchun

$$\overline{p} = |CD| \cdot [\lambda(\overline{AD} - \overline{BC}) + (\lambda+1)\overline{BA}] =$$

$$= |CD| \cdot (\lambda+1)[\frac{\lambda}{\lambda+1}(\overline{AD} - \overline{BC}) + \overline{BA}] = |CD| \cdot (\lambda+1) \cdot \overline{MN}$$

Demak, $\overline{p}$ va $\overline{MN}$ o‘zaro parallel ekan. Isbotlandi.

49. Bu integralni hisoblash uchun, uni I bilan belgilaymiz.
$I = \int \sin(\ln x)dx$ endi buni bo‘laklab integrallash qoidasiga asosan integrallaymiz.

$$u = \sin(\ln x) \quad du = \frac{1}{x}\cos(\ln x)$$
$$dv = dx \quad v = x$$

natijada quyidagi tenglik kelib chiqadi:

$$I = x\sin(\ln x) - \int x\frac{1}{x}\cos(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

$I = x\sin(\ln x) - \int \cos(\ln x)dx$ buni ham bo‘laklab integrallaymiz.

$$u = \cos(\ln x) \quad du = -\frac{1}{x}\sin(\ln x)dx$$
$$dv = dx \quad v = x$$

$$I = x\sin(\ln x) - \left(x\cos(\ln x) + \int \sin(\ln x)dx\right)$$

$$I = x\sin(\ln x) - (x\cos(\ln x) + I)$$

$I = \frac{x\sin(\ln x) - x\cos(\ln x)}{2}$ bu esa integralni javobi. Misol to‘liq yechildi.

50. Bu integralni oldin I deb beldilab olib, so‘ngra ikki marta bo‘laklab integrallash kifoya.

$$I=\int e^x\sin xdx=\begin{vmatrix}u=\sin x, & du=\cos xdx\\ dv=e^x, & v=e^x\end{vmatrix}=e^x\sin x-\int e^x\cos xdx=\begin{vmatrix}u=\cos x, & du=-\sin xdx\\ dv=e^x, & v=e^x\end{vmatrix}=$$

$$=e^x\sin x-e^x\cos x-I \quad\Rightarrow\quad I=\frac{e^x\sin x-e^x\cos x}{2}+C$$

51. $x_1=\frac{1}{2}\left(x_0+\frac{1}{x_0}\right)=\frac{1}{2}\left(2+\frac{1}{2}\right)=\frac{5}{4}$. Ketma-ketlikning har bir hadi musbat bo‘lgani uchun, $x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)\geq\frac{1}{2}\cdot2\sqrt{x_n\cdot\frac{1}{x_n}}\geq1$ tengsizlik o‘rinli. Bu esa ketma-ketlikning quyidan chegaralanganligini bildiradi. Endi quyidagi ayirmani qaraymiz:

$$x_{n+1}-x_n=\frac{1}{2}\left(x_n-x_{n-1}+\frac{1}{x_n}-\frac{1}{x_{n-1}}\right)=\frac{1}{2}\left(x_n-x_{n-1}-\frac{x_n-x_{n-1}}{x_nx_{n-1}}\right)=\frac{1}{2}(x_n-x_{n-1})\left(1-\frac{1}{x_nx_{n-1}}\right)=$$

$$=\frac{1}{2^2}(x_{n-1}-x_{n-2})\left(1-\frac{1}{x_nx_{n-1}}\right)\left(1-\frac{1}{x_{n-1}x_{n-2}}\right)=...=$$

$$=\frac{1}{2^2}\left(1-\frac{1}{x_nx_{n-1}}\right)\left(1-\frac{1}{x_{n-1}x_{n-2}}\right)\cdot...\cdot\left(1-\frac{1}{x_1x_0}\right)(x_1-x_0)<0$$

Chunki ko‘paytmadagi har bir qavs ichidagi ifoda musbat faqat oxirgi qavs ichidagi ayirma $x_1-x_0=\frac{5}{4}-2<0$ manfiy qiymat qabul qiladi. Bundan $x_{n+1}-x_n<0 \Rightarrow x_{n+1}<x_n$ ya’ni ketma-ketlik monoton kamayuvchi. Demak, ketma-ketlik monoton kamayuvchi va quyidan chegaralangan. U holda monoton ketma-ketliklar haqidagi teoremaga ko‘ra, u yaqinlashuvchi.

52. 1-qadam. $n=2$ da tensizlikning chap qismi: $2!=2$

tengsizlikning o‘ng qismi: $\left(\frac{2+1}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}=2,25$.

2 < 2,25 bo‘lganligi sababli 1-qadam isbotlandi.

2-qadam. $n=k$ da tengsizlikning bajarilishi berilgan:

$k!<\left(\frac{k+1}{2}\right)^k,\quad k\geq2.$ n = k+1 da quyidagi tengsizlikning bajarilishini isbotlash lozim:

$$(k+1)!<\left(\frac{k+2}{2}\right)^{k+1}, k\geq2\cdot$$

Isboti. $(k+1)! = k!\cdot(k+1) < \left(\frac{k+1}{2}\right)^k \cdot (k+1) =$

$\left(\frac{k+2}{2}\right)^{k+1}$ songa ko‘paytiramiz va bo‘lamiz:

$$= \left(\frac{k+2}{2}\right)^{k+1} \cdot \frac{(k+1)^k \cdot (k+1)\cdot 2^{k+1}}{2^k \cdot (k+2)^{k+1}} = \left(\frac{k+2}{2}\right)^{k+1} \cdot \frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}} << ,,.deed,,,$$

$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}} < 1$ tengsizlikning bajarilishini isbotlaymiz:

$$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}} = \frac{2}{(\frac{k+2}{k+1})^{k+1}} = 2\cdot\frac{1}{(1+\frac{1}{k+1})^{k+1}}$$

$$\left(1+\frac{1}{k+1}\right)^{k+1} = 1 + \frac{k+1}{k+1} + \underbrace{\frac{(k+1)\cdot k}{2!}\cdot\frac{1}{(k+1)^2} + \ldots + \left(\frac{1}{k+1}\right)^{k+1}}_{>0} > 2\cdot$$

$$\frac{1}{\left(1+\frac{1}{k+1}\right)^{k+1}} < \frac{1}{2} \Rightarrow 2\cdot\left(\frac{k+1}{k+2}\right)^{k+1} < 2\cdot\frac{1}{2} = 1 \qquad k! < \left(\frac{k+2}{2}\right)^{k+1}\cdot 1 = \left(\frac{k+2}{2}\right)^{k+1}.$$

2-qadam isbotlandi.

Matematik induksiya prinsipiga ko‘ra, tengsizlik ixtiyoriy $n \geq 2$ natural son uchun bajariladi.

53. 1-usul. 1-qadam. n = 1 da tenglikning chap qismi $(a+b)^1$ ga teng.

Ushbu tenglikning o‘ng qismi $\sum_{m=0}^{1} C_1^m a^{1-m} b^m = \frac{1!}{0!1!}a + \frac{1!}{1!0!}b = a+b.$

2-qadam. $n=k$ da tenglikning o‘rinli ekanligi berilgan:

$$(a+b)^k = \sum_{m=0}^{k} C_k^m\, a^{k-m} b^m .$$

n = k+1 da tenglikning o‘rinli ekanligini isbotlaymiz:

$$(a+b)^{k+1} = \sum_{m=0}^{k+1} C_{k+1}^m\, a^{k+1-m} b^m .$$

Haqiqatdan, $(a+b)^{k+1} = (a+b)\cdot(a+b)^k = (a+b)\cdot\sum_{m=0}^{k} C_k^m\, a^{k-m} b^m =$

$$= \sum_{m=0}^{k} C_k^m\, a^{k+1-m} b^m + \sum_{m=0}^{k} C_k^m\, a^{k-m} b^{m+1} =$$

Ikkinchi qo‘shiluvchida yig‘indini $m = 1$ da boshlaymiz. Ikkinchi qo‘shiluvchida m ning o‘rniga m-1 olinadi.

$$= \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=1}^{k+1} C_k^{m-1} a^{k-(m-1)} b^{m-1+1} =$$

$$= \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=1}^{k+1} C_k^{m-1} a^{k+1-m} b^m =$$

Birinchi yig‘indida birinchi qo‘shiluvchini, ikkinchi yig‘indida – oxirgi qo‘shiluvchini alohida yozamiz.

Ikkala yig‘indi $m = 1$ dan $m = k$ gacha yig‘iladi. a, b sonlarning darajalari yig‘indisi ushbu belgilar bilan mos tushadi:

$$= C_k^0 a^{k+1} + \sum_{m=1}^{k} (C_k^m + C_k^{m-1}) a^{k+1-m} b^m + C_k^k b^{k+1} =$$

$$C_k^0 = C_{k+1}^0 = C_k^k = C_{k+1}^{k+1} = 1.$$

$$C_k^m + C_k^{m-1} = \frac{k!}{m!(k-m)!} + \frac{k!}{(m-1)!(k-(m-1))!} =$$

$$= \frac{k!}{m(m-1)!(k-m)!} + \frac{k!}{(m-1)!(k+1-m)(k-m)!} =$$

$$= \frac{k!(k+1-m+m)}{m!(k+1-m)!} = \frac{(k+1)!}{m!((k+1)-m)!} = C_{k+1}^m.$$

$$= \sum_{m=0}^{k+1} C_{k+1}^m a^{k+1-m} b^m.$$

2-qadam isbotlandi. 1- va 2- qadamlardan berilgan tenglikning ixtiyoriy n uchun o‘rinli ekanligi kelib chiqadi.

2-usul. $f(x) = (x+b)^n$ funksiyani qaraymiz. Uni yoyib chiqsak, biror $f(x) = A_n x^n + A_{n-1} x^{n-1} + ... + A_k x^k + ... + A_1 x + A_0$ ko‘rinishga keladi. Bu yerda $A_i, \left(i = \overline{0,n}\right)$ lar koeffitsiyentlar. U holda $A_0 = f(0) = b^n$. Endi uning hosilasini qaraymiz. $f'(x) = nA_n x^{n-1} + (n-1)A_{n-1} x^{n-2} + ... + A_1 \Rightarrow A_1 = f'(0) = nb^{n-1}$.

Xuddi shunga o‘xshash $A_k = \dfrac{f^{(k)}(0)}{k!} = \dfrac{n(n-1)...(n-k+1)b^{n-k}}{k!} = C_n^k b^{n-k}$.

Demak, $(x+b)^n = \sum_{k=0}^{n} C_n^k x^k b^{n-k}$. Bu esa $x = a$ da $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$ ga teng.

54. Quyidagi funksiyani qaraymiz:

$g(x) = f(x) - \dfrac{x^2}{2}$ ma’lumki, $g'(x) = f'(x) - x$ va

$g'(0) = f'(0) - 0 = 1$, $g'(1) = f'(1) - 1 = -1$

Demak $g'(x)$ funksiya segmentning chetki nuqtalarida turli ishorali qiymatlar qabul qilyapti. Bolsono-Koshining 1-tearemasiga ko‘ra,

$g'(x)$ funksiya uchun

$\exists c \in (0,1)$ topiladiki, $g'(c) = 0$ bundan $f'(c) - c = 0$, $f'(c) = c$.

55. Quyidagi funksiyani qaraymiz. $g(x) = f(x)e^{-x}$. Ma’lumki, $g'(x) = e^{-x}(f'(x) - f(x))$, $g(0) = g(1) = 0$. Endi $g'(x)$ funksiya Roll teoremasining barcha shartlarini qanoatlantiradi. Roll teoremasiga ko‘ra,

$\exists x \in (0,1)$ uchun $g'(x) = 0$ bo‘ladi. Bundan esa $f'(x) = f(x)$.

56. $P(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$ bo‘lsin u holda

$P'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-1}}{(n-1)!} = P(x) - \frac{x^n}{n!}$ $\quad P(x) = P'(x) + \frac{x^n}{n!}$

$P(x)$ karrali ildizga ega bo‘lsin. U

holda, $\exists \alpha \in R$ $P'(\alpha) = P(\alpha) = 0$, $0 = 0 + \frac{\alpha^n}{n!}$ $\alpha = 0$

Demak, $\alpha = 0$, $P(x)$ ko‘pxadning ildizi. Lekin $P(0) = 1 \neq 0$ ziddiyat $P(x)$ karrali ildizga ega emas.

57. $a_1 = \frac{a_0 + 3}{4} = \frac{3}{4}$, $a_2 = \frac{a_1 + 3}{4} = a_1 = \frac{a_1}{4} + \frac{3}{4}$

$a_3 = \frac{a_2}{4} + \frac{3}{4} = \frac{3}{4^3} + \frac{3}{4^2} + \frac{3}{4}$ demak, $a_n = \frac{3}{4^n} + \frac{3}{4^{n-1}} + ... + \frac{3}{4} = \frac{\frac{3}{4}\left(1 - \frac{1}{4^n}\right)}{1 - \frac{1}{4}} = 1 - \frac{1}{4^n}$ u holda

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1 - \frac{1}{4^n}\right) = 1$

58. $[x] + [2x] + [3x] = 6$ $\quad x = k + \alpha$ bo‘lsin, $k = [x]$, $\alpha = \{x\}$ $k - x$ sonning butun qismi, $\alpha - x$ sonning kasr qismi

$[k + \alpha] + [2k + 2\alpha] + [3k + 3\alpha] = 6$

$k + 2k + [2\alpha] + 3k + [3\alpha] = 6$

$6k + [2\alpha] + [3\alpha] = 6$

Quyidagi hollarni qaraymiz:

1-hol. $0 \leq \alpha < \frac{1}{3}$ $\to 0 \leq 3\alpha < 1, 0 \leq 2\alpha < \frac{2}{3} < 1$

$[2\alpha] = [3\alpha] = 0$, $6k = 6$ $k = 1$

Demak, $x = k + \alpha$ $\quad x = 1 + \alpha.$ $x \in [0 : \frac{1}{3})$

2-hol. $\frac{1}{3} \leq \alpha < \frac{1}{2}$ $\Rightarrow$ $1 \leq 3\alpha < \frac{3}{2}$, $\frac{2}{3} < 2\alpha < 1$ $\quad [3\alpha] = 1, [2\alpha] = 0$ $\quad 6k + 1 = 6$ $k = \frac{5}{6}$

k- butun son bo‘lishi kerak. Demak, bu holda yechim yo‘q.

3-hol. $\frac{1}{2} \le \alpha < \frac{2}{3} \quad \Rightarrow \quad \frac{3}{2} \le 3\alpha < 2,\ 1 \le 2\alpha < \frac{4}{3} \quad \Rightarrow \quad [3\alpha]=1 \quad [2\alpha]=1$

$6k+1+1=6, \quad 6k=4\ k=\frac{4}{6}$

k- butun son bo‘lishi kerak. Demak, bu holda yechim yo‘q.

4-hol. $\frac{2}{3} \le \alpha < 1$ bundan $2 \le 3\alpha < 3,\ \frac{4}{3} \le 2\alpha < 2 \quad [3\alpha]=2, \quad [2\alpha]=1$ va

$6k+2+1=6 \Rightarrow k=\frac{3}{6}$ lekin k- butun son bo‘lishi kerak. Demak, bu holda yechim yo‘q.

Demak, umumiy yechim $x=1+\alpha$, bu yerda $0 \le \alpha < \frac{1}{3}$.

59. $I=\int \frac{\sin x}{\sin x+\cos x+\sqrt{2}}dx \quad G=\int \frac{\cos x}{\sin x+\cos x+\sqrt{2}}dx$ bo‘lsin.

$$I-G=\int \frac{\sin x-\cos x}{\sin x+\cos x+\sqrt{2}}dx=-\int \frac{d(\sin x+\cos x)}{\sin x+\cos x+\sqrt{2}}=-\ln\left(\sin x+\cos x+\sqrt{2}\right)$$

$$I+G=\int \frac{\sin x+\cos x}{\sin x+\cos x+\sqrt{2}}dx=\int\left(1-\frac{\sqrt{2}}{\sin x+\cos x+\sqrt{2}}\right)dx=x-\int \frac{\sqrt{2}dx}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+\sqrt{2}}=$$

$$=x-\int \frac{dx}{\sqrt{2}\cos^2\left(\frac{x}{2}-\frac{\pi}{4}\right)}=\left|\begin{array}{l}\frac{x}{2}-\frac{\pi}{4}=t\\ x=2t+\frac{\pi}{2}\\ dx=2dt\end{array}\right|=x-\int \frac{2dt}{\cos^2 t}=x-2tgt=x-2tg\left(\frac{x}{2}-\frac{\pi}{4}\right)$$

$$\begin{cases}I-G=-\ln\left(\sin x+\cos x+\sqrt{2}\right)\\ I+G=x-2tg\left(\frac{x}{2}-\frac{\pi}{4}\right)\end{cases}$$

bu tengliklarni hadma-had qo‘shib I ni topamiz.

$$I=\frac{x}{2}-tg\left(\frac{x}{2}-\frac{\pi}{4}\right)-\frac{1}{2}\ln\left|\left(\sin x+\cos x+\sqrt{2}\right)\right|+C$$

60. Quyidagi integrallarni hisoblang.

a) $\int \sqrt{tgx}dx$

$$\int \sqrt{tgx}dx = \left| \begin{array}{l} \sqrt{tgx} = t \\ tgx = t^2, dx = \dfrac{2t}{1+t^4}dt \\ x = arctgt^2 \end{array} \right| = \int \frac{2t^2}{t^4+1}dt + \int \frac{(t^2+1)+(t^2-1)}{t^4+1}dt =$$

$$= \int \frac{t^2+1}{t^4+1}dt + \int \frac{t^2-1}{t^4+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{d\left(1-\frac{1}{t}\right)}{\left(1-\frac{1}{t}\right)^2+2} + \int \frac{d\left(1-\frac{1}{t}\right)}{\left(1-\frac{1}{t}\right)^2-2} = \frac{1}{\sqrt{2}}artg\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) +$$

$$+ \ln\left|\frac{t^2 - t\sqrt{2}+1}{t^2+t\sqrt{2}+1}\right| + C = \frac{1}{\sqrt{2}}arctg\left(\frac{tg^2x-1}{\sqrt{2tgx}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{tgx-\sqrt{2tgx}+1}{tgx+\sqrt{2tgx}+1}\right| + C$$

b) $\int \frac{dx}{\sin x} = \int \frac{\sin x dx}{\sin^2 x} = -\int \frac{d(\cos x)}{1-\cos^2 x} = |\cos x = t| = \int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

c) $\int_0^1 xf(x^2)dx$,bu yerda $\int_0^1 f(x)dx = a$.

$$\int_0^1 xf(x^2)dx = \frac{1}{2}\int_0^1 f(x^2)d(x^2) = |x^2 = t| = \frac{1}{2}\int_0^1 f(t)d(t) = \frac{1}{2}a .$$

61. $f(x) = x^3 - 3x^2 + 1$ bo‘lsin.

$f(0.6) = 0{,}6^3 - 0{,}6^2 + 1 = 0{,}216 - 1{,}08 + 1 = 0{,}136 > 0.$

$f(0{,}7) = 0{,}7^3 - 3\cdot 0{,}7^2 + 1 = 0{,}343 - 1{,}47 + 1 = -0{,}127 < 0.$

Balsano-Koshining birinchi teoremasiga ko‘ra, shunday $x_0 \in (0.6, 0.7)$ nuqta topiladiki , $f(x_0) = 0$ bo‘ladi. Demak, x_0 bu tenglamaning ildizidir.

62. $f(0) = 1 > 0$
$f(1) = -6 < 0$

Bolsano-Koshining birinchi teoremasiga ko‘ra, shunday $x_0 \in (0,1)$ topiladiki, $f(x_0) = 0.$ [0,1] segmentni o‘nta teng bo‘lakka bo‘lamiz.

$$[0,1] = \bigcup_{k=1}^{10}\left[\frac{k-1}{10}; \frac{k}{10}\right]$$

Endi har bir segmentning chegaralarida funksiyaning qiymatlari ko‘paytmasi manfiy bo‘ladiganini topamiz. [0;0,1] da tekshiramiz.

$f(0) = 0 - 8\cdot 0 + 1 > 0,$ $f(0{,}1) = 0{,}01 - 0{,}8 + 1 = 0{,}21 > 0$ $[0;0{,}1]$ da yechim yo‘q, chunki $f(0)\cdot f(0{,}1) > 0$ $[0{,}1;0{,}2]$ da tekshiramiz.

f(0,1)=0,21>0, f(0,2)=0,04-1,6+1=-0,56<0.

Demak, bu segmentda f(x) Bolsano-Koshining birinchi teoremasining shartlarini qanotlantiradi. Bunga ko‘ra, $\exists x_0 \in (0{,}1; 0.2)$ topilib,

$f(x_0)=0 \quad 0{,}1 < x_0 < 0{,}2$.

Demak, x_0 bu tenglamaning ildizi va verguldan keyingi birinchi raqami 1, ya’ni $x_0 \approx 0{,}1$.

63. Quyidagi hollarni qaraymiz:

1-hol. $x_1 + x_2 + ...x_n = 0 \qquad 0 \le |x_1| + ...|x_n|$.Ravshanki bu tengsizlik hamisha o‘rinli, chunki $\forall k \in \{1,2..n\}$ *da* $|x_k| \ge 0$.

2-hol. $x_1 + x_2 + ... + x_n \ne 0$ bu holda

$$1 = \frac{x_1 + x_2 + ... + x_n}{x_1 + x_2 + ... + x_n} = \frac{x_1}{x_1 + x_2 + ... + x_n} + \frac{x_2}{x_1 + x_2 + ... + x_n} + ... + \frac{x}{x_1 + x_2 + ... + x_n} \le$$

$$\le \left|\frac{x_1}{x_1 + x_2 + ... + x_n}\right| + \left|\frac{x_2}{x_1 + x_2 + ... + x_n}\right| + ... + \left|\frac{x_n}{x_1 + x_2 + ... + x_n}\right| = \frac{|x_1| + |x_2| + ...|x_n|}{x_1 + x_2 + ... + x_n}$$

Bundan quyidagi tengsizlikka ega bo‘lamiz:

$$|x_1 + x_2 + ... + x_n| \le |x_1| + |x_2| + ... + |x_n|$$

64. 2^{nx} sonning raqamlari soni p , 5^{nx} niki esa q bo‘lsin. U holda 2^{nx} va 5^{nx} sonlari ketma-ket yozilsa, $p+q$ xonali son hosil bo‘ladi. Demak, $p+q$ ni topishimiz kerak. Ravshanki,

$$10^{p-k} < 2^{nx} < 10^{p}$$

$$10^{q-1} < 5^{nx} < 10^{q}$$

Bu tengsizliklarni hadma-had ko‘paytirib topamiz:

$10^{p+q-2} < 10^{nx} < 10^{p+q} \Rightarrow p+q-2 < nx < p+q$

nx butun son ekanidan $nx = p+q-1$ kelib chiqadi. Demak, $p+q = nx+1$.

65. $f(x) \in c[a,b]$ bo‘lsin, Veyershtrasning 2-teoremasiga ko‘ra, f(x) funksiyaning [a, b] da eng katta va eng kichik qiymatlari mavjud.

$$m = \min_{x\in[a,b]} f(x), \quad M = \max_{x\in[a,b]} f(x) \text{ va } \forall x \in [a.b] \text{ uchun } m \le f(x) \le M \text{ bo‘ladi.}$$

$x_1\, x_2\, x_3 \in [a.b]$ bo‘lsin. U holda

$$m = \frac{m+m+m}{3} \le \frac{f(x_1) + f(x_1) + f(x_3)}{3} \le \frac{M+M+M}{3} \le M.$$

Demak, $\frac{f(x_1) + f(x_1) + f(x_3)}{3} \in (m, M)$. Bolsano-Koshining birinchi teoremasiga ko‘ra, $\exists \xi \in (a,b)$ topilib, $f(\xi) = \frac{f(x_1) + f(x_1) + f(x_3)}{3}$ tenglik o‘rinli bo‘ladi.

66. $f(x) \in C^1[0,\infty)$, $|f'(x)| \le M$ bo‘lsin u holda $\forall x', x'' \in [0,\infty)$ nuqta

Olaylik Lagranj teoremasiga ko‘ra, $\exists C \in (x', x'')$ topilib,
$f(x') - f(x'') = f'(c)(x' - x'')$ u holda quyidagiga ega bo‘lamiz:
$|f(x') - f(x'')| = |f'(c)||x' - x''| \le M|x' - x''| \qquad |x' - x''| < \delta$
bo‘lsa, $|f(x') - f(x'')| < M\delta$ bo‘ladi $M\delta = \varepsilon$ deb olinib, $\delta = \frac{\varepsilon}{M}$ ekanini topamiz. Demak, $f(x)$ funksiya uchun $\forall \varepsilon > 0$ uchun $\exists \delta(\varepsilon) = \frac{\varepsilon}{M}$ topilib, $|x' - x''| < \delta$ tengsizlikni qanoatlantiruvchi $\forall x', x'' \in [0, \infty$ uchun
$|f(x') - f(x'')| < \varepsilon$ o‘rinli.
Demak, $f(x)$ $[0, \infty)$ da tekis uzliksiz.

67. $i = cos\left(\frac{\pi}{2} + 2\pi k\right) + i\,sin\left(\frac{\pi}{2} + 2\pi k\right) = e^{i\left(\frac{\pi}{2} + 2\pi k\right)}$

$i^i = e^{i^2\left(\frac{\pi}{2} + 2\pi k\right)} = e^{-\frac{\pi}{2} + 2\pi k} \qquad k \in Z$

Demak, i^i ning qiymati cheksiz ko‘p qiymatli

68. Quyidagi hollarni qaraymiz:

1-hol. $z_1 + z_2 = 0 \quad 0 \le |z_1| + |z_2|$ ravshanki, bu tenglik doimo o‘rinli.

2-hol. $z_1 + z_2 \ne 0 \quad \forall z \in C$ uchun $Re\, z \le |z|$

$$1 = Re 1 = Re\left(\frac{z_1}{z_1 + z_2}\right) + Re\left(\frac{z_2}{z_1 + z_2}\right) \le \frac{|z_1|}{|z_1 + z_2|} + \frac{|z_1|}{|z_1 + z_2|}$$

Bu yerdan

$|z_1 + z_2| \le |z_1| + |z_2|$ kelib chiqadi.

69. 68- misolga qarang.

70. Ikkinchi tur Eyler integrali $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx$ Gamma funksiya va $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ ligidan

$$\int_0^{+\infty} e^{-x^2} dx = \left| \begin{array}{l} x^2 = y \\ x = \sqrt{y} \\ dx = \frac{1}{2} y^{-\frac{1}{2}} dy \end{array} \right| = \frac{1}{2} \int_0^{+\infty} y^{-\frac{1}{2}} e^{-y} dy = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$ ekani kelib chiqadi.

71. Buning uchun Dirixle funksiyasidan foydalanamiz. Dirixle funksiyasi quyidagicha edi. $D(x) = \begin{cases} 1, \ agar \ \ x - ratsional \ son \ bo'lsa, \\ 0, \ agar \ \ x - irratsional \ son \ bo'lsa. \end{cases}$ u holda quyidagi funksiya $f(x)=(x-1)(x-2)...(x-2013)D(x)$ masala shartini qanoatlantiradi. Bunday funksiyalardan ko‘plab topish mumkin.

72. Buning uchun quyidagi ikkita funksiya haqida gapirish yetarli. $f(x)=x-\frac{1}{x}$ va $g(x)=\sin x+\frac{1}{x}$ funksiyalarning $[0,1]$ segmentdagi aniq integrali mavjud emas. $f(x)+g(x)=x+\sin x$ yig‘indining esa $[0,1]$ segmentda aniq integrali mavjud. O‘zlari integrallanuvchi emas, lekin yig‘indisi integrallanuvchi bo‘lgan funksiyalarga juda ko‘plab misol topish mumkin.

73. $f'(\sin^2 x)=1+\cos^2 x=1+1-\sin^2 x=2-\sin^2 x$ endi $\sin^2 x=t$ deb belgilab olamiz. $f'(t)=2-t \Rightarrow f(t)=2t-\frac{t^2}{2}+C \Rightarrow f(x)=2x-\frac{x^2}{2}+C$

74. Lemma. Simmetrik oraliqda toq funksiyani integrali nolga teng.

Isboti. Bizga $[-a,a]$ oraliqda aniqlangan, uzluksiz va toq bo‘lgan $f(x)$ funksiya berilgan bo‘lsin. U holda

$$\int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_{0}^{a} f(x)dx=F(0)-F(-a)+F(a)-F(0)=-F(a)+F(a)=0$$ bu

yerda $F(x)$-juft funksiya.

U holda $f(t)=\frac{t}{t^2+x^2}\cos x$ funksiya t ga nisbatan toq funksiya ekanidan, yuqoridagi lemmaga ko‘ra, limit ichidagi integral nolga teng. Bu esa limitning ham nol ekanini ko‘rsatadi.

75. Buning uchun $f_1(x)=x^2$ funksiyaga tegishli bo‘lgan ixtiyoriy nuqtani olib, bu nuqtadan $f_2(x)=x-1$ to‘g‘ri chiziqqacha bo‘lgan masofani topamiz. Masalan, (1,1) nuqta birinchi funksiyani qanoatlantiradi. U holda nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofani formulasiga ko‘ra,

$$d=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}=\frac{|1-1-1|}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

76. $\frac{1}{x\sqrt{2}}=a,\ \frac{1}{y\sqrt{3}}=b,\ \frac{1}{2z}=c$ deb belgilash kiritsak, u holda

$Q(a,b,c)=2a^2+6b^2+12c^2$ ifodaning eng katta qiymatini topsak, masala yechiladi. a,b,c musbat sonlar quyidagi shartlarni qanoatlantiradi:

$$\max\{a,b,\}<c\le\frac{1}{2} \qquad (1)$$

$$c\sqrt{2}+a\sqrt{3}\ge 2\sqrt{6}ac \qquad (2)$$

$$c\sqrt{2}+b\sqrt{5}\ge 2\sqrt{10}bc \qquad (3)$$

(2) dan

$$\frac{\sqrt{2}}{a}+\frac{\sqrt{3}}{c}\geq 2\sqrt{6}\Rightarrow \frac{2}{a^2}+\frac{3}{c^2}\geq 12\Rightarrow \frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)\geq 2a^2,$$

Bundan

$$a^2+c^2=2a^2+c^2-a^2\leq \frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+c^2\left(1-\frac{a^2}{c^2}\right)\leq \frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+\frac{1}{2}\left(1-\frac{a^2}{c^2}\right)=\frac{5}{6}$$

Xuddi shunday (1) va (3) dan $b^2+c^2\leq \frac{7}{10}$ ekanligini topamiz.

Shunday qilib, $Q(a,b,c,)=2(a^2+c^2)+6(b^2+c^2)+4c^2\leq \frac{118}{15}$

tenglik $Q(a,b,c)=Q\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{2}}\right)=\frac{118}{15}$ bajariladi va

$a=\frac{1}{\sqrt{3}},\ b=\frac{1}{\sqrt{5}},\ c=\frac{1}{\sqrt{2}}$ qiymatlar (1)-(2)-(3) shartlarni qanoatlantiradi.

Bundan $\max P(x,y,z)=\max Q(a,b,c)=\frac{118}{15}$.

77. O‘rta arifmetik va o‘rta geometrik miqdorlar haqidagi Koshi tengsizligini qo‘llaymiz:

$$1+\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+...+\left(1-\frac{1}{n}\right)=1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{n-1}{n}\geq n\sqrt[n]{1\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}...\frac{n-1}{n}}=$$

$$=n\sqrt[n]{\frac{1}{n}}=n^{\frac{n-1}{n}}.$$

78. $\frac{1}{1+x_i^2}=y_i$ (i=1,2,…2011) deb belgilash kiritsak, u holda $y_1+y_2+\ldots+y_{2011}=1$ bo‘ladi. O‘rta arifmetik va o‘rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo‘llasak,

$$1-y_i=y_1+y_2+\ldots+y_{2002}-y_i\geq 2001\sqrt[2001]{\frac{y_1y_2...y_{2002}}{y_i}}\quad (i=1,2,\ldots 2011)$$

ekanligini topamiz va bu tengsizliklarni hadma-had ko‘paytirib,

$$\prod_{i=1}^{2002}(1-y_i)\geq \prod_{i=1}^{2002}2001\sqrt[2001]{\frac{y_1y_2...y_{2002}}{y_i}}=2001^{2002}y_1y_2...y_{2002},\quad \prod_{i=1}^{2002}\frac{1-y_i}{y_i}\geq 2001^{2002}$$

yoki $\prod_{i=1}^{2002}x_i\geq 2001^{1001}$ tengsizlikni hosil qilamiz.

79. $y_i=\frac{1998}{x_i+1998}$ almashtirish kiritamiz.

Ravshanki, $y_i \geq 0, \quad i=1,2,...,n$ va $y_1+y_2+...+y_n=1$.

Demak, $1-y_i=\sum_{j\neq i} y_j$.

Koshi tengsizligiga koʻra, $1-y_i \geq (n-1)\sqrt[n-1]{\prod_{j\neq i} y_j}$.

Bu tengsizliklarni barchasini koʻpaytirsak,

$\prod_{i=1}^{n}(1-y_i) \geq (n-1)^n \prod_{i=1}^{n} y_i$ yoki $\prod_{i=1}^{n}\frac{1-y_i}{y_i} \geq (n-1)^n$ tengsizlikni hosil qilamiz.

$\frac{1-y_i}{y_i}=\frac{x_i}{1998}$ boʻlgani uchun bundan $x_1x_2...x_n \geq 1998^n(n-1)^n$ tengsizlikni hosil qilamiz.

80. Oʻrta arifmetik va oʻrta geometrik miqdorlar haqidagi Koshi tengsizligi va $x \in \left(0;\frac{\pi}{2}\right)$ uchun $\sin x(\sin 2x-1) \leq a$ yoki $\sin x \leq \sqrt{\frac{tgx}{2}}$ tengsizliklarni oʻrinli ekanligini e'tiborga olib,

$$\sin x_1 \cdot \sin x_2 \cdot ... \cdot \sin x_n \leq \left(\frac{\sin x_1+\sin x_2+...+\sin x_n}{n}\right)^n \leq$$

$$\leq \left(\frac{\sqrt{tgx_1}+\sqrt{tgx_2}+...+\sqrt{tgx_n}}{n}\right)^n \cdot 2^{-\frac{n}{2}} \leq$$

$$\leq \left(\sqrt{\frac{tgx_1+tgx_2+...+tgx_n}{n}}\right)^n \cdot 2^{-\frac{n}{2}} \leq 2^{-\frac{n}{2}}$$

munosabatni hosil qilamiz.

81. Istalgan natural n uchun $\frac{1}{2n-1}>\frac{1}{2n}$ ekanligini e'tiborga olsak, $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}>\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ munosabat oʻrinlidir. Endi

$\underbrace{\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}}_{n}>\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ yoki $\frac{1}{2}>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}$ ekanligidan foydalansak, U holda

$$1+\frac{1}{3}+...+\frac{1}{2n-1}=\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n-1}>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}+\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)=$$

$$=\frac{n+1}{n}\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)$$

boʻladi.

82. a) umumiylikni chegaralamasdan $a \ge b \ge c$ deb olib, uchburchak tengsizligini qo‘llasak, $1 = a + b + c > 2a \Rightarrow b \le a < \frac{1}{2}$ va bundan

$$a^n + b^n < \frac{1}{2^n} + \frac{1}{2^n} = \frac{2}{2^n} \Rightarrow \left(a^n + b^n\right)^{\frac{1}{n}} < \frac{2^{\frac{1}{n}}}{2} \quad (*) \ .$$

Endi qo‘yidagini qaraymiz:

$$\left(b + \frac{c}{2}\right)^n = b^n + \frac{n}{2} c\, b^{n-1} + ... + \frac{c^n}{2^n} > b^n + c^n \text{ (chunki } \frac{n}{2} cb^{n-1} > c^n \text{).}$$

Xuddi shunday,

$$\left(a + \frac{c}{2}\right)^n > a^n + c^n \ .$$

Demak, $\left(b^n + c^n\right)^{\frac{1}{n}} + \left(a^n + b^n\right)^{\frac{1}{n}} < b + \frac{c}{2} + a + \frac{c}{2} = 1. \qquad (**)$

(*) va (**) larni hadma-had qo‘shib, isboti talab etilgan tengsizlikni hosil qilamiz;

b) α, β, γ uchburchak burchaklari uchun $1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2\cos\alpha \cos\beta \cos\gamma$ tenglikdan foydalanib,

$$\frac{\cos\alpha}{\cos\beta\cos\gamma} \cdot \frac{\cos\beta}{\cos\gamma\cos\alpha} \cdot \frac{\cos\gamma}{\cos\alpha\cos\beta} = \frac{\cos\alpha}{\cos\beta\cos\gamma} + \frac{\cos\beta}{\cos\gamma\cos\alpha} + \frac{\cos\gamma}{\cos\alpha\cos\beta} + 2$$

ifodani hosil qilamiz.

$$x = \frac{\cos\alpha}{\cos\beta\cos\gamma}, \ y = \frac{\cos\beta}{\cos\gamma\cos\alpha}, \ z = \frac{\cos\gamma}{\cos\alpha\cos\beta}$$

deb belgilash kiritib, $\cos\alpha + \cos\beta + \cos\gamma \le \frac{3}{2}$ tengsizlikdan foydalansak,

$$\frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{yz}+\sqrt{zx}} \frac{1}{} \le \frac{3}{2} \Leftrightarrow \ 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) \le 3\sqrt{xyz} \Leftrightarrow$$

$$4(x + y + z + 2(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})) \le 9xyz \Leftrightarrow$$

$$8(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}) \le 9(x + y + z + 2) - 4(x + y + z) = 5(x + y + z) + 18$$

83. Buni isbotlash uchun tenglikning o‘ng tomonidan uning chap tomonini keltirib chiqaramiz:

$$(a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right) = a^n + a^{n-1}b + a^{n-2}b^2 + ... + ab^{n-1} - a^{n-1}b - a^{n-2}b^2 - ... - b^n = a^n - b^n$$

bu yerda o‘rtadagi hadlarning barchasi ixchamlashib ketadi.

84. $\{a_n\}_{n=1}^{\infty}$ ketma-ketlik chegaralanganligi uchun, quyidagi quyi va yuqori limitlar chekli bo‘ladi:

$$\underline{\lim}_{n\to\infty}(a_n - a_{n+1}) = a, \quad \overline{\lim}_{n\to\infty}(a_n - a_{n+1}) = b.$$

Quyi va yuqori limitlarning ta’rifiga ko‘ra, $a \le b$ bo‘ladi. Biz $a = b = 0$ bo‘lishini isbotlaymiz. $b > 0$ deb faraz qilaylik. Yuqori limitning ta’rifiga ko‘ra, shunday $a_{n_k} - a_{n_k+1}$ qismiy ketma-ketlik borki, u b ga intiladi. Bu qismiy ketma-ketlikni $a_{n_k} - a_{n_k+1} > \frac{b}{2}$ tengsizlik bajariladigan qilib tanlash oson, buning uchun bu tengsizlik bajarilmaydigan cheklita hadni tashlab yuborish kifoya.

$\lim\limits_{n\to\infty}(a_n - 2a_{n+1} + a_{n+2}) = 0$ bo‘lgani uchun limitning ta’rifiga ko‘ra, ixtiyoriy N natural son uchun shunday $n_0 = n_0(N)$ nomer topiladiki, bunda $n > n_0$ bulganda ushbu

$(a_n - a_{n+1}) - (a_{n+1} - a_{n+2}) < \frac{b}{2N}$ tengsizlik o‘rinli bo‘ladi. Agar biz $n_k > n_0(N)$ deb olsak, quyidagi tengsizliklar o‘rinli bo‘ladi:

$$(a_{n_k+1} - a_{n_k+2}) > (a_{n_k} - a_{n_k+1}) - \frac{b}{2N} > \frac{b}{2} - \frac{b}{2N} = \frac{b(N-1)}{2N},$$

$$(a_{n_k+2} - a_{n_k+3}) > (a_{n_k+1} - a_{n_k+2}) - \frac{b}{2N} > \frac{b(N-1)}{2N} - \frac{b}{2N} = \frac{b(N-2)}{2N},$$

....

$$(a_{n_k+N-1} - a_{n_k+N}) > \frac{b}{2N}.$$

Bu tengsizliklarni qo‘shib, quyidagiga ega bo‘lamiz:

$$a_{n_k+1} - a_{n_k+N} > \frac{b}{2N}((N-1) + (N-2) + ... + 1) = \frac{b(N-1)}{4}.$$

Bunga ko‘ra, N ning ixtiyoriyligi $\{a_n\}_{n=1}^{\infty}$ ning chegaralanganligiga zid keladi. Demak, farazimiz noto‘g‘ri, ya’ni $b \le 0$ ekan. $a \ge 0$ bo‘lishi ham xuddi shunday isbot qilinadi. Demak, $0 \le a \le b \le 0$, ya’ni $a = b = 0$ ekan, bu esa, quyidagi limit mavjud va uning qiymati nolga teng bo‘lishini bildiradi: $\lim\limits_{n\to\infty}(a_n - a_{n+1}) = 0$. Isbotlandi.

85. Matematik induksiya metodini qo‘llaymiz.

$n = 1$ da tenglik bajariladi.

$n = 2$ da $|\sin 2x| = 2|\sin x||\cos x| \le 2|\sin x|$, chunki $|\cos x| \le 1$.

Bu tasdiqni $n = k$ da $|\sin kx| \le k|\sin x|$ ni to‘g‘ri deb olib, $n = k+1$ da to‘g‘riligini isbotlaymiz.

$n = k+1$ da

$$|\sin(k+1)x| = |\sin(kx + x)| = |\sin kx\cos x + \cos kx\sin x| \le |\sin kx\cos x| + |\cos kx\sin x| \le$$

$$\le |\sin kx| + |\sin x| \le k|\sin x| + |\sin x| = (k+1)|\sin x|$$

Demak, ixtiyoriy natural n lar uchun $|\sin nx| \le n|\sin x|$ tengsizlik o‘rinli ekan.

86. $S_k = \frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{3k+1}$ orqali belgilaymiz.

1-qadam. n = 1 da $S_1 = \frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{3\cdot 1+1} = \frac{13}{12} > 1$ ga ega bo‘lamiz. 1-qadam isbotlandi.

2-qadam. n = k da quyidagi tengsizlikning bajarilishi berilgan:

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{3k+1} > 1.$$

Quyidagi tengsizlikning bajarilishini isbotlaymiz:

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + ... + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > 1.$$

Isboti.

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + ... + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} + \left(\frac{1}{k+1} - \frac{1}{k+1}\right) =$$

$$= \underbrace{\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + ... + \frac{1}{3k+1}}_{=S_k > 1} + \underbrace{\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} - \frac{1}{k+1}}_{>0} > 1.$$

“ k> 0 ” tengsizlik quyidagicha kelib chiqadi:

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} - \frac{1}{k+1} = \frac{1}{3k+2} + \frac{1}{3k+4} - \frac{2}{3k+3} =$$

$$= \frac{(3k+4)(3k+3) + (3k+2)(3k+3) - (6k+4)(3k+4)}{(3k+2)(3k+3)(3k+4)} =$$

$$= \frac{2}{(3k+2)(3k+3)(3k+4)} > 0.$$ 2-qadam isbotlandi.

87. 1-qadam. n = 6 da: $\frac{5^6}{6!} = \frac{5^5}{5!}\left(\frac{5}{6}\right)^{6-5}$ ega bo‘lamiz. 1-qadam isbotlandi.

2-qadam. $\frac{5^k}{k!} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}$, $k \ge 6$ tengsizlikning bajarilishi berilgan. Quyidagi tengsizlikning bajarilishini isbotlash lozim:

$$\frac{5^{k+1}}{(k+1)!} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

Isboti. $n = k \ge 6$ da quyidagiga ega bo‘lamiz:

$$\frac{5^{k+1}}{(k+1)!} = \underbrace{\frac{5^k}{k!}}_{\le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}} \cdot \underbrace{\frac{5}{k+1}}_{<\frac{5}{6}} \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}\frac{5}{6} = \frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

2-qadam isbotlandi. Matematik induksiya prinsipiga ko‘ra, berilgan tengsizlik ixtiyoriy $n \ge 6$ natural son uchun bajariladi.

88. n = 1, 2, 3 da yig‘indini topamiz:

$$S_1 = arctg\frac{1}{2} = arctg\frac{1}{1+1};$$

$$S_2 = arctg\frac{1}{2} + arctg\frac{1}{8} = arctg\frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{8}} = arctg\frac{2}{3} = arctg\frac{2}{2+1};$$

$$S_3 = arctg\frac{2}{3} + arctg\frac{1}{18} = arctg\frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\cdot\frac{1}{18}} = arctg\frac{3}{4} = arctg\frac{3}{3+1}.$$

Quyidagini isbotlaymiz:

$$S_n = arctg\frac{1}{2} + arctg\frac{1}{8} + ... + arctg\frac{1}{2n^2} = arctg\frac{n}{n+1}, \quad \forall n \in N. \qquad (*)$$

Matematik induksiya metodi bilan formulani isbotlaymiz.

1-qadam. n=1 qiymatda hisoblanganda, S_1 isbotlandi.

2-qadam. $n = k$ da (*) tenglikning bajarilishi berilgan. n=k+1 da $S_{k+1} = arctg\frac{k+1}{k+1+1}$ tenglikning bajarilishini isbotlaymiz:

$$S_{k+1} = \underbrace{arctg\frac{1}{2} + arctg\frac{1}{8} + .. + arctg\frac{1}{2k^2}}_{=S_k} + arctg\frac{1}{2(k+1)^2} =$$

$$= S_k + arctg\frac{1}{2(k+1)^2} = arctg\frac{k}{k+1} + arctg\frac{1}{2(k+1)^2} =$$

$$= arctg\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{k+1}\cdot\frac{1}{2(k+1)^2}} = arctg\frac{k+1}{k+2}.$$

Bu yerda

$$\frac{\dfrac{k}{k+1}+\dfrac{1}{2(k+1)^2}}{1-\dfrac{k}{k+1}\cdot\dfrac{1}{2(k+1)^2}}=\frac{(2k^2+2k+1)\cdot 2\cdot(k+1)^3}{2\cdot(k+1)^2\cdot(2k^3+6k^2+5k+2)}==\frac{(2k^2+2k+1)\cdot(k+1)}{(2k^3+6k^2+5k+2)}=$$

$$\frac{k+1}{k+2}.$$

Oxirgi tenglik quyidagidan kelib chiqadi:

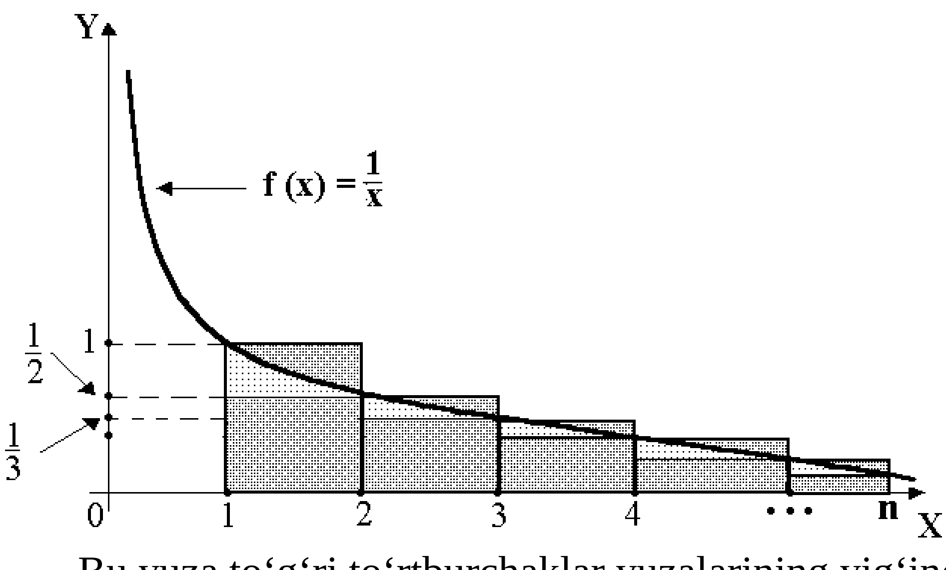
$$\begin{array}{r|l} -\;2k^3+6k^2+5k+2 & 2k^2+2k+1 \\ \underline{2k^3+2k^2+k} & k+2 \\ -\;4k^2+4k+2 & \\ \underline{4k^2+4k+2} & \\ 0 & \end{array}$$

2-qadam isbotlandi. Matematik induksiya prinsipiga ko‘ra, (*) tenglikning ixtiyoriy *n* natural sonlarda bajarilishi kelib chiqadi.

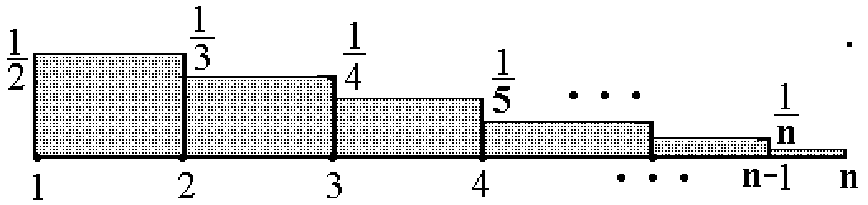
89. Dastlab, quyidagi tengsizlikni isbotlaymiz:

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}<\int_1^n\frac{dx}{x}=\ln n<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-1},\ \forall\, n\in N,\ n\geq 2.$$

$\int_1^n\frac{dx}{x}=\ln n$ integral $f(x)=\frac{1}{x}$ egri chiziq, x = 1, x = n va y=0 chiziqlar bilan chegaralangan shakl yuzasiga teng.

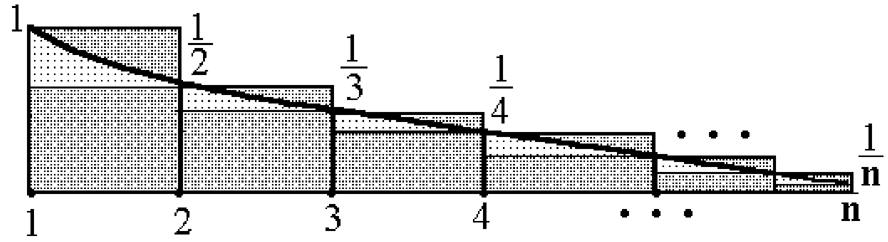


Bu yuza to‘g‘ri to‘rtburchaklar yuzalarining yig‘indisidan kattadir.



$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}<\ln n \quad \forall\, n\in N,\ \ n\geq 2.$$

Bu yuza esa to‘rtburchaklar yuzalarining yig‘indisidan kichikdir.



$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n-1}>\ln n,\ \ \forall\, n\in N,\ \ n\geq 2\,.$$

Matematik induksiya metodi bilan tengsizliklarni isbotlaymiz.

1-qadam. Uchta rasm va [0, 1] oraliqda aniqlangan integral xossasidan quyidagi tengsizlik kelib chiqadi: $\frac{1}{2}\cdot 1<\int\limits_1^2\frac{dx}{x}=\ln 2<1\cdot 1.$

Bundan n = 2 da tengsizliklarning o‘rinli ekanligi tasdiqlanadi. 1-qadam isbotlandi.

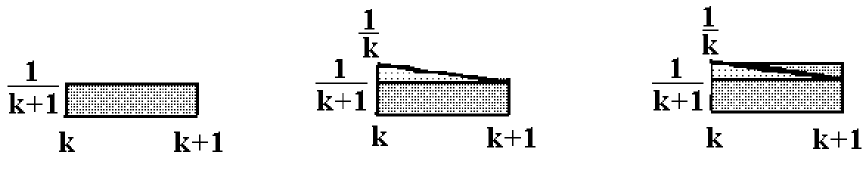
2-qadam. $n = k$ quyidagi tengsizlikning bajarilishi berilgan

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}<\ln k<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1},\quad k\geq 2\,.$$

n = k+1 da ushbu tengsizlikning bajarilishini isbotlash lozim:

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}+\frac{1}{k+1}<\ln(k+1)<1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1}+\frac{1}{k},\quad k\geq 2\,. \qquad (2)$$

Isboti. Quyidagi shakllarning S_1 , S_2 , S_3 yuzalarini taqqoslaymiz:



$$S_1=\frac{1}{k+1}\cdot 1 \quad < \quad S_2=\int\limits_k^{k+1}\frac{dx}{x}=\ln\frac{k+1}{k} \quad < \quad S_3=\frac{1}{k}\cdot 1$$

(2) tengsizlikning chap qismini taqqoslaymiz:

$$\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k}+\frac{1}{k+1}<\ln k+\frac{1}{k+1}\cdot 1<\ln k+\int\limits_k^{k+1}\frac{dx}{x}=$$

$= \ln k + \ln(k+1) - \ln k = \ln(k+1), \quad k \geq 2$.

(2) tengsizlikning o‘ng qismini taqqoslaymiz:

$\ln(k+1) = \ln k + \ln(k+1) - \ln k =$

$$= \ln k + \underbrace{\int_k^{k+1} \frac{dx}{x}}_{<\frac{1}{k}} < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k-1} + \frac{1}{k}, \quad k \geq 2.$$

2-qadam isbotlandi. 1 va 2-qadamlarning isbotidan ixtiyoriy natural $n \geq 2$ uchun tasdiq o‘rinli ekanligi kelib chiqadi.

90. 1-qadamni isbotiga o‘xshash to‘rt marotaba x ni t ga almashtirish va t bo‘yicha 0 dan $x \in \left(0, \frac{\pi}{2}\right)$ gacha mos tengsizlikni integrallash protsedurasini bajarish kerak. (1) tengsizlikning o‘ng qismida x ni t ga almashtirish va t bo‘yicha 0 dan $x \in \left(0, \frac{\pi}{2}\right)$ gacha ushbu tengsizlikni integrallab, quyidagini hosil qilamiz:

$$1 - \cos x = \int_0^x \sin t dt \leq \int_0^x \left(t - \frac{t^3}{3!} + \ldots + \frac{t^{2k+1}}{(2k+1)!} \right) dt = \frac{x^2}{2} - \frac{x^4}{4!} + \ldots + \frac{x^{2k+2}}{(2k+2)}.$$

$\Rightarrow \cos x \geq 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots - \frac{x^{2k+2}}{(2k+2)}$. Ikkinchi marotaba x ni t ga almashtirish va t bo‘yicha 0 dan $x \in \left(0, \frac{\pi}{2}\right)$ gacha integtallab, quyidagiga ega bo‘lamiz:

$$\sin x = \int_0^x \cos t dt \geq \int_0^x \left(1 - \frac{t^2}{2} + \ldots - \frac{t^{2k+2}}{(2k+2)!} \right) dt = x - \frac{x^3}{3!} + \ldots - \frac{t^{2k+3}}{(2k+3)!}.$$

Bu quyidagiga teng kuchli

$$\sin x \geq x - \frac{x^3}{3!} + \ldots - \frac{t^{2k+3}}{(2k+3)!}, \quad x \in \left(0, \frac{\pi}{2}\right). \qquad (2)$$

(1) tengsizlikning chap qismi isbotlandi. (2) tengsizlikni uchinchi marotaba x ni t ga almashtirish va t bo‘yicha $x \in \left(0, \frac{\pi}{2}\right)$ gacha integrallab, quyidagiga ega bo‘lamiz:

$$1-\cos x=\int_0^x \sin t dt \geq \int_0^x \left(t-\frac{t^3}{3!}+...-\frac{t^{2k+3}}{(2k+3)!}\right)dt=\frac{x^2}{2}-\frac{x^4}{4!}+...-\frac{x^{2k+4}}{(2k+4)!}.$$

Bundan $\cos x \leq 1-\frac{x^2}{2}+\frac{x^4}{4!}-...+\frac{x^{2k+4}}{(2k+4)!}$. To‘rtinchi marotaba ushbu tengsizlikni X ni T ga almashtirish va T bo‘yicha $x\in\left(0,\frac{\pi}{2}\right)$ gacha integrallab, quyidagiga ega bo‘lamiz:

$$\sin x=\int_0^x \cos t dt \leq \int_0^t \left(1-\frac{t^2}{2}+...+\frac{t^{2k+4}}{(2k+4)!}\right)dt=x-\frac{x^3}{3!}+...+\frac{x^{2k+5}}{(2k+5)!}.$$

Ya’ni quyidagi tengsizlik bajariladi:

$$\sin x \leq x-\frac{x^3}{3!}+...+\frac{x^{2k+5}}{(2k+5)!}.$$

(1) tengsizlikning o‘ng qismi, ushbu tengsizlik va (2) tengsizlikdan (1) tengsizlik hosil qilinadi. 2-qadam isbotlandi.

91. Ushbu masalani yechishda matematik induksiya metodidan foydalanmasdan isbotlash mumkin. $10^n+10^{n-1}+...+1$ yig‘indi q=10 maxrajli n+1 ta haddan iborat geometrik progressiyani anglatadi. U holda

$$\left(10^n+10^{n-1}+...+1\right)\cdot\left(10^{n+1}+5\right)+1=\left(\frac{10^{n+1}-1}{10-1}\right)\cdot\left(10^{n+1}+5\right)+1=$$

$$=\frac{\left(10^{n+1}\right)^2-10^{n+1}+5\cdot 10^{n+1}-5+9}{9}=\frac{\left(10^{n+1}\right)^2+2\cdot 10^{n+1}\cdot 2+2^2}{9}=\left(\frac{10^{n+1}+2}{3}\right)^2$$

ixtiyoriy n natural son uchun o‘rinli bo‘ladi.

92. Quyidagi belgilash kiritamiz: $f(x)=x^{3m}+x^{3n+1}+x^{3p+2}$ va $g(x)=x^2+x+1$. Ma’lumki, $f(x)$ ko‘phad $g(x)$ ko‘phadga bo‘linishi uchun $g(x)$ ning ildizlari $f(x)$ning ham ildizlari bo‘lishi kerak. Demak, biz $g(x)=0$ tenglamaning yechimlari $f(x)$ ning ham ildizlari bo‘lishini ko‘rsatsak bo‘ldi.

$$g(x)=0 \Rightarrow x^2+x+1=0 \Rightarrow x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} \quad \text{hamda}$$

$$x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$

Endi $\begin{cases} f(x_1)=0 \\ f(x_2)=0 \end{cases}$ munosabatning o'rinli ekanini ko'rsatamiz. Buning uchun Muavr formulasiga ko'ra, $x_1^3=1$ *va* $x_2^3=1$ ekanligini e'tiborga olsak, $f(x_1)=x_1^{3m}+x_1^{3n+1}+x_1^{3p+2}=(x_1^3)^m+(x_1^3)^n\cdot x_1+(x_1^3)^p\cdot x_1^2=$
$=1+x_1+x_1^2=0 \Rightarrow f(x_1)=0$

Xuddi shunday,

$$f(x_2)=x_2^{3m}+x_2^{3n+1}+x_2^{3p+2}=(x_2^3)^m+(x_2^3)^n\cdot x_2+(x_2^3)^p\cdot x_2^2=$$
$$=1+x_2+x_2^2=0 \Rightarrow f(x_2)=0$$

Demak, $g(x)=x^2+x+1$ ko'phadning ildizlari $f(x)=x^{3m}+x^{3n+1}+x^{3p+2}$ ko'phadning ham ildizlari ekan, ya'ni $\forall m,n,p\in N$ larda $x^{3m}+x^{3n+1}+x^{3p+2}$ ko'phad x^2+x+1 ko'phadga qoldiqsiz bo'linar ekan.

93. Quyidagi belgilash kiritamiz : $f(x)=x^{3m}-x^{3n+1}+x^{3p+2}$ va $g(x)=x^2-x+1$. Biz $g(x)$ ning ildizlari $f(x)$ ni ham ildizi ekanligini ko'rsatsak, yetarli. $g(x)=0\Rightarrow x^2-x+1=0\Rightarrow \begin{cases} x_1=\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3} \\ x_2=\frac{1}{2}-i\frac{\sqrt{3}}{2}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3} \end{cases}$

Bulardan ko'rinadiki, Muavr fo'rmulasiga ko'ra, $x_1^3=x_1^3=-1$ bo'ladi hamda $x_1^2=x_1-1$ *va* $x_2^2=x_2-1$ ekanligini e'tiborga olib, bularni $f(x)$ ga qo'yamiz:

$$f(x_1)=x_1^{3m}-x_1^{3n+1}+x_1^{3p+2}=(x_1^3)^m-(x_1^3)^n\cdot x_1+(x_1^3)^p\cdot x_1^2=$$
$$=(-1)^m-(-1)^n x_1+(-1)^p x_1^2=(-1)^m-(-1)^n x_1+(-1)^p(x_1-1)=$$
$$=\left[(-1)^m-(-1)^p\right]+x_1\left[(-1)^p-(-1)^n\right]$$

Bu oxirgi ifoda nol bo'lishi uchun qo'shilivchi qavslarining har biri nol bo'lishi kerak, ya'ni $(-1)^m=(-1)^p=(-1)^n$ bo'lishi kerak. Bu munosabat esa m,n,p lar bir vaqtda juft yoki bir vaqtda toq bo'lganda bajariladi.

$f(x_2)=0$ bo'lishi uchun xuddu shu shart kelib chiqadi.

Demak, biz quyidagi xulosaga keldik: $x^{3m}-x^{3n+1}+x^{3p+2}$ ko'phad x^2-x+1 ko'phadga qoldiqsiz bo'linishi uchun m,n,p lar bir vaqtda juft yoki bir vaqtda toq bo'lishi kerak.

94. $g(x)=x^4+x^2+1$ ko'phadni o'zaro tub ko'paytuvchilarga ajratamiz:

$$g(x)=x^4+x^2+1=(x^2-x+1)(x^2+x+1)$$

114-misolda ixtiyoriy natural $\forall m,n,p\in N$ larda $x^{3m}+x^{3n+1}+x^{3p+2}$ ko‘phad x^2+x+1 ko‘phadga qoldiqsiz bo‘linadi. Demak, biz x^2-x+1 ni ildizlari $f(x)=x^{3m}+x^{3n+1}+x^{3p+2}$ ni ham ilzdizlari bo‘ladigan m,n,p larni topishimiz kerak.

$$g(x)=0\Rightarrow x^2-x+1=0\Rightarrow\begin{cases}x_1=\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\\ x_2=\frac{1}{2}-i\frac{\sqrt{3}}{2}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\end{cases}$$

Bulardan ko‘rinadiki, Muavr fo‘rmulasiga ko‘ra, $x_1^3=x_1^3=-1$ bo‘ladi hamda $x_1^2=x_1-1\ va\ x_2^2=x_2-1$ ekanligini e’tiborga olib, bularni $f(x)$ ga qo‘yamiz:

$$f(x_1)=x_1^{3m}+x_1^{3n+1}+x_1^{3p+2}=(x_1^3)^m+(x_1^3)^n\cdot x_1+(x_1^3)^p\cdot x_1^2=$$
$$=(-1)^m+(-1)^n x_1+(-1)^p x_1^2=(-1)^m+(-1)^n x_1+(-1)^p(x_1-1)=$$
$$=\left[(-1)^m-(-1)^p\right]+x_1\left[(-1)^p-(-1)^{n+1}\right]$$

Demak, $f(x_1)=0$ bo‘lishi uchun $(-1)^m=(-1)^p=(-1)^{n+1}$ bo‘lishi, ya’ni $m,p,n+1$ sonlari bir vaqtda toq yoki bir vaqtda juft bo‘lishi kerak.

$f(x_2)=0$ bo‘lishi uchun xuddu shu shart kelib chiqadi.

Bulardan quyidagi xulosaga kelamiz: $x^{3m}+x^{3n+1}+x^{3p+2}$ ko‘phad x^4+x^2+1 ko‘phadga qoldiqsiz bo‘linishi uchun $m,p,n+1$ sonlari bir vaqtda toq yoki bir vaqtda juft bo‘lishi kerak.

95. Biz $g(x)=x^2+x+1$ ko‘phad ildizlari $f(x)=x^{2m}+x^m+1$ ko‘phadning ham ildizi bo‘ladigan m larni topishimiz kerak.

$g(x)=0\ \Rightarrow x^2+x+1=0\Rightarrow x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ hamda $x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$. Bularni $f(x)$ ga qo‘yib, quyidagiga ega bo‘lamiz:

$$f(x_1)=x_1^{2m}+x_1^m+1=\cos\frac{4\pi m}{3}+i\sin\frac{4\pi m}{3}+\cos\frac{2\pi m}{3}+i\sin\frac{2\pi m}{3}+1=$$
$$=2\cos\pi m\cos\frac{\pi m}{3}+i\sin\pi m\cos\frac{\pi m}{3}+1=2(-1)^m\cos\frac{\pi m}{3}+1$$

Bu oxirgi ifoda, ya’ni $f(x_1)=0$ bo‘lishi uchun m soni 3 ga bo‘linmasligi kerak, ya’ni $m=3k+1$ yoki $m=3k+2$ ko‘rinishda bo‘lishi kerak. $f(x_2)=0$ bo‘lishi uchun ham shu shart kelib chiqadi. Bulardan quyidagi xulosaga kelamiz: $x^{2m}+x^m+1$ ko‘phad x^2+x+1 ko‘phadga bo‘linishi uchun m soni $m=3k+1$ yoki $m=3k+2$ ko‘rinishda bo‘lishi kerak.

96. Buning uchun $g(x)=x^{k-1}+x^{k-2}+...+1$ ko‘phadni ildizi $f(x)=x^{ka_1}+x^{ka_2+1}+...+x^{ka_k+k-1}$ ko‘phadni ham ildizi bo‘lishini ko‘rsatamiz.

$$x^{k-1}+x^{k-2}+...+1=0 \qquad \Big|\cdot(x-1)$$

$$x^k-1=0$$

$$x^k=1$$

Demak, $g(x)$ ko‘phadning ildizlari 1 soning $k-$darajali ildizlaridir. Bu ildizlarni x_i $i=\overline{1,k}$ bilan belgilasak. Berilishiga ko‘ra, har x_i uchun

$$x_i^k=1$$

bo‘ladi. Bulardan

$$f(x_i)=x_i^{ka_1}+x_i^{ka_2+1}+...+x_i^{ka_k+k-1}=\left(x_i^k\right)^{a_1}+\left(x_i^k\right)^{a_2}\cdot x_i+...+\left(x_i^k\right)^{a_k}\cdot x^{k-1}=$$

$$=x_i^{k-1}+x_i^{k-2}+...+1=0$$

Isbot tugadi.

97. Biz $g(x)=x^2+x+1$ ko‘phad ildizi $f(x)=(x+1)^m-x^m-1$ ko‘phadning ham ildizi bo‘ladigan m larni topishimiz kerak.

$$g(x)=0 \Rightarrow x^2+x+1=0 \Rightarrow x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$. Bularni hamda $1+x_1=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ shu bilan birga, Muavr formulasini e’tiborga olsak, quyidagilarga ega bo‘lamiz:

$$f(x_1)=(x_1+1)^m-x_1^m-1=\cos\frac{\pi m}{3}+i\sin\frac{\pi m}{3}-\cos\frac{2\pi m}{3}-i\sin\frac{2\pi m}{3}-1$$

Kosinus va sinus funksiyalarning davri $2\pi k\ k\in Z$ bo‘lgani uchun $\frac{\pi m}{3}=2\pi k \Rightarrow m=6k$ larni e’tiborga olib, quyidagi hollarni qarab chiqamiz:

$m=6n \Rightarrow f(x_1)=\cos 2\pi n+i\sin 2\pi n-\cos 4\pi n-i\sin 4\pi n-1=-1\neq 0$

$m=6n+1 \Rightarrow f(x_1)=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}-\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}-1=0$

$m=6n+2 \Rightarrow f(x_1)=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}-\cos\frac{4\pi}{3}-i\sin\frac{4\pi}{3}-1=i\sqrt{3}-1\neq 0$

$m=6n+3 \Rightarrow f(x_1)=\cos\pi+i\sin\pi-\cos 2\pi-i\sin 2\pi-1=-3$

$m=6n+4 \Rightarrow f(x_1)=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}-\cos\frac{8\pi}{3}-i\sin\frac{8\pi}{3}-1=-i\sqrt{3}-1\neq 0$

$m=6n+5 \Rightarrow f(x_1)=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}-\cos\frac{10\pi}{3}-i\sin\frac{10\pi}{3}-1=0$

Demak, $f(x_1)=0$ bo‘lishi uchun m natural soni $m=6n+1$ *va* $m=6n+5$ $(n\in N)$ ko‘rinishda bo‘lishi kerak. $f(x_2)=0$ bo‘lishi uchun ham xuddi shu shart kelib chiqadi. Bulardan quyidagi xulosaga kelamiz: $(x+1)^m-x^m-1$ ko‘phad x^2+x+1 ko‘phadga bo‘linishi uchun m natural soni $m=6n+1$ *va* $m=6n+5$ $(n\in N)$ ko‘rinishda bo‘lishi kerak.

98. Biz $g(x)=x^2+x+1$ ko‘phad ildizi $f(x)=(x+1)^m+x^m+1$ ko‘phadni ham ildizi bo‘ladigan m larni topishimiz kerak.

$g(x)=0 \Rightarrow x^2+x+1=0 \Rightarrow x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$. Bularni hamda $1+x_1=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, shu bilan birga, Muavr formulasini e’tiborga olsak, quyidagilarga ega bo‘lamiz:

$$f(x_1)=(x_1+1)^m+x_1^m+1=\cos\frac{\pi m}{3}+i\sin\frac{\pi m}{3}+\cos\frac{2\pi m}{3}+i\sin\frac{2\pi m}{3}+1$$

Kosinus va sinus funksiyalarning davri $2\pi k$ $k\in Z$ bo‘lgani uchun, $\frac{\pi m}{3}=2\pi k \Rightarrow m=6k$ larni e’tiborga olib, quyidagi hollarni qarab chiqamiz:

$m=6n \Rightarrow f(x_1)=\cos 2\pi n + i\sin 2\pi n + \cos 4\pi n + i\sin 4\pi n + 1 = 3 \neq 0$

$m=6n+1 \Rightarrow f(x_1)=\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + 1 = i\sqrt{3}+1 \neq 0$

$m=6n+2 \Rightarrow f(x_1)=\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + 1 = 0$

$m=6n+3 \Rightarrow f(x_1)=\cos\pi + i\sin\pi + \cos 2\pi + i\sin 2\pi + 1 = 1 \neq 0$

$m=6n+4 \Rightarrow f(x_1)=\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} + 1 = 0$

$m=6n+5 \Rightarrow f(x_1)=\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} + \cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3} + 1 = -i\sqrt{3}+1 \neq 0$

Demak, $f(x_1)=0$ bo‘lishi uchun m natural soni $m=6n+2$ *va* $m=6n+4$ $(n\in N)$ ko‘rinishda bo‘lishi kerak. $f(x_2)=0$ bo‘lishi uchun ham xuddi shu shart kelib chiqadi. Bulardan quyidagi xulosaga kelamiz: $(x+1)^m + x^m + 1$ ko‘phad x^2+x+1 ko‘phadga bo‘linishi uchun m natural soni $m=6n+2$ *va* $m=6n+4$ $(n\in N)$ ko‘rinishda bo‘lishi kerak.

99. $(x^2+x+1)^2=(x-x_1)^2(x-x_2)^2$. Yuqoridagi misollarda ko‘rsatilgandi:

$$x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

x_1 *va* x_2 sonlari $(x^2+x+1)^2$ ko‘phadning karrali ildizlari. Bu ildizlari $f(x)$ ni ham ildizi bo‘lishi uchun m natural soni $m=6n+1$ *va* $m=6n+5$ $(n\in N)$ ko‘rinishda bo‘lishi kerak edi. Lekin bu yerda x_1 ildiz karrali ildiz bo‘lishi uchun $f'(x_1)=m\left((1+x_1)^{m-1}-x^{m-1}\right)=0$ ham nol bo‘ladigan m larni topamiz. Bunda faqat ikkita hol bo‘lishi mumkin:

$m=6n+1 \Rightarrow f'(x_1)=m(\cos 2\pi n + i\sin 2\pi n - \cos 4\pi n - i\sin 4\pi n)=0$

$m=6n+5 \Rightarrow f'(x_1)=$

$=m(\cos(2\pi n+\pi)+i\sin(2\pi n+\pi)-\cos(4\pi n+2\pi)-i\sin(4\pi n+2\pi))=-2m\neq 0$

$f(x_2)=0$ bo‘lishi uchun ham xuddi shunday shart kelib chiqadi.

Demak, bulardan quyidagi xulosaga kelamiz: $(x+1)^m - x^m - 1$ ko‘phad $(x^2+x+1)^2$ ko‘phadga bo‘linishi uchun m natural soni $m=6n+1$ $(n\in N)$ ko‘rinishda bo‘lishi kerak.

100. $\left(x^2+x+1\right)^2=\left(x-x_1\right)^2\left(x-x_2\right)^2$. Yuqoridagi misollarda ko‘rsa-tilgandi:

$$x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

x_1 *va* x_2 sonlari $\left(x^2+x+1\right)^2$ ko‘phadning karrali ildizlari. Bu ildizlari $f(x)$ ni ham ildizi bo‘lishi uchun m natural soni $m=6n+2$ *va* $m=6n+4$ $(n\in N)$ ko‘rinishda bo‘lishi kerak edi. Lekin bu yerda x_1 ildiz karrali ildiz bo‘lishi uchun $f'(x_1)=m\left((1+x_1)^{m-1}+x_1^{m-1}\right)=0$ ham nol bo‘ladigan m larni topamiz. Bunda faqat ikkita hol bo‘lishi mumkin:

$$m=6n+2\Rightarrow f'(x_1)=m\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=im\sqrt{3}\neq 0$$

$$m=6n+5\Rightarrow f'(x_1)=m(\cos\pi+i\sin\pi+\cos 2\pi+i\sin 2\pi)=0$$

$f(x_2)=0$ bo‘lishi uchun ham xuddi shunday shart kelib chiqadi.

Demak, bulardan quyidagi xulosaga kelamiz: $(x+1)^m+x^m+1$ ko‘p-had $\left(x^2+x+1\right)^2$ ko‘phadga bo‘linishi uchun m na-tural soni $m=6n+4$ $(n\in N)$ ko‘rinishda bo‘lishi kerak.

101. Bu tengsizlikni isbotlash uchun, avval, uni quyidagi ko‘rinish-ga keltirib olamiz:

$$\left(1+\frac{1}{n}\right)^{n+1}<e\left(1+\frac{1}{2n}\right)$$

Quyidagi yordamchi funksiyani qaraymiz:

$f(x)=x+x\ln\left(1+\dfrac{x}{2}\right)-(1+x)\ln(1+x)$ $\left(0<x\le\dfrac{1}{n}\right)$ funksiyani o‘suvchi yoki kamayuvchi ekanini uni hosilasi yordamida tekshiramiz. Agar biz $x>0$ larda $\ln x\le x-1$ tengsizlikni ham e’tiborga olsak,

$$f'(x)=\frac{x}{x+2}-\ln\frac{1+x}{1+\frac{x}{2}}>\frac{x}{x+2}-\frac{1+x}{1+\frac{x}{2}}+1=0\Rightarrow x\in\left(0,\frac{1}{n}\right)\quad f'(x)>0$$

ekanligi ya’ni funksiya $\left(0,\dfrac{1}{n}\right)$ oraliqda o‘suvchi ekanligi kelib chiqadi.

Bundan $\forall x\in\left(0,\dfrac{1}{n}\right)$ *da* $f\left(\dfrac{1}{n}\right)>f(0)=0\Rightarrow\left(1+\dfrac{1}{n}\right)^{n+1}<e\left(1+\dfrac{1}{2n}\right)$

Demak, $\forall n \in N$ lar uchun $\frac{e}{2n+1} < e - \left(1 + \frac{1}{n}\right)^n$ tengsizlik o‘rinli ekan.

102. Isboti. 4-shartga ko‘ra, $\forall \varepsilon > 0$ $\frac{\varepsilon}{4n_0 M}$ ga ko‘ra,

$\exists n_0(\varepsilon) \in N$ $\forall n \in N$ topiladiki, $\forall n > n_0$ da $|x_n - a| < \frac{\varepsilon}{2}$ o‘rinli bo‘ladi. $\lim_{x\to\infty} x_n = a$ dan $|x_n| < M$ $(M > 0)$, $\forall n \in N$ $|x_n - a| < |x_n| + |a| < M + M < 2M$

3- shartga ko‘ra, $\forall \varepsilon > 0, \frac{\varepsilon}{4n' M}$ ga ko‘ra, $\exists n' \in N$ $\forall n > n'$ lar uchun $|P_{n_k}| < \frac{\varepsilon}{4n' M}$ quyidagini qaraymiz. $\forall$ $\varepsilon > 0$ uchun $\max(n', n_0) = \overline{n}, \exists \overline{n} \in N, \forall n > \overline{n}$

$$\left|\sum_{k=1}^{n} P_{n_k} x_k - a\right| = \left|\sum_{k=1}^{n} P_{n_k} x_k - \sum_{k=1}^{n} P_{n_k}(x_k - k)\right| = \left|\sum_{k=1}^{n} P_{n_k}(x_k - a)\right| =$$

$$= \left|P_{n_1}(x_1 - a) + P_{n_2}(x_2 - a) + \ldots + P_{n_{n_0}}(x_{n_0} - a) + P_{n_{n_0+1}}(x_{n_0+1} - a) + \ldots. + P_{n_n}(x_n - a)\right| <$$

$$< \frac{\varepsilon}{4n_0 M} \cdot 2M \cdot n_0 + \frac{\varepsilon}{2} \cdot 1 < \varepsilon$$

Bundan $\lim_{n\to\infty} \sum_{k=1}^{n} P_{n_k} x_k = a$ ekani kelib chiqadi.

103. Ushbu

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2n}$$

belgilashni kiritib olamiz. U holda

$$a_n^2 = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots \cdot (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot \ldots (2n)^2} = \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{5 \cdot 7}{6^2} \ldots \frac{(2n-3)(2n-1)}{(2(n-1))^2} \cdot \frac{(2n-1)(2n+1)}{(2n)^2} \cdot \frac{1}{2n+1} =$$

$$= \frac{2^2 - 1}{2^2} \cdot \frac{4^2 - 1}{4^2} \cdot \frac{6^2 - 1}{6^2} \ldots \frac{\left(2(n-1)^2\right) - 1}{(2(n-1))^2} \cdot \frac{(2n)^2 - 1}{(2n)^2} \cdot \frac{1}{2n+1} < 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot 1 \cdot \frac{1}{2n+1} = \frac{1}{2n+1}$$

bo‘ladi. Demak,

$$0 < a_n < \frac{1}{\sqrt{2n+1}}, \quad n = 1,2,\ldots$$

Ikkita militsioner haqidagi lemmaga ko‘ra,

$$0 \le \lim_{n\to\infty} \le \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$$

Javob: 0.

104.

$$\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x.....\cos nx}{x^2}=$$

$$\lim_{x\to 0}[\frac{1-\cos x}{x^2}+\frac{(1-\cos 2x)\cos x}{x^2}+\frac{(1-\cos 3x)\cos x\cos 2x}{x^2}$$

$$\frac{(1-\cos 4x)\cos x\cos 2x\cos 3x}{x^2}+......$$

$$......+\frac{(1-\cos nx)\cos x\cos 2x\cos 3x...\cos nx}{x^2}]=$$

$$\lim_{x\to 0}\frac{1-\cos kx}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{kx}{2}}{\left(\frac{kx}{2}\right)^2}\cdot\left(\frac{k}{2}\right)^2=\frac{2k^2}{4}=\frac{k^2}{2}\Rightarrow$$

$$\Rightarrow\ \lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x.....\cos nx}{x^2}=\frac{1^2}{2}+\frac{2^2}{2}+....+\frac{n^2}{2}=$$

$$\frac{1^2+2^2+..+n^2}{4}=\frac{n(n+1)(2n+1)}{12}$$

105. 3-misoldagi kabi, $x\ge e$ da $f(x)=\frac{\ln x}{x}$ funksiya kamayuvchi va $\pi>e$ ekanidan $f(\pi)<f(e)\ \Rightarrow\frac{\ln\pi}{\pi}<\frac{\ln e}{e}\ \Rightarrow\ \pi^e<e^\pi$ ekani kelib chiqadi.

106. Biz yana $f(x)=\frac{\ln x}{x}$ funksiyani $x>1$ oraliqda qaraymiz. 3-misoldan ma'lumki, bu funksiya monoton kamayuvchi. U holda $a>b>1$ ga funksiyani ta'sir ettirsak, $f(a)<f(b)\Rightarrow\frac{\ln a}{a}<\frac{\ln b}{b}\Rightarrow a^b<b^a$ (1)

Endi (1) ga funksiyani yana bir marta qo'llasak, $a^{b^a}>b^{a^b}$ hosil bo'ladi.

107. $f(x)=\log_x(x+1)\quad(x\ge 2)$ funksiyani qaraymiz.

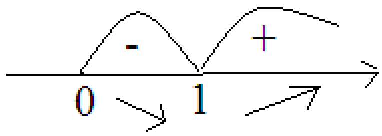
$$f'(x)=\left(\frac{\ln(x+1)}{\ln x}\right)'=\frac{\frac{1}{x+1}\cdot\ln x-\frac{1}{x}\cdot\ln(x+1)}{\ln^2 x}=\frac{\ln x^{\frac{1}{x+1}}-\ln(x+1)^{\frac{1}{x}}}{\ln^2 x}\qquad x>1 \text{ ekanidan}$$

$(x+1)^{\frac{1}{x}}>x^{\frac{1}{x}}>x^{\frac{1}{x+1}}$ tengsizlik o'rinli. Demak, $f'(x)<0$. U holda $f(x)$ funksiya $x\ge 2$ da kamayuvchi. $n+1>n\ \Rightarrow\ f(n+1)<f(n)\ \Rightarrow\log_{n+1}(n+2)<\log_n(n+1)$.

108. Biz $\frac{\mathrm{a}}{\mathrm{b}}=x>0$ deb olib, $f(x)=(1+x)^m+\left(1+\frac{1}{x}\right)^m,\ (x>0)$ funksiyani qaraymiz. Uning hosilasini tekshiramiz:

$$f'(x)=m(1+x)^{m-1}-\frac{m}{x^2}\left(1+\frac{1}{x}\right)^{m-1}=m(1+x)^{m-1}\left(1-\frac{1}{x^2}\cdot\frac{1}{x^{m-1}}\right)=m(1+x)^{m-1}\cdot\frac{x^{m+1}-1}{x^{m+1}},\ (x>0)$$

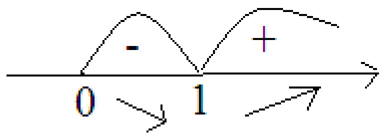
1-hol. $m+1>0$ bo'lsa,



Ya’ni, $0 < x < 1$ da kamayuvchi va $x > 1$ da o‘suvchi. U holda

$f(x) > f(1) \;\Rightarrow\; (1+x)^m + \left(1+\frac{1}{x}\right)^m > 2^{m+1}$

2-hol. $m+1<0$ bo‘lsa,



Bunda ham $0 < x < 1$ oraliqda kamayuvchi va $x > 1$ bo‘lganda o‘suvchi. U holda $f(x) > f(1) \;\Rightarrow\; (1+x)^m + \left(1+\frac{1}{x}\right)^m > 2^{m+1}$

3-hol. $m+1=0 \Rightarrow m=-1$ bo‘lsa,

$(1+x)^m + \left(1+\frac{1}{x}\right)^m = (1+1)^m + (1+1)^m = 2^m + 2^m = 2^{m+1}$ bu holda tenglik belgisi bajariladi. Biz tengsizlikni umumiy hol uchun isbotladik.

109. Masala shartiga ko‘ra $a \le x_i \le b,\ (i=1,2,3,...,n)$. U holda

$0 \ge (x_i - a)(x_i - b)$ tengsizlik o‘rinli. Bu tengsizlikni har bir x_i lar uchun yozib chiqamiz:

$$\begin{cases} x_1(a+b) \ge x_1^2 + ab \\ x_2(a+b) \ge x_2^2 + ab \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_n(a+b) \ge x_n^2 + ab \end{cases} \Rightarrow \begin{cases} a+b \ge x_1 + \dfrac{ab}{x_1} \\ a+b \ge x_2 + \dfrac{ab}{x_2} \\ \dots\dots\dots\dots\dots\dots \\ a+b \ge x_n + \dfrac{ab}{x_n} \end{cases} \Rightarrow n(a+b) \ge (x_1 + x_2 + x_n) + ab\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right) \ge$$

$$\ge 2\sqrt{ab(x_1 + x_2 + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right)} \Rightarrow (x_1 + x_2 + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right) \le \frac{n^2(a+b)^2}{4ab}$$

110.

$$(a+b-c)^2 > 0 \Rightarrow a^2 + b^2 + c^2 + 2(ab - bc - ac) > 0 \Rightarrow \frac{5}{3} + 2(ab - bc - ac) > 0 \Rightarrow bc + ac - ab < \frac{5}{6} < 1$$

oxirgi tengsizlikning ikkala tarafini $abc > 0$ ga bo‘lsak, $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} < \frac{1}{abc}$ tengsizlik hosil bo‘ladi.

111. $P(x)$ ko‘phad n turli ildizga ega ekanligidan uni quyidagicha ko‘rinishda yozish mumkin:

$$P(x) = (x - x_1)(x - x_2)\dots(x - x_n)$$

$$\forall j\in\overline{1,n}\quad P(x_j)=0$$

Endi $m<n$ ekanligidan $\dfrac{Q(x)}{P(x)}$ ni oddiy kasrlarga yoyamiz:

$$\frac{Q(x)}{P(x)}=\sum_{k=1}^{n}\frac{A_k}{x-x_k}\qquad (*)$$

Bu oxirgi tenglikni ikkala tarafini ham $\lfloor P(x)-P(x_j)\rfloor$ ga ko‘paytiramiz, u holda quyidagi ifodaga kelamiz:

$$Q(x)=\sum_{k=1}^{n}\frac{A_k\lfloor P(x)-P(x_j)\rfloor}{x-x_j}\qquad (**)$$

Endi oxirgi tenglikda $x\to x_j$ *da* limitga o‘tsak hamda $\lim\limits_{x\to x_j}\dfrac{P(x)-P(x_j)}{x-x_j}=P'(x_j)$ ekanligini e’tiborga olsak,

$$\lim_{x\to x_j}Q(x)=\lim_{x\to x_j}\sum_{k=1}^{n}\frac{A_k\lfloor P(x)-P(x_j)\rfloor}{x-x_j}\ \Rightarrow\ Q(x_j)=A_jP'(x_j)\Rightarrow A_j=\frac{Q(x_j)}{P'(x_j)}$$

munosabatlarga ega bo‘lamiz. Bularni $(*)$ ga olib borib qo‘yasak va shu $(*)$ ifodani ikkala tarafini ham x ga ko‘paytirsak,

$$\frac{xQ(x)}{P(x)}=\sum_{k=1}^{n}\frac{xQ(x_k)}{P'(x_k)(x-x_k)}\qquad (***)$$

Bu $(***)$ ifodada $x\to\infty$ da limitga o‘tsak, unda biz quyidagi ikki holga ega bo‘lishimiz mumkin:

1-hol. $m=n-1$ bo‘lganda

$$\lim_{x\to\infty}\frac{xQ(x)}{P(x)}=\lim_{x\to\infty}\sum_{k=1}^{n}\frac{xQ(x_k)}{P'(x_k)(x-x_k)}\ \Rightarrow\ 1=\sum_{k=1}^{n}\frac{Q(x_k)}{P'(x_k)}$$

2-hol, $m<n-1$ bo‘lsa,

$$\lim_{x\to\infty}\frac{xQ(x)}{P(x)}=\lim_{x\to\infty}\sum_{k=1}^{n}\frac{xQ(x_k)}{P'(x_k)(x-x_k)}\ \Rightarrow\ 0=\sum_{k=1}^{n}\frac{Q(x_k)}{P'(x_k)}$$ bo‘ladi.

112. Biz $f(x)=(1+x)\ln\left(1+\dfrac{1}{x}\right),\ (x>0)$ funksiyani qaraymiz. Bu funksiya uchun $f'(x)=\ln\left(1+\dfrac{1}{x}\right)-\dfrac{1}{x^2}\cdot\dfrac{1+x}{1+\dfrac{1}{x}}=\ln\left(1+\dfrac{1}{x}\right)-\dfrac{1}{x}\le\dfrac{1}{x}-\dfrac{1}{x}=0$ ekanini ko‘rish qiyin emas. Bu esa funksiyaning berilgan oraliqda monoton kamayuvchi ekanini bildiradi. U holda $n<n+1$ ga funksiyani ta’sir ettirsak,

$$f(n) > f(n+1) \Rightarrow (1+n)\ln\left(1+\frac{1}{n}\right) > (1+n+1)\ln\left(1+\frac{1}{n+1}\right) \Rightarrow \left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$$

tengsizlik hosil bo‘ladi. Shuni isbotlash talab etilgan edi.

113. $f(x)$ funksiya $[a,b]$ segmentda uzluksizligi hamda qavariqligidan shunday $\xi \in [0, b-a]$ topiladiki,

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{a+\xi}{2} + \frac{b-\xi}{2}\right) \geq \frac{1}{2}\left(f(a+\xi) + f(b-\xi)\right)$$

Bu oxirgi munosabatni ξ bo‘yicha $[0, b-a]$ da integrallasak hamda $a+\xi = t$ *va* $b-\xi = z$ o‘zgartirish kiritsak, u holda

$(b-a)f\left(\frac{a+b}{2}\right) \geq \frac{1}{2}\left(\int_0^{b-a} f(a+\xi)d\xi + \int_0^{b-a} f(b-\xi)d\xi\right) = \int_a^b f(x)dx$ kelib chiqadi,

ya’ni talab qilinayotgan munosabatni o‘ng tarafi kelib chiqadi.

Endi $[a,b]$ segmentda quyidagicha bo‘laklash olamiz:

$P = \left\{x_i + i\frac{b-a}{n}; i = \overline{1,n}\right\}$ va $\xi_i = x_i$ desak, $\Delta x_i = \frac{b-a}{n}$ ekanini e’tiborga olsak, u holda $S_P(f) = \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a + i\frac{b-a}{n}\right) = \frac{b-a}{n}\left(\sum_{i=0}^{n-1} f\left(\left(1-\frac{i}{n}\right)a + \frac{i}{n}b\right)\right)$ ekani kelib chiqadi. $f(x)$ funksiya $[a,b]$ segmentda qavariqligidan

$$S_P(f) \geq \frac{b-a}{n}\sum_{i=0}^{n-1}\left(\left(1-\frac{i}{n}\right)f(a) + \frac{i}{n}f(b)\right) = \frac{b-a}{n}\left(\frac{n+1}{2}f(a) + \frac{n-1}{2}f(b)\right)$$

Agar oxirgi munosabatdan $n \to \infty$ limitga o‘tsak, u holda

$\int_a^b f(x)dx \geq \frac{(b-a)}{2}\left(f(a) + f(b)\right)$ kelib chiqadi. Bular talab qilinayotgan munosabatni to‘liq isbotlaydi.

114. Limit ichidagi integral $tg\varphi = \frac{a}{b}, \left(0 < \varphi < \frac{\pi}{2}\right)$ desak,

$I_n = \sqrt{a^2+b^2}\int_0^{2\pi}|\sin(\varphi - nx)|dx$ ko‘rinishga keladi. Integral ostidagi

funksiyaning davri $T = \frac{\pi}{n}$ ga teng bo‘lgani uchun

$$I_n = 2n\sqrt{a^2+b^2}\int_0^{\pi n}|\sin(\varphi - nx)|dx = \left|\begin{matrix} nx - \varphi = t \\ ndx = dt \end{matrix}\right| = 2\sqrt{a^2+b^2}\int_{-\varphi}^{\pi-\varphi}|\sin t|dt = 2\sqrt{a^2+b^2}\int_0^{\pi}|\sin t|dt = 4\sqrt{a^2+b^2}$$

tenglik o‘rinli.

115. Agar $x \neq y$ va $x, y \in N$ bo‘lsa, $x > y$ yoki $x < y$ bo‘lishi aniq. Faraz qilaylik, $x > y$ bo‘lsin. U holda tenglik bajarilishi uchun x ga y bo‘linishi kerak. $x = yn, (n \in N)$ deb olaylik. Tenglikka qo‘ysak,

$$(yn)^y = y^{yn} \Rightarrow yn = y^n \Rightarrow y^{n-1} = n \Rightarrow y = n^{\frac{1}{n-1}} \text{ va } x = n^{\frac{n}{n-1}}$$

ekani kelib chiqadi. Oxirgi tengliklar esa $n = 2$, ya’ni, $x = 2, y = 4$ bo‘lganda bajariladi. Xuddi shunga o‘xshash $x < y$ hol uchun $y = 2, x = 4$ javoblarga ega bo‘lamiz. Demak, yechim $(2,4),(4,2)$ ga teng.

116. 1 sonining n-darajali ildizi quyidagi ko‘rinishda $z^{n-1} + z^{n-2} + ... + z + 1 = (z - w)(z - w^2)...(z - w^{n-1})$, bu yerda, $w = \exp(2\pi i / n)$ ekani bizga ma’lum. U holda $z = 1$ bo‘lsa, $n = (1 - w)(1 - w^2)...(1 - w^{n-1})$ tenglik o‘rinli bo‘ladi.

$1 - w^k = 1 - \exp(2\pi ik / n) = -\exp(\pi ik / n), \left(k = \overline{1, n-1}\right)$ va

$\exp(\pi ik / n) - \exp(-\pi ik / n) = -2i \exp(\pi ik / n) \sin \frac{\pi k}{n}$ ga ko‘ra,

$$\prod_{k=1}^{n-1}(1 - w^k) = 2^{n-1}(-1)^{n-1} i^{n-1} \exp\left(\frac{\pi i}{n}(1 + 2 + ... + n - 1)\right) \prod_{k=1}^{n-1} \sin\frac{\pi k}{n} = 2^{n-1}(-1)^{n-1} i^{n-1} \exp\left(\frac{\pi i}{2}(n-1)\right) \prod_{k=1}^{n-1} \sin\frac{\pi k}{n} =$$

$$= 2^{n-1}(-1)^{n-1} i^{n-1} i^{n-1} \prod_{k=1}^{n-1} \sin\frac{\pi k}{n} = 2^{n-1}(-1)^{n-1}(i^2)^{n-1} \prod_{k=1}^{n-1} \sin\frac{\pi k}{n} = 2^{n-1} \prod_{k=1}^{n-1} \sin\frac{\pi k}{n}$$

u holda $\prod_{k=1}^{n-1} \sin\frac{\pi k}{n} = \frac{n}{2^{n-1}}$ tenglik o‘rinli.

117. *Ko‘rsatma*: a) $x'_n = \frac{1}{n}$ va $x''_n = \frac{1}{n+1}$ ketma-ketliklarga tekis uzluksizlikning Geyne ta’rifidan foydalaning;

b) *ko‘rsatma*: $x'_n = \pi n$ va $x''_n = \pi n + \frac{1}{n}$ ketma-ketliklarga tekis uzluksizlikning Geyne ta’rifidan foydalaning;

c) *ko‘rsatma*: $x'_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$ va $x''_n = \frac{1}{\pi n}$ ketma-ketliklarga tekis uzluksizlikning Geyne ta’rifidan foydalaning.

118. Teskarisini faraz qilamiz. $\det A = 0$ bo‘lsin. U holda $AX = 0$ tenglama noldan farqli yechimga ega.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0 \end{cases}$$

tenglamalar sistemasining noldan farqli yechimlari biror $\alpha_1, \alpha_2, ..., \alpha_n$ sonlari bo‘lsin. Ular tenglamalar sistemasidagi har bir tenglikni ayniyatga aylantiradi, ya’ni:

$$\begin{cases} a_{11}\alpha_1 + a_{12}\alpha_2 + ... + a_{1n}\alpha_n = 0 \\ a_{21}\alpha_1 + a_{22}\alpha_2 + ... + a_{2n}\alpha_n = 0 \\ .... \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + ... + a_{nn}\alpha_n = 0 \end{cases}$$

$\max(|\alpha_1|, |\alpha_2|, ..., |\alpha_n|) = |\alpha_k|$ desak, $a_{k1}\alpha_1 + a_{k2}\alpha_2 + ... + a_{kn}\alpha_n = 0, \quad \alpha_k \neq 0$ ga ko‘ra

$a_{kk}\alpha_{ki} = \sum_{i \neq k}^{n} a_{ki}\alpha_i \Rightarrow |a_{kk}| = \dfrac{\sum_{i \neq k}^{n} |a_{ki}||\alpha_i|}{|\alpha_{ki}|} \leq \left|\sum_{i \neq k}^{n} |a_{ki}|\right| < |a_{kk}|$ o‘rinli. Bu esa ziddiyat.

Demak, $\det A \neq 0$ ya’ni, A matritsa teskarilanuvchi.

119. $[1, n]$ segmaentni teng n ta bo‘lakka bo‘lib, quyidagilarga ega bo‘lamiz:

$$\int_1^n f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + ... + \int_{n-1}^n f(x)dx < f(2)(2-1) + f(3)(3-2) + ... + f(n)(n-(n-1)) =$$
$$= f(2) + f(3) + ... + f(n)$$

$$\int_1^n f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + ... + \int_{n-1}^n f(x)dx > f(1) + f(2) + ... + f(n-1).$$ Demak,

$$f(1) + f(2) + ... + f(n-1) < \int_1^n f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx + ... + \int_{n-1}^n f(x)dx < f(2) + f(3) + ... + f(n)$$

Bularni aniqlashda funksiyaning monoton o‘suvchi ekanidan foydalanildi.

120. $x_n - x_{n-1} = f(x_{n-1}) - f(x_{n-2})$ ga Lagranj teoremasini qo‘llaymiz: $\exists c_n \in (a_n, b_n)$ topiladiki, bu yerda $a_n = \min(x_n, x_{n-1}), b_n = \max(x_n, x_{n-1})$

$|x_n - x_{n-1}| = |f'(c_n)(x_{n-1} - x_{n-2})| \leq q|x_{n-1} - x_{n-2}|$ tengsizlik o‘rinli bo‘ladi. Lagranj teoremasini ketma-ket qo‘llab, quyidagilarni topamiz:

$|x_n - x_{n-1}| \leq q|x_{n-1} - x_{n-2}| \leq q^2|x_{n-2} - x_{n-3}| \leq ... \leq q^{n-2}|x_2 - x_1|$ demak, $\forall n \in N$ uchun

$|x_n - x_{n-1}| \leq \dfrac{q^n|x_2 - x_1|}{q^2}$. Endi $x_1 + \sum_{n=1}^{\infty}(x_{n+1} - x_n)$ funksional qatorni tekshiramiz.

Bu qator $|x_n - x_{n-1}| \leq \dfrac{q^n|x_2 - x_1|}{q^2} \Rightarrow |x_{n+1} - x_n| \leq \dfrac{q^{n+1}|x_2 - x_1|}{q^2} \Rightarrow \sum_{n=1}^{\infty} q^{n+1}\dfrac{|x_2 - x_1|}{q^2}$

Veyershtrass alomatiga ko‘ra yaqinlashuvchi. U holda uning yig‘indisi $S_n = x_n$ yaqinlashuvchi.

121. Teoremani ikki hol uchun isbotlaymiz.

1-hol. l-chekli bo‘lsin.

3-shartga ko‘ra, $\forall\varepsilon>0$ soni uchun $\frac{\varepsilon}{2}$ ga ko‘ra, ketma-ketlikning shunday N nomerli hadi topiladi barcha $n\geq N$ tengsizlikni qanoatlantiruvchi hadlar uchun $\left|\frac{x_{n+1}-x_n}{y_{n+1}-y_n}-l\right|<\frac{\varepsilon}{2}$ tengsizlik bajariladi. Buni boshqacha ko‘rinishda quyidagicha yozish mumkin:

$l-\frac{\varepsilon}{2}<\frac{x_{n+1}-x_n}{y_{n+1}-y_n}<l+\frac{\varepsilon}{2} \quad \Leftrightarrow \left(l-\frac{\varepsilon}{2}\right)(y_{n+1}-y_n)<(x_{n+1}-x_n)<\left(l+\frac{\varepsilon}{2}\right)(y_{n+1}-y_n)$ oxirgi tengsizlikni har bir $n\geq N$ lar uchun yozib chiqamiz:

$$\begin{cases}\left(l-\frac{\varepsilon}{2}\right)(y_{N+1}-y_N)<(x_{N+1}-x_N)<\left(l+\frac{\varepsilon}{2}\right)(y_{N+1}-y_N)\\ \left(l-\frac{\varepsilon}{2}\right)(y_{N+2}-y_{N+1})<(x_{N+2}-x_{N+1})<\left(l+\frac{\varepsilon}{2}\right)(y_{N+2}-y_{N+1})\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \left(l-\frac{\varepsilon}{2}\right)(y_n-y_{n-1})<(x_n-x_{n-1})<\left(l+\frac{\varepsilon}{2}\right)(y_n-y_{n-1})\end{cases}$$

Bularni qoshish natijasida $\left|\frac{x_n-x_N}{y_n-y_N}-l\right|<\frac{\varepsilon}{2}$ (1) ga ega bo‘lamiz.

Endi quyidagi yordamchi ifodalarni qaraymiz:

$$\frac{y_n-y_N}{y_n}=1-\frac{y_N}{y_n}<1 \qquad (2)$$

chunki, N-tayinlangan son va $y_n\to+\infty,\ (n\to+\infty)$.

$\forall\varepsilon>0$ soni uchun $\frac{\varepsilon}{2}$ ga ko‘ra, shunday N_0 nomer topiladiki, barcha $n\geq N_0$ tengsizlikni qanoatlantiruvchi hadlar uchun $\left|\frac{x_N-ly_N}{y_n}-l\right|<\frac{\varepsilon}{2}$ (3)

tengsizlik bajariladi. Chunki, x_N, l, y_N chekli sonlar va $y_n\to+\infty,\ (n\to+\infty)$.
U holda $\forall\varepsilon>0$ soni uchun shunday $\overline{N}=\max\{N,N_0\}$ nomer topiladiki, barcha $n\geq\overline{N}$ lar uchun

$$\left|\frac{x_n}{y_n}-l\right|=\left|\frac{x_N-ly_N}{y_n}+\frac{y_n-y_N}{y_n}\left(\frac{x_n-x_N}{y_n-y_N}-l\right)\right|\leq\left|\frac{x_N-ly_N}{y_n}\right|+\left|\frac{y_n-y_N}{y_n}\right|\left|\frac{x_n-x_N}{y_n-y_N}-l\right|<\frac{\varepsilon}{2}+1\cdot\frac{\varepsilon}{2}<\varepsilon$$

Bu esa $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=l$ ekanini bildiradi.

2-hol. l-cheksiz bo‘lsin.

U holda $\lim\limits_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=\infty \quad \Rightarrow \lim\limits_{n\to\infty}\frac{y_{n+1}-y_n}{x_{n+1}-x_n}=0$. Oxirgi tenglikda quyidagi shartlar o‘rinli:

1) $x_{n+1}>x_n\ (n\in N)$ chunki, limiti $+\infty$ ga teng;

2) $\lim\limits_{n\to\infty} x_n = +\infty, \qquad \lim\limits_{n\to\infty} y_n = +\infty,$;

3) $\lim\limits_{n\to\infty} \dfrac{y_{n+1}-y_n}{x_{n+1}-x_n} = 0$.

Bu teoremaning l-chekli holiga tushadi. Bundan $\lim\limits_{n\to\infty}\dfrac{y_n}{x_n}=0 \;\Rightarrow \lim\limits_{n\to\infty}\dfrac{x_n}{y_n}=\infty$ tenglikning o‘rinli ekanligi kelib chiqadi.

122. $\begin{cases} 2^n = (1+1)^n = C_n^0 + C_n^1 + C_n^2 + C_n^3 + C_n^4 + \ldots \\ 0^n = (1-1)^n = C_n^0 - C_n^1 + C_n^2 - C_n^3 + C_n^5 - \ldots \end{cases}$ bu tengliklarni qo‘shsak,

$2\left(C_n^0 + C_n^2 + C_n^4 + \ldots\right) = 2^n \Rightarrow C_n^0 + C_n^2 + C_n^4 + \ldots = 2^{n-1}$ ga ega bo‘lamiz.

Yuqoridagi birinchi tenglikdan ikkinchisini ayirsak,

$2\left(C_n^1 + C_n^3 + C_n^5 + \ldots\right) = 2^n \Rightarrow C_n^1 + C_n^3 + C_n^5 + \ldots = 2^{n-1}$ ga ega bo‘lamiz.

123. $(1+i)^n = C_n^0 + C_n^1 i + C_n^2 i^2 + C_n^3 i^3 + C_n^4 i^4 + \ldots C_n^{n-1} i^{n-1} + C_n^n i^n$ lekin, ikkinchi tomondan, $(1+i)^n = \left(\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)\right)^n = 2^{\frac{n}{2}}\left(\cos\dfrac{\pi n}{4} + i\sin\dfrac{\pi n}{4}\right)$ bu ikkita ifodani tenglashtirsak,

$$2^{\frac{n}{2}}\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right) = \left(C_n^0 - C_n^2 + C_n^4 - C_n^6 + C_n^8 - C_n^{10} + \ldots\right) + i\left(C_n^1 - C_n^3 + C_n^5 - C_n^7 + \ldots\right)$$

ayniyatga ega bo‘lamiz, bu tenglikning haqiqiy va mavhum qismlarini tenglashtirib, quyidagilarni hosil qilamiz:

$$C_n^0 - C_n^2 + C_n^4 - C_n^6 + C_n^8 - C_n^{10} + \ldots = 2^{\frac{n}{2}}\cos\frac{\pi n}{4} \text{ va } C_n^1 - C_n^3 + C_n^5 - C_n^7 + C_n^9 \ldots = 2^{\frac{n}{2}}\sin\frac{\pi n}{4}$$

124. a) $z = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$ deb belgilab olaylik. U holda quyidagilarga egamiz:

$$1 + z = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = 1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$1 + z^2 = 1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = 1 + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = 1 - \frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$z^{3n} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{3n} = \cos\frac{6\pi n}{3} + i\sin\frac{6\pi n}{3} = 1$$

$$1 + z^{2n} + z^{4n} = \cos\frac{4\pi n}{3} + i\sin\frac{4\pi n}{3} + \cos\frac{8\pi n}{3} + i\sin\frac{8\pi n}{3} = 0, \quad (n \in N)$$

Bularni bilgan holda, quyidagi ifodalarni qaraymiz:

$(1+z)^n = C_n^0 + C_n^1 z + C_n^2 z^2 + C_n^3 z^3 + C_n^4 z^4 + C_n^5 z^5 + C_n^6 z^6 + \ldots$

$(1+z^2)^n = C_n^0 + C_n^1 z^2 + C_n^2 z^4 + C_n^3 z^6 + C_n^4 z^8 + C_n^5 z^{10} + C_n^6 z^{12} + \ldots$ bundan tashqari, 131-misolga ko‘ra, $2^n = C_n^0 + C_n^1 + C_n^2 + C_n^3 + C_n^4 + \ldots$ ekanini hisobga olib va bu uchala tengliklarni qo‘shsak,

$$2^n+(1+z)^n+(1+z^2)^n=2^n+\cos\frac{\pi n}{3}+i\sin\frac{\pi n}{3}+\cos\frac{5\pi n}{3}+i\sin\frac{5\pi n}{3}=3C_n^0+C_n^1(1+z+z^2)+$$
$$+C_n^2(1+z^2+z^4)+C_n^3(1+z^3+z^6)+C_n^4(1+z^4+z^8)+C_n^5(1+z^5+z^{10})+C_n^6(1+z^6+z^{12})+\ldots\Rightarrow$$
$$\Rightarrow 2^n+\cos\frac{\pi n}{3}+\cos\left(2\pi n-\frac{\pi n}{3}\right)+2i\sin\pi n\cdot\cos\frac{2\pi n}{3}=3C_n^0+3C_n^3+3C_n^6+\ldots\Rightarrow$$
$$\Rightarrow C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+\cos\frac{\pi n}{3}\right)$$
tenglikka ega bo‘lamiz;

b) $\begin{array}{l}(1+z)^n=C_n^0+C_n^1z+C_n^2z^2+C_n^3z^3+C_n^4z^4+C_n^5z^5+C_n^6z^6+\ldots\\(1+z^2)^n=C_n^0+C_n^1z^2+C_n^2z^4+C_n^3z^6+C_n^4z^8+C_n^5z^{10}+C_n^6z^{12}+\ldots\end{array}$ tengliklarning birinchisini z ga va ikkinchisini z^2 ga ko‘paytiramiz. Natijada

$\begin{array}{l}z(1+z)^n=C_n^0z+C_n^1z^2+C_n^2z^3+C_n^3z^4+C_n^4z^5+C_n^5z^6+C_n^6z^7+\ldots\\z^2(1+z^2)^n=C_n^0z^2+C_n^1z^4+C_n^2z^6+C_n^3z^8+C_n^4z^{10}+C_n^5z^{12}+C_n^6z^{14}+\ldots\end{array}$ ifodalarga ega bo‘lamiz. Yana $2^n=C_n^0+C_n^1+C_n^2+C_n^3+C_n^4+\ldots$ ekanini hisobga olib va bu uchala tengliklarni qo‘shsak,

$$2^n+z(1+z)^n+z^2(1+z^2)^n=2^n+\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^2\cdot\left(\cos\frac{n}{3}+i\sin\frac{\pi}{3}\right)^n+$$
$$+\left(\cos\frac{n}{3}+i\sin\frac{\pi}{3}\right)^4\cdot\left(\cos\frac{n}{3}+i\sin\frac{\pi}{3}\right)^{5n}=2^n+\cos\frac{(n+2)\pi}{3}+i\sin\frac{(n+2)\pi}{3}+$$
$$+\cos\frac{(5n+4)\pi}{3}+i\sin\frac{(5n+4)\pi}{3}=C_n^0(1+z+z^2)+C_n^1(1+z^2+z^4)+C_n^2(1+z^3+z^6)+$$
$$+C_n^3(1+z^4+z^8)+C_n^4(1+z^5+z^{10})+C_n^5(1+z^6+z^{12})+C_n^6(1+z^7+z^{14})+\ldots\Rightarrow$$
$$\Rightarrow 2^n+\cos\left(2\pi+\frac{\pi(n-4)}{3}\right)+\cos\left(2\pi n-\frac{\pi(n-4)}{3}\right)+2i\sin2(n+1)\pi\cdot\cos\frac{(2n+1)\pi}{3}=$$
$$=3C_n^2+3C_n^5+3C_n^8+\ldots\Rightarrow C_n^2+C_n^5+C_n^8+\ldots=\frac{1}{3}\left(2^n+\cos\frac{\pi(n-4)}{3}\right)$$

c) yuqoridagilar asosida ushbu $\begin{cases}2^n=C_n^0+C_n^1+C_n^2+C_n^3+C_n^4+\ldots\\C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+\cos\frac{\pi n}{3}\right)\\C_n^2+C_n^5+C_n^8+\ldots=\frac{1}{3}\left(2^n+\cos\frac{\pi(n-4)}{3}\right)\end{cases}$

tengliklarga egamiz. Ikkinchi tenglikka uchinchi tenglikni qo‘shib, birinchi tenglikdan ayirsak,

$$2^n-\frac{1}{3}\left(2^n+\cos\frac{\pi n}{3}+2^n+\cos\frac{\pi(n-4)}{3}\right)=\left(C_n^0+C_n^1+C_n^2+C_n^3+C_n^4+\ldots\right)-$$

$$-\left(\left(C_n^0+C_n^3+C_n^6+\ldots\right)+\left(C_n^2+C_n^5+C_n^8+\ldots\right)\right)\Rightarrow C_n^1+C_n^4+C_n^7+\ldots=$$

$$=\frac{1}{3}\left(2^n-2\cos\frac{\pi(n-2)}{3}\cos\frac{2\pi}{3}\right)\Rightarrow C_n^1+C_n^4+C_n^7+\ldots=\frac{1}{3}\left(2^n+\cos\frac{\pi(n-2)}{3}\right)$$

ekanligini topishimiz mumkin.

125. a) 122- va 123-misollarga ko‘ra, $\begin{cases} C_n^0+C_n^2+C_n^4+C_n^6+C_n^8+C_n^{10}+\ldots=2^{n-1} \\ C_n^0-C_n^2+C_n^4-C_n^6+C_n^8-C_n^{10}+\ldots=2^{\frac{n}{2}}\cos\frac{\pi n}{4} \end{cases}$ tengliklar o‘rinli. Bu tengliklarni qo‘shsak,

$$2\left(C_n^0+C_n^4+C_n^8+\ldots\right)=2^{n-1}+2^{\frac{n}{2}}\cos\frac{\pi n}{4}\Rightarrow C_n^0+C_n^4+C_n^8+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{\pi n}{4}\right)$$ ga

ega bo‘lamiz;

b) 122- va 123-misollarga ko‘ra, $\begin{cases} C_n^1+C_n^3+C_n^5+C_n^7+\ldots=2^n \\ C_n^1-C_n^3+C_n^5-C_n^7+\ldots=2^{n-1}\sin\frac{\pi n}{4} \end{cases}$ tengliklar o‘rinli. Bu tengliklarni qo‘shsak,

$$2\left(C_n^1+C_n^5+C_n^9+\ldots\right)=2^n+2^{n-1}\sin\frac{\pi n}{4}\Rightarrow C_n^1+C_n^5+C_n^9+\ldots=\frac{1}{2}\left(2^n+2^{n-1}\sin\frac{\pi n}{4}\right)$$

ekanligi kelib chiqadi;

c) 122- va 123-misollarga ko‘ra, $\begin{cases} C_n^0+C_n^2+C_n^4+C_n^6+C_n^8+C_n^{10}+\ldots=2^{n-1} \\ C_n^0-C_n^2+C_n^4-C_n^6+C_n^8-C_n^{10}+\ldots=2^{\frac{n}{2}}\cos\frac{\pi n}{4} \end{cases}$ ifodadagi birinchi tenglikdan ikkinchisini ayirsak,

$$2\left(C_n^2+C_n^6+C_n^{10}+\ldots\right)=2^{n-1}-2^{\frac{n}{2}}\cos\frac{\pi n}{4}\Rightarrow C_n^0+C_n^4+C_n^8+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\cos\frac{\pi n}{4}\right)$$ ga

ega bo‘lamiz;

d) 122- va 123-misollarga ko‘ra, $\begin{cases} C_n^1+C_n^3+C_n^5+C_n^7+\ldots=2^n \\ C_n^1-C_n^3+C_n^5-C_n^7+\ldots=2^{n-1}\sin\frac{\pi n}{4} \end{cases}$ ifodadagi birinchi tenglikdan ikkinchisini ayirsak,

$$2\left(C_n^3+C_n^7+C_n^{11}+\ldots\right)=2^n-2^{n-1}\sin\frac{\pi n}{4}\Rightarrow C_n^3+C_n^7+C_n^{11}+\ldots=\frac{1}{2}\left(2^n-2^{n-1}\sin\frac{\pi n}{4}\right)$$

ekanligi kelib chiqadi.

6-§. Mustaqil yechish uchun misol va masalalar

1. Agar $f(x)=\int\limits_x^{x+1}\sin t^2dt$ bo‘lsa, $xf(x)$ funksiyani $x\to+\infty$ da yuqori va quyi limitini hisoblang.

2. $f'(x)=f\left(\frac{1}{x}\right)$, $x>0$ tenglamani yeching.

3. $\sum\limits_{n=0}^{\infty}\frac{2^n}{n!}\frac{d^n}{dy^n}\left(\frac{1}{1+\sqrt{y}}\right)\Bigg|_{y=x}=\frac{1}{3}$ tenglamani yeching.

4. Aylananing AB vatarining o‘rtasi bo‘lgan C nuqtadan ikkita KL va MN vatarlar o‘tkazilgan (K va M nuqtalar AB vatardan bir tomonda yotadi). Agar Q nuqta AB va KN hamda P nuqta AB va ML vatarlarning kesishish nuqtalari bo‘lsa, $QC=CP$ tenglikni isbotlang.

5. Agar $a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n$ ko‘phadning barcha ildizlari haqiqiy bo‘lsa, u holda $a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ ko‘phadning ham barcha ildizlari haqiqiy bo‘lishini isbotlang.

6. A matristaning xos qiymatlarini bilgan holda $f(A)$ matristaning determinantini hisoblang. Bunda $f(A)=a_nA^n+a_{n-1}A^{n-1}+...+a_0$ matritsaviy ko‘phad.

7. To‘rt yoqli burchakning ixtiyoriy tekis burchagi qolgan uchtasining yig‘indisidan kichik bo‘lishini isbotlang.

8. $[a,b]$ segmentda aniqlangan $f(x)$ funksiya uzluksiz differensiallanuvchi bo‘lib, $\Delta_n=\int\limits_a^b f(x)dx-\frac{b-a}{n}\sum\limits_{\nu=1}^{n}f\left(a+\nu\frac{b-a}{n}\right)$ bo‘lsa, u holda $\lim\limits_{n\to\infty}n\Delta_n$ ni hisoblang.

9. $f(x)$ va $g(x)$ funksiyalar $[a,b]$ segmentda uzluksiz bo‘lib,
$\left|\int\limits_a^b f(x)g(x)dx\right|^2=\int\limits_a^b|f(x)|^2dx\int\limits_a^b|g(x)|^2dx$ tenglikni qanoatlantirsa, u holda $g(x)\equiv cf(x)$ ekanligini isbotlang.

10. Ushbu $x^2+x+1=py$ tenglama p parametrning cheksiz ko‘p tub qiymatlarida butun yechimlarga ega bo‘lishini isbotlang.

11. ABC uch burchakning AB tomoniga teng tomonli ABC_1 uch burchak shunday yasalganki, uning C va C_1 uchlari AB to‘g‘ri chiziqning bir tomonida joylashgan. Ushbu $|CC_1|^2 = \frac{1}{2}(a^2+b^2+c^2) - 2S\sqrt{3}$ tenglikni isbotlang. Bu yerda $a, b, c,$ uchburchak tomonlari va S ucburchak yuzasi.

12. $SABC$ tetraedrning har bir qirrasi va bu qirralarga qarama-qarshi bo‘lgan qirralar o‘rtalaridan tekisliklar o‘tkazilgan. Bu o‘tkazilgan barcha tekisliklar bir nuqtada kesishishini isbotlang. Bu tasdiqlarni kesishish nuqtasini K bilan belgilab, $\overrightarrow{SK}$ vektorni $\overrightarrow{SA} = \vec{a}$, $\overrightarrow{SB} = \vec{b}, \overrightarrow{SC} = \vec{c}$ vektorlar orqali ifodalang.

13. Quyidagi ayniyatlarni isbotlang.

a) $\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \frac{1}{2}\sum_{i,j}(a_k b_j - a_j b_k)^2$;

b) $\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n}\left(a_k \frac{a_1 b_1 + ... + a_n b_n}{a_1^2 + ... + a_n^2} - b_k\right)^2$.

14. $\overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty}\sup_{k>n}(x_k)$ va $\underline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty}\inf_{k>n}\{x_k\}$ munosabatlarni isbotlang.

15. $ctg^2\frac{\pi}{2n+1} + ctg^2\frac{2\pi}{2n+1} + ... + ctg^2\frac{n\pi}{2n+1} = \frac{n(2n-1)}{3}$ ayniyatni isbotlang.

16. $\forall n \in N$ uchun $\frac{\pi^2}{6}\left(1 - \frac{6n+1}{(2n+1)^2}\right) < 1 + \frac{1}{2^2} + ... + \frac{1}{n^2} < \frac{\pi^2}{6}\left(1 - \frac{1}{(2n+1)^2}\right)$ qo‘sh tengsizlikning bajarilishini isbotlang.

17. $\int_0^{+\infty}\frac{\sin x}{x}dx$ xosmas integralni hisoblang.

18. $\lim_{x\to\infty}\frac{\int_0^x \cos t^2 dt}{x}$ limitni hisoblang.

19. $\{x_n\}$ ketma-ketlik quyidagicha berilgan: $x_1 = \frac{1}{2}$, $x_{n+1} = x_n - x_n^2$, $n \ge 1$. U holda $\lim_{n\to\infty} n x_n = 1$ ekanligini isbotlang.

20. $f(x)$ funksiya $[a;b]$ segmentda musbat, uzluksiz va monoton kamayuvchi bo‘lsa, u holda quyidagi tengsizlikni isbotlang.

$$\int_a^b x f^2(x)dx \le \frac{1}{2}\left(\int_a^b f(x)dx\right)^2$$

21. Agar natural n sonini $n=\frac{x^2-1}{y^2-1}$ ko‘rinishida tasvirlab bo‘lmasa, uni “Sehrli” son deb ataymiz, bu yerda $x, y \in N$ va $x, y > 1$. “Sehrli” sonlar cheksiz ko‘pmi?

22. $A=(a_{ij})$ $a_{ij} \in R$ bo‘lsa, $|\det A|^2 \le \prod_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2$ ekanini isbotlang.

23. ABC uch burchak ichidan ixtiyoriy M nuqta olingan. Agar uch burchakning yuzi S ga teng bo‘lsa, quyidagini isbotlang:

$$4S \le AM \cdot BC + BM \cdot AC + CM \cdot AB$$

24. $f(x)$ va $g(x)$ ko‘phadlar berilgan. Agar $\deg f < \deg g$ va $g(x_i)=0,\ i=\overline{1,n}$ bo‘lib, $x_1, x_2, ..., x_n$ lar turli haqiqiy sonlar bo‘lsa, $\frac{f(x)}{g(x)}=\sum_{j=1}^{n}\frac{f(x_j)}{g'(x_j)}\cdot\frac{1}{x-x_j}$ tenglikni isbotlang.

25. Ellipsning yarim o‘qlarini toping. $45x^2-72xy+80y^2=4$

26. Quyidagi ketma-ketliklar uchun x_n ning faqat n bog‘liqlik formulasini keltirib chiqaring.

1) $x_n=5x_{n-1}-4x_{n-2}$ bu yerda, $x_1=1, x_2=19$;

2) $x_n=x_{n-1}+x_{n-2}$ bu yerda, $x_1=1, x_2=2$ bu ketma-ketlik Fibonachchi sonlarini ifodalaydi;

3) $x_n=6x_{n-1}-9$ bu yerda, $x_1=3$.

27. λ ning shungay qiymatlarini topingki, natijada

$$rang\begin{pmatrix}3 & 1 & 1 & 4\\ \lambda & 4 & 10 & 1\\ 1 & 7 & 17 & 3\\ 2 & 2 & 4 & 3\end{pmatrix}$$ eng kichik bo‘lsin.

28. $x_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n$ ketma-ketlikning yaqinlashuvchi ekanini isbotlang.

29. (Urganch, 2011) $\{x_n\}$ ketma-ketlik uchun $x_{m+n} \le x_m + x_n$ $(m, n \in N)$ o‘rinli bo‘lsa, $\left\{\frac{x_n}{n}\right\}$ yaqinlashuvchi ketma-ketlik isbotlang.

30. Agar $A_n=\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}^n=\|a_{ij}(n)\|$ bo‘lsa, quyidagi $\frac{a_{12}(n)}{a_{22}(n)}$ nisbatning $n\to\infty$ da limiti mavjudligini isbotlang va uni toping.

7-§. Olimpiadalarda taklif qilingan testlar

1. ABC to‘g‘ri burchakli uchburchakda AB va BC katetlar mos ravishda 3 va 4 ga teng. BC katetda shunday D nuqta olinganki, BD:DC=3:2 ga teng.

AB va AD+AC vektorlarning skalyar ko‘paytmasini toping.

A) 12; B) 42; C) 9; D) 24.

2. Muntazam tetraedrning qirrasi 1 ga teng. Tetraedr ichidagi ixtiyoriy nuqtadan uning yoqlarigacha bo‘lgan masofalar yig‘indisini toping.

A) 1; B) $\frac{\sqrt{6}}{3}$; C) $\frac{\sqrt{6}}{2}$; D) $\sqrt{12}$.

3. $f_1(x)=x^2$ va $f_1(x)=x-1$ funksiyalar grafiklari orasidagi eng qisqa masofani toping.

A) 1; B) $\frac{\sqrt{3}}{4}$; C) $\frac{3\sqrt{2}}{3}$; D) $\frac{3\sqrt{2}}{8}$.

4. $1+2\cdot 3+3\cdot 3^2+...+2010\cdot 3^{2009}+2011\cdot 3^{2010}+2012\cdot 3^{2011}$ yig‘indini hisoblang.

A) $1006\cdot 3^{2012}-1$; B) $\frac{1}{4}\left(4023\cdot 3^{2012}+1\right)$; C) $\frac{1}{4}\left(1006\cdot 3^{2012}+1\right)$; D) 2012.

5. $\int\frac{dx}{\sin(x+a)\sin(x+b)}$ ni hisoblang:

A) $\frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|+C$; B) $\frac{2}{\sin(a-b)}\ln\left|\frac{\cos(x+a)}{\cos(x+b)}\right|+C$;

C) $\frac{2}{\sin(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|+C$; D) $\frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right|+C$.

6. Quyidagi limitni hisoblang.

$$\lim_{x\to 0}\left(\frac{m}{1-(x+1)^m}-\frac{n}{1-(x+1)^n}\right)$$

A) 0; B) mn; C) $\frac{m+n}{mn}$; D) $\frac{m-n}{2}$.

7. Qatorning yaqinlashish radiusini toping.

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$$

A) $R=4$; B) $R=\frac{1}{4}$; C) $R=3$; D) $R=\frac{1}{3}$.

8. Tengsizlikni yeching. $|x^2-2x-3|<3x-3$

A) (2; 4) ; B) (2; 5); C) (1; 6); D) (3; 4).

9. $y=\dfrac{2}{x-3}$ egri chiziqning assipmtotalarini toping.

A) $y=0$; B) $x=0$; C) $x=3$; D) $x=3,\ y=0$.

10. ABC ucburchakning B va C burchaklari ayirmasi $\dfrac{\pi}{2}$ ga teng. Agar b va c tomonlari yig‘indisi k ga, A uchidan tushirilgan balandlik h ga teng bo‘lsa, uchburchakning C burchagini toping.

A) $2\arcsin 2hk$; B) $\arccos\dfrac{k}{h}$; C) $\dfrac{1}{2}\arcsin\dfrac{2h}{k}$;

D) $\dfrac{1}{2}\arcsin\dfrac{2h(h+\sqrt{h^2+k^2}}{k^2}$.

11. ABC ucburchakning balandliklari $AA_1=h_a$, $BB_1=h_b$, va C burchagining bissektrisasi $CD=l$ ga teng bo‘lsa, uchburchakning C burchagini toping.

A) $\arccos\dfrac{lh_b}{h_a^2+h_d^2}$; B) $arctg\,\dfrac{2lh_a}{h_a^2+h_d^2}$; C) $2\arcsin\dfrac{h_a h_b}{l(h_a+h_d)}$;

D) $\dfrac{1}{2}\arcsin\dfrac{h_a}{l(h_a+h_d)}$.

12. ABC yucburchakning B va C burchaklari nisbati 3:1, A burchagi bissektrisasi uchburchakning yuzini 2:1 nisbatda bo‘lsa, uchburchak burchaklarini toping.

A) $\dfrac{\pi}{3},\dfrac{\pi}{3},\dfrac{\pi}{3}$; B) $\dfrac{\pi}{2},\dfrac{\pi}{2},0$; C) $\dfrac{\pi}{4},\dfrac{\pi}{2},\dfrac{\pi}{4}$; D) $\dfrac{\pi}{3},\dfrac{\pi}{6},\dfrac{\pi}{2}$.

13. ABC uchburchakda AM va BN – bissektrisalari O-nuqtada kesishadi. Agar AO:OM=$\sqrt{3}:1$, BO:ON=$1:(\sqrt{3}-1)$ bo‘lsa, uchburchakning A, B, C burchaklarini toping.

A) $\dfrac{\pi}{4},\dfrac{\pi}{3},\dfrac{7\pi}{2}$; B) $\dfrac{\pi}{4},\dfrac{\pi}{2},\dfrac{3\pi}{4}$; C) $\dfrac{\pi}{3},\dfrac{\pi}{2},\dfrac{\pi}{6}$; D) $\dfrac{\pi}{6},\dfrac{\pi}{6},\dfrac{2\pi}{3}$.

14. Agar $f(x)=x^2+14x+42$ bo‘lsa, $f(f(f(f(x))))=0$ tenglamani yeching.

A) ildizi yo‘q; B) $\pm\sqrt[16]{7}-7$; C) $\pm\sqrt[32]{7}+7$; D) $\pm\sqrt[16]{7}+7$.

15. $A=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}$ bo‘lsa, $\det e^{A}$ ni toping.

A) e^3; B) $2e^4$; C) e^4; D) $-2e^2$.

16. $A=\begin{pmatrix}4 & -15 & 6\\ 1 & -4 & 2\\ 1 & -5 & 3\end{pmatrix}$ bo‘lsa, $\det(\ln A)$ ni toping.

A) $\ln 4$; B) $\ln 2$; C) 1; D) 0.

17. $A=\begin{pmatrix}\pi-1 & 1\\ -1 & \pi+1\end{pmatrix}$ bo‘lsa, $\det\sin A$ ni toping.

A) 0; B) 1; C) $\sin 1$; D) -1.

18. (a,b,c) nuqtadan koordinata o‘qlaridan a, b va c uzunlikdagi kesmalar ajratuvchi tekislikkacha bo‘lgan masofani toping.

A) $\frac{abc}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}$; B) $\frac{abc}{2\sqrt{a^2b^2+a^2c^2+b^2c^2}}$; C) $\frac{4abc}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}$;

D) $\frac{2abc}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}$.

19. $\lim\limits_{n\to\infty}\sin^2\left(\pi\sqrt{n^2+n}\right)$ ni hisoblang.

A) 0; B) 1; C) $\frac{1}{2}$; D) $\frac{1}{3}$.

20. $\lim\limits_{n\to\infty}\underbrace{\sin\sin\ldots\sin x}_{n\ ta}$ ni hisoblang.

A) 1; B) 0; C) $\frac{1}{2}$; D) $\frac{1}{3}$.

21. $\lim\limits_{n\to\infty}\sin\left(\pi\sqrt{n^2+1}\right)$ ni hisoblang.

A) 0; B) 1; C) $\frac{1}{2}$; D) $\frac{1}{3}$.

22. $\lim\limits_{x\to 0} x\cdot\left[\frac{1}{x}\right]$ ni hisoblang.

A) 0; B) 1; C) $\frac{1}{2}$; D) $\frac{1}{3}$.

23. $\lim\limits_{x\to 0} x\cdot\sqrt{\cos\frac{1}{x}}$ ni hisoblang.

A) $\frac{1}{2}$; B) ∞; C) 1; D) 0.

24. Agar ABC uchburchakda A burchagi 120^0 bo‘lsa, $\cos 3A+\cos 3B+\cos 3C$ ni toping.

A) 0; B) 1; C) $\frac{1}{2}$; D) $\frac{3}{2}$.

25. Agar n soni 1 dan katta bo‘lgan natural son bo‘lsa, $\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+...\cos\frac{2n\pi}{n}$ yig‘indi nimaga teng?

A) 0; B) 1; C) $\frac{1}{2}$; D) $\frac{3}{2}$.

26. Qaysi katta? a=127^{23} 513^{18}=b.

A) a>b; B) a<b; C) a=b; D) 3a=b.

27. Integralni hisoblang. $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}}$ $(n>1)$.

A) $\frac{\pi}{\sin\frac{\pi}{n}}$; B) $\frac{n\pi}{\sin\frac{\pi}{n}}$; C) $\frac{\pi}{\pi n\sin\frac{\pi}{n}}$; D) $\frac{\pi}{n\sin\frac{\pi}{n}}$.

28. Integralni hisoblang. $\int_0^{+\infty} x^{2n}e^{-x^2}dx$ $(n - musbat\ butun\ son)$.

A) $\frac{(2n-1)!!}{2^{n+1}}\sqrt{\pi}$; B) $\frac{(2n-1)!!}{2^{n}}\sqrt{\pi}$; C) $\frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}$; D) $\frac{(2n-1)!}{2^{n+1}}\sqrt{\pi}$.

29. Hisoblang. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}\cdot\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}$.

A) $\frac{\pi}{2}$; B) $\frac{\pi}{3}$; C) $\frac{\pi}{4}$; D) $\frac{\pi}{6}$.

30. Hisoblang. $\int_{(1,\pi)}^{(2,\pi)} \left(1-\frac{y^2}{x^2}\cos\frac{y}{x}\right)dx+\left(\sin\frac{y}{x}+\frac{y}{x}\cos\frac{y}{x}\right)dy$ $(x\neq 0)$.

A) $\frac{\pi}{2}+1$; B) $\pi+1$; C) $\frac{\pi}{3}+1$; D) $2\pi+1$.

31. A va B lar n- tartibli kvadrat matritsalar bo‘lib, $\det(AB)\neq 0$ bo‘lsa, u holda quyidagilardan qaysi biri noto‘gri.

1) $(AB)^{-1}=B^{-1}A^{-1}$ 2) $(AB)^T=B^TA^T$ 3) $rangA=rangA^T$ 4) $(AB)^T=A^TB^T$

A) 1 va 2; B) 1 va 4; C) 4; D) 3.

32. Quyidagi matritsaviy tenglama yechimining determinantini toping.

$$\begin{pmatrix} a & 1 & a \\ -1 & a & 1 \\ a & -1 & a \end{pmatrix} X = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 4 & 5 \end{pmatrix}$$

A) $4a$; B) $-2a$; C) $-\frac{1}{4}a$; D) $\frac{1}{4a}$.

33. Determinatni hisoblang. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2009 & 2010 & 2011 & 2012 \\ 2009^2 & 2010^2 & 2011^2 & 2012^2 \\ 2009^3 & 2010^3 & 2011^3 & 2012^3 \end{vmatrix}$

A) 6; B) 0; C) 12; D) 3.

34. Limitni hisoblang. $\lim\limits_{x\to 0,\, y\to 0}(x^2+y^2)^{x^2y^2}$

A) 1; B) 0; C) $e-1$; D) 2.

35. $y=0,\ x^2+y^2=1$ chiziqlar orasidagi yuzani toping.

A) π; B) $\frac{\pi}{4}$; C) $\frac{\pi}{2}$; D) $\frac{\pi}{6}$.

36. Agar α,β,γ $x^3+px+q=0$ tenglamaning ildizlari bo‘lsa, $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ ni hisoblang.

A) 1; B) aniqlab bo‘lmaydi; C) 0; D) $\alpha+\beta+\gamma$.

37. $\int \frac{1}{\sqrt{x-x^2}}dx$ integralni hisoblang.

A) $\frac{1}{2\sqrt{2-x^2}}+c$; B) $\arcsin\frac{x}{2}+c$; C) $\arccos\frac{x}{2}+c$; D) $\arcsin(2x-1)+c$.

38. $x^2+y^2+ay=0$ $(a>0)$ aylana markazidan $y=2(a-x)$ to‘g‘ri chiziqcha bo‘lgan masofani toping.

A) $\frac{a\sqrt{5}}{4}$; B) $\frac{a\sqrt{3}}{2}$; C) $\frac{a\sqrt{5}}{2}$; D) $\frac{\sqrt{5}}{2a}$.

39. Agar $f(x)=x^2+12x+30$ bo‘lsa, $f(f(f(f(f(x)))))=0$ tenglamani yeching.

A) ildizlari ko‘p, umumiy yechimini yozib bo‘lmaydi;
B) ildizlari yo‘q;
C) $\pm\sqrt[32]{6}-6$;
D) $\pm\sqrt[32]{6}+6$.

40. Berilgan $f(x)$ funksiya uchun $f_2(x)=f(f(x)),\ f_3(x)=f(f(f(x)))$ *va hokazo* $f_k(x)=f(f(f\ldots f(x))\ldots)$ deb belgilaymiz. Agar $f(x)=\frac{1}{\sqrt{1+x^2}}$ bo‘lsa, u holda $\lim\limits_{n\to\infty}\left(\sqrt{n}f_n(x)\right)$ ni hisoblang $(x>0)$

A) mavjud emas; B) -1; C) 0; D) ∞.

41. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsga ichki chizilgan kvadrat yuzasini toping.

A) a,b; B) a^2+b^2; C) $\frac{ab}{2(a^2+b^2)}$; D) $\frac{4a^2b^2}{a^2+b^2}$.

42. Agar $a+b=1$ tenglik o‘rinli bo‘lsa, a^4+b^4 ifodaning eng kichik qiymatini toping.

A) $\frac{1}{64}$; B) $\frac{1}{8}$; C) $\frac{1}{128}$; D) $\frac{1}{256}$.

43. Limitni hisoblang.

$\lim\limits_{n\to+\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n}\right)$

A) 2; B) $\ln 2$; C) $\ln 3$; D) 0.

44. λ va μ larning qanday qiymatlarida $\bar{a}=(\lambda+1,\ 1,\ 2),\ \bar{b}=(\mu,\ 2,\ \lambda-1)$ vektorlar uchun $[\bar{a}\times\bar{b}]=\bar{0}$ bo‘ladi? Bu yerda $[\bar{a}\times\bar{b}]$ vektor ko‘paytmani bildiradi.

A) $\lambda=-5,\ \mu=0$; B) $\lambda=5,\ \mu=2$; C) $\lambda=5,\ \mu=12$; D) $\lambda=-5,\ \mu=12$.

45. Agar $\sum\limits_{i=1}^{n}a_i=\sum\limits_{i=1}^{n}b_i=1$ bo‘lsa, $\sum\limits_{i=1}^{n}\frac{a_i^2}{a_i+b_i}$ ning eng kichik qiymatini toping.

A) $-\infty$; B) aniqlab bo‘lmaydi; C) n; D) 1.

46. $y=2x-6,\ y=0,\ x=5$ chiziqlar bilan chegaralangan figuraning OY o‘qi atrofida aylanishdan hosil bo‘lgan jismning hajmini toping.

A) 34π; B) $34\frac{2}{3}\pi$; C) 33π; D) 36π.

47. $y''+m^2y=0$ differensial tenglama yechimining nollari orasidagi masofani hisoblang.

A) π; B) $\frac{\pi}{m}$; C) $m\pi$; D) aniqlab bo‘lmaydi.

48. Agar $x_1^2+x_2^2\le 1$ bo‘lsa, $f(\psi_1,\psi_2)=\max(\psi_1x_1+\psi_2x_2)$ funksiyani toping.

A) $|\psi_1|+|\psi_2|$; B) $\sqrt{\psi_1^2+\psi_2^2}$; C) $\psi_1^2+\psi_2^2$; D) $|\psi_1|\cdot|\psi_2|$.

49. Agar $|x_1|\le 1,\ |x_2|\le 1$ bo‘lsa, $f(\psi_1,\psi_2)=\max(\psi_1x_1+\psi_2x_2)$ funksiyani toping.

A) $\psi_1^2+\psi_2^2$; B) $|\psi_1|+|\psi_2|$; C) $|\psi_1|\cdot|\psi_2|$; D) $\sqrt{|\psi_1|+|\psi_2|}$.

50. Limitni hisoblang. $\lim\limits_{n\to+\infty}\frac{\sqrt[n]{n!}}{n}$

A) 0; B) $\frac{1}{e}$; C) 1; D) e.

51. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = ?$

A) $(b-a)(c-a)(c-b)$; B) $a^2b^2c^2$; C) 0; D) $b^2 - a^2 + c^2$.

52. $\int_{-1}^{1} \frac{x}{\cos^3 x} dx = ?$

A) 1; B) $\frac{2}{\cos 1}$; C) $\frac{1}{\cos 2}$; D) 0.

53. $f(y) = x^2 \sin x - x^3 \cos x$ funksiya uchun $x \to 0$ da $f(x) \sim kx^m$ bo‘lsa, k va m ni toping.

A) $k = \frac{1}{3},\ m = 5$; B) $k = 1,\ m = 3$; C) $k = \frac{1}{3}, m = 3$; D) $k = -\frac{1}{2}, m = 2$.

54. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ellipsga $M(5;\ 4)$ nuqtadan o‘tkazilgan urinmalarning urinish nuqtalari orasidagi masofani toping.

A) 5; B) 16; C) $\sqrt{38}$; D) $\sqrt{41}$.

55. $\lim_{t \to 0} \int_{-1}^{1} \frac{t}{t^2 + x^2} \cos x dx = ?$

A) 1; B) 0; C) π; D) $2\cos 1$].

56. Uchlari $A(1;2;3)$ $B(5;2;1)$ $C(0;4;4)$ nuqtalarda bo‘lgan uch burchakning yuzasini hisoblang.

A) $\sqrt{21}$; B) 5; C) 1; D) $\sqrt{12}$.

57. $rang\left\{\begin{pmatrix} 1 & 1 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right\} = ?$

A) 1; B) 2; C) 3; D) 0.

58. $A = \begin{pmatrix} 2001 & 2002 \\ 2003 & 2004 \end{pmatrix}$; $B = \begin{pmatrix} 2005 & 2006 \\ 2007 & 2008 \end{pmatrix}$.

$\det(A^{-1}B^2) = ?$

A) 0; B) -2; C) 1; D) 2009.

59. $\varphi(n)$ orqali n dan kichik va n bilan o‘zaro tub sonlar sonini belgilasak, $\varphi(1996)$ ni toping.

A) 1995; B) 1996; C) 996; D) 1001.

60. 13^{20} ni 19 ga bo‘lganda qoldiqni toping.

A) 5; B) 12; C) 13; D) 17.

61. $x^2+2xy+3y^2-6x-4y+16=0$ egri chiziq markazining koordinatalari topilsin.
A) $(3;-1)$; B) $(1{,}5;-0{,}5)$; C) $(3{,}5;-0{,}5)$; D) $(3;\ 2)$.

62. $\lim\limits_{n\to\infty}\dfrac{1^3+2^3+3^3+...+n^3}{n^4}=?$
A) ∞; B) 0; C) 0.25; D) $\dfrac{1}{3}$.

63. $\lim\limits_{n\to\infty}\sqrt[n]{2n+1}$ hisoblang.
A) 1; B) 0 ; C) ∞; D) limit mavjud emas.

64. ABCD muntazam tetraedr qirrasi 4 ga teng P, Q, M, N lar mos ravishda AB, AC, BD, CD qirralarining o‘rtalari. $\vec{AD}\cdot\vec{AB}+\vec{PQ}\cdot\vec{MN}=?$
A) 0; B) 12; C) 4; D) 16.

65. $\begin{cases}2p^2+k^2-2pk=25\\ 2pq-q^2=25\end{cases}$ bo‘lsa, $\dfrac{p+q}{k^2}$ ni toping.
A) aniqlab bo‘lmaydi; B) 1; C) $0{,}25$; D) $\pm 0{,}4$.

66. $A=\begin{pmatrix}3 & 4\\ 5 & 7\end{pmatrix}$ bo‘lsa, quyidagi ko‘phadlarning qaysi birisi uchun $P(A)=0$ bo‘ladi?
A) $P(\lambda)=\lambda^2+2\lambda+9$;
B) $P(\lambda)=\lambda^2+10\lambda+1$;
C) $P(\lambda)=\lambda^2-10\lambda+1$;
D) $P(\lambda)=\lambda^2-10\lambda+15$.

67. $u_1:\dfrac{x-2}{4}=\dfrac{y+1}{1}=\dfrac{z-1}{-1}$, $u_2:\dfrac{x+u}{2}=\dfrac{y-2}{-2}=\dfrac{z+2}{-3}$ to‘g‘ri chiziqlarga perpendikular $\vec{u}$ vektorni toping.
A) $\vec{u}=(-5,10,-10)$; B) $\vec{u}=(-2,5,5)$;
C) $\vec{u}=(4,1,-1)$; D) $\vec{u}=(2,-2,-3)$.

68. λ ning qanday qiymatlarida $x^3+16x+\lambda=0$ kubik tenglama barcha ildizlari haqiqiy bo‘ladi.
A) $\lambda<-12$ B) ixtiyoriy qiymatlarida;
C) $\lambda=0$ D) hech bir qiymatlarida.

69. ABC uchburchakning medianalari 3, 4 va 5 ga teng. Uch burchaklarning yuzini hisoblang.
A) 5; B) 10; C) 8; D) 25.

70. ABC uchburchakning yuzi 48 ga teng. Uning 10 va 12 teng medianalari bu uchburchakning uchta uchburchakka va bitta to‘rtburchakka ajratildi. Hosil bo‘lgan to‘rtburchakning yuzini toping.
A) 12; B) 8; C) 10; D) 16.

71. $f = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ kvadrat formani kanonik ko‘rinishga keltirganda nechta kvadrat qatnashadi?
A) 4 ; B) 3; C) 1; D) 2.

72. $f(x) = \dfrac{\cos x}{1 + \sin x}$ bo‘lsa, $f'\left(\dfrac{\pi}{2}\right) = ?$

A) 0; B) $-\dfrac{1}{2}$; C) 1; D) $\dfrac{1}{2}$.

73. $f(x) = \left|x^2 - 3x - 4\right|, \quad f'(0) = ?$

A) 3; B) -3; C) mavjud emas; D) 4.

74. a ning qanday qiymatlarida quyidagi $f(x)$ funksiya uzluksiz bo‘ladi?

$$f(x) = \begin{cases} \dfrac{x^2 \sin x + tg^3 x}{x^3}, & x > 0 \\ e^{x+a}, & x \le 0 \end{cases}$$

A) $a = 0$; B) $a = 1$;
C) hech bir qiymatida uzluksiz bo‘lmaydi; D) $a = \ln 2$.

75. $A = \begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}$ matritsa barcha xos qiymatlarining yig‘indisini toping.
A) –5; B) 1; C) –1; D) 5.

76. Quyidagi parallel tekisliklar orasidagi masofani toping.
$\Pi_1 : 4x + 3y - 5z - 8 = 0$
$\Pi_2 : 4x + 3y - 5z + 12 = 0$
A) 20; B) 4; C) $2\sqrt{2}$; D) 10.

77. $y^2 = 2x$ parabolaga muntazam uchburchak ichki chizilgan. Bu uchburchakning yuzini hisoblang.
A) $3\sqrt{3}$; B) $12\sqrt{3}$; C) 12; D) $6\sqrt{3}$.

78. Quyidagini hisoblang.

$$\left(\frac{1 + i\sqrt{3}}{1 - i}\right)^{20}$$

A) $2^9(1-i\sqrt{3})$; B) $2^{10}(1-i\sqrt{3})$; C) $-2^9(1-i\sqrt{3})$; D) $2^9(1+i\sqrt{3})$.

79. $\int\limits_0^{100\pi} \sqrt{1-\cos 2x}\,dx = ?$

A) 100; B) 0; C) $100\sqrt{2}$; D) $200\sqrt{2}$.

80. Quyidagi yig‘indini hisoblang.

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots \qquad \text{bu yerda } (|x|<1)$$

A) $\ln(1+x)$; B) $\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$; C) $\ln(1-x)$; D) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

81. Quyidagi qator yig‘indisini toping.

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$$

A) $\sqrt{2}-1$; B) 1; C) $\sqrt{2}$; D) $1-\sqrt{2}$.

82. Quyidagi funksiyaning eng kichik qiymatini toping.

$$f(x,y)=2^x+2^{1-x-y}+2^y$$

A) 2; B) 6; C) 4; D) 5.

83. $2^{99}+2^9$ sonini 49 ga bo‘lgandagi qoldiqni toping.

A) 0; B) 40; C) 39; D) 1.

84. Hosilani hisoblang. $y(x)=x^x+arctg(x^2+1)$.

A) $x^x(1+2\ln x)+\frac{2x}{1+(x^2+1)^2}$;

B) $x^x(1+\ln x)+\frac{x}{1+(x^2+1)^2}$;

C) $x^x(1+\ln x)+\frac{2x}{1+(x^2+1)^2}$;

D) $x^x(1+\ln x)+\frac{x}{2(1+(x^2+1))^2}$.

85. Qaysi (a,b,c) uchlik uchun $\begin{cases} x+y=a \\ x^2+y^2=b \\ x^3+y^3=c \end{cases}$ tenglamalar sistemasi yechimga ega?

A) $(1;1;1)$; B) $(-1;1;-1)$; C) $(1;-1;-1)$; D) $(-1;1;1)$

86. $\lim\limits_{x\to 0-0} \dfrac{1}{e^{\frac{2}{x}}+1}$ limitni hisoblang.

A) 0,5; B) 1; C) -1; D) ∞.

87. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n^2+3n+2)}$ qator yig'indisini hisoblang.

A) $\ln 2$; B) $\ln 2 - \dfrac{1}{2}$; C) e; D) $\dfrac{1}{4}$.

88. $\int\limits_{-5}^{5} x^5 |x| dx$ integralni hisoblang.

A) 0; B) 5; C) -7; D) $\dfrac{2}{7} \cdot 5^7$.

89. Agar $y = F(x)$ funksiya $y = f(x)$ funksiya uchun boshlang'ich funksiya bo'lsa, u holda $y = 5f(-5x)$ funksiyaning boshlang'ich funksiyasini toping.

A) $-5F(-5x)+C$; B) $\dfrac{-1}{5}F(-5x)+C$; C) $F(-5x)+C$; D) $-F(-5x)+C$.

90. $\int \dfrac{dx}{\sin x}$ integralni hisoblang.

A) $\ln\dfrac{1+\cos x}{1+\sin x}+C$; B) $\dfrac{1}{2}\ln\dfrac{1-\cos x}{1+\cos x}+C$; C) $\ln\dfrac{1+\cos x}{\sin x}+C$; D) $\ln\sin x + C$.

91. $|\sin x| > |\cos x|$ tengsizlikni yeching.

A) $\left(\dfrac{\pi}{4}+\pi n; \dfrac{3\pi}{4}+\pi n\right)$; B) $A)\left(\dfrac{\pi}{4}+2\pi n; \dfrac{3\pi}{4}+2\pi n\right)$; C) $A)\left(-\dfrac{\pi}{4}+\pi n; \dfrac{\pi}{4}+\pi n\right)$;

D) $A)\left(\pi n; \dfrac{\pi}{4}+\pi n\right)$.

92. $\lim\limits_{n\to\infty} \dfrac{1^3+2^3+...+n^3}{5n^4}$ limitni hisoblang.

A) $\dfrac{1}{10}$; B) 2; C) $\dfrac{1}{20}$; D) 0.

93. $\int \dfrac{dx}{e^x-1}$ aniqmas integralni hisoblang.

A) $\ln\left|\dfrac{e^x-1}{e^x}\right|+C$; B) $\ln\left|\dfrac{e^x}{e^x-1}\right|+C$; C) $\ln\left|\dfrac{e^x+1}{e^x-1}\right|+C$; D) $\ln\left|\dfrac{e^x-1}{e^x+1}\right|+C$.

94. $f(x) = (x^2+x)^{100}$ ko'phadning barcha koeffitsiyentlari yig'indisini toping.

A) 200; B); 3 C) 2^{100}; D) 2^{200}.

95. $f(x)$ ko‘phadni $x-c$ ga bo‘lgandagi qoldiqni toping.
A) 0; B) $f(0)$; C) $f(-c)$; D) $f(c)$.

96. Agar $\vec{a},\vec{b},\vec{c}$ lar birlik vektorlar bo‘lib, $\vec{a}+\vec{b}+\vec{c}=0$ bo‘lsa, $\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{a}\vec{c}$ ning qiymatini toping.
A) 0; B) $\frac{3}{2}$; C) $-\frac{3}{2}$; D) -1.

97. $\vec{a}$ vektor $\vec{b}=(1;2;3)$ va $\vec{c}=(-2;4;1)$ vektorlarga perpendikulyar bo‘lib, ushbu $\vec{a}\cdot(\vec{i}-2\vec{j}+\vec{k})=-6$ shartni qanoatlantirsa, $\vec{a}$ ni toping.
A) $\vec{a}=(1;2;-1)$; B) $\vec{a}=\left(5;\frac{7}{2};-4\right)$; C) $\vec{a}=(5;1;-4)$; D) $\vec{a}=\left(5;\frac{7}{4};-4\right)$.

98. $(x^2-x-3)^4$ ifoda yoyilmasida x ning juft darajalari oldidagi koeffitsiyentlar yig‘indisini toping.
A) 40; B) 41; C) 42; D) 43.

99. $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{10}$ matritsa elementlari yig‘indisini toping.
A) 12; B) 13; C) 14; D) 15.

100. $z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ bo‘lsa, $z^4=?$
A) -1; B) 4; C) -2; D) -4.

101. $f(x)=\sum_{i=1}^{5}\frac{|x-i|}{x-i}$ funksiyaning qiymatlar sohasi nechta butun sondan iborat?
A) 4; B) 5; C) 6; D) 7.

102. $\arccos x$ funksiyaning juft va toq funksiyalar yig‘indisi ko‘rinishida yozib, juft qismini ko‘rsating.
A) $\frac{\pi}{2}$; B) $\frac{\arccos x+\arcsin x}{2}$; C) $\arcsin x$; D) $\frac{\arccos x-\arcsin x}{2}$.

103. Agar $M-$chegaralangan ketma-ketliklar to‘plami, $C-$yaqinlashuvchi ketma-ketliklar to‘plami, C_0-limiti nolga teng bo‘lgan yaqinlashuvchi ketma-ketliklar to‘plami bo‘lsa, l_1-absolut qiymati bilan yig‘iluvchi ketma-ketliklar to‘plami $\left(\sum_{k=1}^{\infty}|x_k|\right)$ bo‘lsa, u holda quyidagilardan qaysilari noto‘gri:
1) $l_1\subset C_0$;
2) $C_0\subset C$;

3) $M \subset l_1$;
4) $M \subset C_0 \subset l_1$;
5) $M \subset C$.
A) 1, 5;   B) 1, 2, 3; C) 1, 3, 4;   D) 3, 4, 5.

104. λ- soning $\bar{a} = (\lambda + 2\sqrt{2}, 0, 2)$, $\bar{b} = (4, \lambda - 2\sqrt{2}, 0)$, $\bar{c} = (0, \lambda, 1)$ vektorlar komplanar bo‘ladigan barcha qiymatlari ko‘paytmasini toping.
A) $2\sqrt{2} - 2$; B) -8; C) $2\sqrt{2} + 2$; D) 0.

105. Agar $(2\sqrt{2} - x)(2\sqrt{2} + x) = \left(\frac{x}{x-1}\right)^2$ bo‘lsa, u holda $\frac{2x^2}{x-1}$ ni toping.
A) 2; B) $\frac{8}{2\sqrt{2} - 1}$; C) 16;   D) 8.

106. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{k^2 + 3k + 1}{(k+2)!}$ limitni hisoblang.
A) 2;   B) 1; C) 0;   D) 8.

107. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ellips berilgan. $(-2;1)$ nuqta orqali shu nuqtada teng ikkiga bo‘linuvchi vatar o‘tkazilsin.
A) $8x - 9y + 25 = 0$;   B) $8x + 9y + 25 = 0$;   C) $9x - 8y + 25 = 0$; D) $9x + 8y + 24 = 0$.

108. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ellipsning $x - y + 1 = 0$ to‘g‘ri chiziqqa parallel bo‘lgan urinmalarini toping.
A) $x - y + \sqrt{3} = 0,\ \ x - y - \sqrt{3} = 0$; B) $x - y + 3 = 0,\ \ x - y - 3 = 0$;
C) $x - y + \sqrt{10} = 0,\ \ x - y - \sqrt{10} = 0$; D) $x - y + 10 = 0,\ \ x - y - 10 = 0$.

109. AC va BD diagonallari o‘zaro perpendikulyar bo‘lgan $ABCD$ to‘rtburchakka radiusi 2 ga teng bo‘lgan aylana tashqi chizilgan. Agar $AB = 3$ bo‘lsa, CD ni toping.
A) $\sqrt{3}$; B) $\sqrt{10}$; C) $\sqrt{5}$; D) $\sqrt{7}$.

110. Hisoblang.
$$\lim_{n\to\infty}\left(1 + \sqrt{2} + \frac{(\sqrt{2})^2}{2!} + \frac{(\sqrt{2})^3}{3!} + \frac{(\sqrt{2})^4}{4!} + \frac{(\sqrt{2})^5}{5!} + \ldots + \frac{(\sqrt{2})^n}{n!}\right).$$
A) e;   B) e^{-1};   C) $e^{\frac{\sqrt{2}}{2}}$;   D) $e^{\sqrt{2}}$.

111. Agar $f(x)=\frac{x}{\sqrt{1+x^2}}$ bo‘lsa, $f(f(f(f...f(2008)...)))$ in hisoblang.

A) 0; B) $\frac{1}{\sqrt{2009}}$; C) $\frac{2008}{\sqrt{1+2008^3}}$; D) $\frac{2008}{\sqrt{1+2008^2}}$.

112. Limitni hisoblang. $\lim\limits_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3}$

A) 1; B) $\frac{1}{2}$; C) -2; D) $-\frac{1}{2}$.

113. a parametrning qanday qiymatlarida $f(x)=\begin{cases} e^x, & x<0 \\ a+x, & x\geq 0 \end{cases}$ funksiya uzluksiz bo‘ladi?

A) $a=e$; B) $a=1$; C) $a=0$; D) $a=-1$.

114. Limitni hisoblang. $\lim\limits_{x\to e}\frac{e^{e^e}-x^{e^e}}{x-e^e}$

A) $e^{e^{e+1}}(e-1)$; B) 0; C) $e^{e^e}(e-1)$; D) $e^{e^e}(e+1)$.

115. $\int(2^x+3^x)^2dx$ aniqmas integralni hisoblang.

A) $\frac{6^x}{\ln 6}+2+C$; B) $\frac{4^x}{\ln 4}+2\frac{6^x}{\ln 6}+\frac{9^x}{\ln 9}+C$; C) $\frac{4^x}{\ln 5}+\frac{6^x}{\ln 6}+\frac{9^x}{\ln 9}+C$;
D) $\frac{4^x}{\ln 5}+\frac{6^x}{\ln 6}+\frac{9^x}{\ln 9}+C$.

116. $\int \mathrm{tg}x dx$ aniqmas integralni hisoblang.

A) $-\ln|\cos x|+C$; B) $\ln|\cos x|+C$; C) $\ln|\sin x|+C$; D) $-\ln|\sin x|+C$.

117. $\int_0^2|1-x|dx$ hisoblang.

A) e; B) 2008; C) 1; D) 0.

118. Limitni hisoblang.

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+...+\frac{n}{n^2+n^2}\right)$$

A) $\frac{\pi}{4}$; B) $\frac{\pi}{6}$; C) $\frac{\pi}{2}$; D) $\frac{3\pi}{4}$.

119. Limitni hisoblang.

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$$

A) $\frac{2}{3}(2\sqrt{2}-1)$; B) $\frac{2}{3}(2\sqrt{2}-2)$; C) $\frac{2}{3}(3\sqrt{2}-1)$; D) $\frac{2}{3}(2\sqrt{2}-3)$.

120. Hisoblang. $\frac{d}{dx}\int\limits_0^{x^2}\sqrt{1+t^2}dt$

A) $\sqrt{1+x^4}$; B) $2x\sqrt{1+x^4}$; C) $2x+\sqrt{1+x^4}$; D) 1.

121. Qator yig‘indisini toping.

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)\cdot(3n+1)}+\ldots+$$

A) $\frac{1}{2}$; B) $\frac{1}{3}$; C) $\frac{1}{6}$; D) 1.

122. Ko‘paytmani toping.

$$\prod_{n=2}^{\infty}(1-\frac{1}{n^2})$$

A) $\frac{1}{2}$; B) 2; C) 0; D) 1.

123. Hisoblang. $\cos\frac{\pi}{5}-\cos\frac{2\pi}{5}$

A) $-\frac{1}{2}$; B) $\frac{1}{2}$; C) $\frac{1}{4}$; D) $\frac{1}{8}$.

124. $\varphi(x)=\sin^6 x+\cos^6 x$ funksiyaning eng katta qiymatini toping.

A) 1; B) $\frac{1}{4}$; C) $\frac{1}{2}$; D) $\frac{1}{3}$.

125. Hisoblang. $\ln \mathrm{tg}1^0\cdot\ln \mathrm{tg}2^0\cdot\ln \mathrm{tg}3^0\cdot\ldots\cdot\ln \mathrm{tg}89^0$

A) 1; B) 0; C) $\frac{1}{2}$; D) $-\frac{1}{2}$.

126. ABC uchburchakda AC tomonga tushirilgan balandligi 2 ga AB tomoni 5 ga, ABC uchburchakka tashqi chizilgan aylana radiusi 5 ga teng bo‘lsa, BC tomonining uzunligini toping.

A) 2; B) 5; C) 4; D) $\sqrt{21}$.

127. Gipotenuzasi c ga teng bo‘lgan to‘g‘ri burchakli uchburchakning o‘tkir burchaklarining kosinuslari yig‘indisi q ga teng. Uchburchakning yuzini hisoblang.

A) $\frac{c^2(q^2-1)}{4}$; B) cq; C) c^2q^2; D) $\frac{c^2(q^2+1)}{4}$.

128. Uchlari A(1;1), B(2;2), C(3;1) bo‘lgan uchburchakning medianalar kesishgan nuqtaning koordinatalarini toping.

A) (1; 2) ; B) (2; $\frac{4}{3}$) ; C) ($\frac{1}{3}$; 3) ; D) (2;1) .

129. $|\vec{a}|=2$, $|\vec{b}|=3$, $\vec{a}$ va $\vec{b}$ orasidagi burchak 60^0 bo‘lsa, $2\vec{a}(3\vec{a}-4\vec{b})$ in hisoblang.

A) 24; B) 0; C) 12; D) -12.

130. Uchlari A(0; 0), B(2; 0), C(0; -4) bo‘lgan uchburchakka tashqi chizilgan aylana markazining koordinatalari topilsin.

A) (1; -3); B) (1; -2); C) (-2; 1); D) (3; 4).

131. Yon tomoni a gat eng bo‘lgan teng yonli uchburchakning asosi qanday bo‘lganda, uning yuzi eng katta bo‘ladi?

A) $a\sqrt{2}$; B) $a\sqrt{3}$; C) $2a$; D) a.

132. $\vec{a}$ vektor $\vec{b}(1;2;3)$ vektorga kollinear bo‘lib, $\vec{a}\cdot\vec{b}=28$ bo‘lsa, $|\vec{a}|$ ni toping.

A) $\sqrt{44}$ B) $\sqrt{56}$ C) 3 D) 14

133. M(-3; -5) nuqtadan o‘tib $7x+4y+3=0$ to‘g‘ri chiziqqa parallel bo‘lgan chiziq tenglamasini tuzing.

A) $-3x-5y+1=0$; B) $7x+4y+41=0$; C) $4x+7y+41=0$; D) $4x+7y-41=0$.

134. a va b ning qanday qiymatlarida $ax-2y-1=0$ va $6x-4y-b=0$ to‘g‘ri chiziqlar kesishmaydi?

A) $a=3;\ b=2$; B) $a=3;\ b\neq 2$; C) $a\neq 3;\ b\neq 2$; D) $a=2;\ b=3$;

135. $x^2+y^2-4x+8y-6=0$ egri chiziq bilan chegaralangan figuraning yuzini hisoblang.

A) 26π; B) 10π; C) 6π; D) 8π.

136. Matritsaning rangini toping. $\begin{pmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{pmatrix}$

A)1; B) 2; C) 3; D) 4;

137. Matritsaning rangini toping. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

A) -1; B) 3; C) 4; D) 1.

138. Ikkinchi darajali haqiqiy o‘zgaruvchili $f(x)$ ko‘phad topingki, uning uchun $f(1)=8,\ f(-1)=2,\ f(2)=14$ o‘rinli bo‘lsin.

A) x^2+3x+4; B) x^2-2x+3; C) x^2+3x-4; D) x^2-4x-3.

139. Quyidagi determinantni hisoblang.

$$\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}$$

A) $\sin(\alpha-\beta)+\sin(\gamma+\beta)$; B) 1;

C) $\sin(\alpha+\gamma)$; D) $\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$.

140. Determinantni hisoblang. $\begin{vmatrix} 1 & 0 & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1 \end{vmatrix}$

A) 3; B) -1; C) -2; D) 2.

141. Hisoblang. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2$

A) $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ B) $\begin{pmatrix} 2 & 0 & 3 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ C) $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix}$ D) $\begin{pmatrix} 9 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

142. $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ matritsaga teskari matritsaning determinantini hisoblang.

A) -2; B) -$-\frac{1}{2}$; C) -3; D) -5.

143. $A=\begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix}$ matritsaning xos qiymatlari yig‘indisini toping.
A) -3; B) 7; C) -10; D) 1.
144. A matritsaning rangi 5 ga teng bo‘lsa, A^T matritsaning rangini toping.
A) 10; B) 2,5; C) 5; D) -5.
145. Agar x^3-mx^2+nx-5 ko‘phad x^2-1 ko‘phadga qoldiqsiz bo‘linsa, $n+m$ in hisoblang.
A) -6; B) 4; C) 3; D) -5.
146. x^5-7x^2-5x-8 ko‘phadni $x-2$ ko‘phadga bo‘lgandagi qoldig‘ini toping.
A) -8; B) -5; C) -7; D) -14.
147. i va k larning qanday qiymatlarida $a_{62}a_{i5}a_{33}a_{k4}a_{46}a_{21}$ ko‘paytma oltinchi tartibli determinantga minus ishora bilan kiradi.
A) $i=2,k=6$; B) $i=1,k=5$; C) $i=6,k=2$; D) $i=5,k=1$.
148. $\det A=2, \det B=3$ bo‘lsa, $\det(A^{-1}\cdot B^2)$ in hisoblang.
A) $\frac{11}{2}$; B) 2; C) 0; D) $\frac{9}{2}$.
149. $\sum_{n=1}^{2008} i^n$ hisoblang. (i – mavhum birlik).
A) 0; B) 2008; C) $1004i$; D) 1004.
150. Sistemani tekshiring va uning umumiy yechimini λ parametrga bog‘liq ravishda aniqlang.

$$\begin{aligned} \lambda x_1+x_2+x_3&=1 \\ x_1+\lambda x_2+x_3&=1 \\ x_1+x_2+\lambda x_3&=1 \end{aligned}$$

A) $\lambda=-1,3; x_1=x_2=x_3=\frac{1}{\lambda}; \lambda=-1$ da $x_1=3-x_2-x_3, \lambda=3$ da $x_2=x_1-3x_2$;
B) $\lambda=1,-2; x_1=x_2=x_3=\frac{1}{\lambda+2}; \lambda=1$ da $x_1=1-x_2-x_3, \lambda=-2$ da sistema yechimga ega emas;
C) to‘g‘ri javob yo‘q;
D) $\lambda=4,2; x_1=x_2=x_3=\frac{4}{\lambda+1}; \lambda=4$ da $4x_1=1-x_2-x_3, \lambda=-2$ da sistema yechimga ega emas.

Test javoblari

№	0	1	2	3	4	5	6	7	8	9
0		A	B	D	B	D	D	B	B	D
1	D	C	D	C	B	C	D	A	D	B
2	B	A	B	D	B	A	B	D	A	C
3	B	C	C	A	A	C	C	D	C	C
4	D	D	B	B	C	A	B	B	B	B
5	B	A	D	A	D	C	A	B	B	C
6	D	D	C	A	B	D	C	A	A	C
7	D	A	B	A	D	D	C	B	A	C
8	B	D	A	A	C	A	B	D	A	D
9	A	A	C	A	C	D	C	B	B	A
10	D	C	A	C	B	D	A	C	C	B
11	D	C	B	B	B	B	A	D	A	A
12	B	B	A	B	B	B	C	A	B	B
13	B	A	B	B	B	A	B	C	A	D
14	C	A	B	A	C	A	D	D	D	A
15	B									

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MUNDARIJA

Qaydlar uchun

Qaydlar uchun

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MASALA QANDAY YECHILADI?

(Uslubiy qo‘llanma)

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